UNIVERSITY of RWANDA

COLLEGE OF SCIENCE AND TECHNOLOGY

SCHOOL OF SCIENCE

DEPARTMENT OF MATHEMATICS

MODERN THEORY OF SEPARABLE HAMILTONIAN SYSTEMS

Maniraguha Jean de Dieu

Master of Science

in

Applied Mathematics

2020

UNIVERSITY of RWANDA

COLLEGE OF SCIENCE AND TECHNOLOGY

SCHOOL OF SCIENCE

DEPARTMENT OF MATHEMATICS

MODERN THEORY OF SEPARABLE HAMILTONIAN SYSTEMS

By

Maniraguha Jean de Dieu

Student number: 214001999

Thesis Submitted in Partial Fulfillment of the Academic Degree of

Master of Science

in

Applied Mathematics

(Mathematical Modeling and Scientific Computing)

Supervisors: Dr. Krzysztof MARCINIAK; Dr. Célestin KURUJYIBWAMI

Kigali-Rwanda

June 9, 2020

Declaration

I, **Jean de Dieu MANIRAGUHA**, hereby declare and confirm that this research thesis entitled "*MODERN THEORY OF SEPARABLE HAMILTONIAN SYSTEMS*" is my original work and it has never been submitted or published anywhere by any person to any University or other institution of higher learning for academic publication for any award. This thesis was conducted under the supervision of **Dr. Krzysztof MARCINIAK** from Linköping University, Sweden and **Dr. Célestin KURUJYIBWAMI** from University of Rwanda-College of Science and Technology, Rwanda.

June 9, 2020

MANIRAGUHA Jean de Dieu

Dedication

To my family and my supervisors ***Dr. Krzysztof MARCINIAK****;* ***Dr. Célestin KURUJYIBWAMI.***

Abstract

In this thesis we present a modern, geometric (invariant) approach to the theory of autonomous (i.e. not explicitly depending on time) Hamiltonian integrable systems that are separable in the sense of Hamilton-Jacobi theory. We start with an elementary exposition of theory of Poisson manifolds, since Hamiltonian systems are dynamical systems defined on Poisson manifolds, and for that purpose we need to introduce some basic definitions from differential geometry, such as smooth (real) manifolds and tensor fields on smooth manifolds.

In the next part of the thesis we introduce the reader to some basic facts about Hamiltonian systems and then about completely integrable (in the sense of Liouville) systems, presenting the famous Liouville-Arnold theorem describing the geometry of completely integrable systems and the existence of the so called action-angle variables that linearize the flow of any completely integrable system in a neighbourhood of any invariamt tori. We explain the basic ideas of Hamilton-Jacobi theory of finding solutions to a Hamiltonian system by looking for additively separable solutions of the related Hamilton-Jacobi equation. We put this theory in the modern language of separation relations on Poisson manifolds. Next we focus on a very important class of separable systems that are called Stäckel systems, that is separable systems generated by separation relations that are linear in Hamiltonians and quadratic in momenta (and on their subclass generated by a single hyperelliptic separation (spectral) curve). Of special interest to us are Stäckel systems of Benenti type that is Stäckel systems with the Stäckel matrix in the form of a Vandermonde matrix.

For Stäckel systems of Benenti type and generated by a single separation curve, we investigate the problem of finding canonical maps that turn the Hamiltonians of these systems into polynomials. We present two such maps: to the so called Viète coordinates (which was previously known) and to what we call in this thesis Newton coordinates, i.e. coordinates generated by sums of powers of separation coordinates. The second possibility was discovered only very recently and in this thesis we present our own, alternative proof of Buchstaber and Mikhailov result; we also analyze in detail all the features of Benenti Hamiltonians in Newton coordinates, such as their

Killing tensors, their special conformal Killing tensor and their pseudoriemannian metric. These results are new.

We finish the thesis by pointing out some interesting directions for the future research.

The thesis is richly furnished with examples.

Keywords and phrases: Hamiltonian systems, integrable systems, Hamilton-Jacobi theory, sepration relations, separation curves, Stäckel separable systems, Benenti class,

AMS 2010 Subject Classification: 70H20, 53D17, 37K10, 70H06, 70G45

Acknowledgment

First of all I owe my gratitude to my supervisor **Dr. Krzysztof MARCINIAK**, Linköping University, Sweden, for his invaluable continuous support, for his time, his explanation, for sharing his ideas with me, for his corrections, encouragement and guidance. Without him this thesis would not be achieved.

A special thank of mine goes to my co-supervisor **Dr. Célestin KURUJYIBWAMI** who introduced me to some basic key elements of this research thesis. You have helped me by explaining me the tasks I was about to do, by suggesting improvements to my work and by helping me establishing Skype connection with my supervisor. I want to thank you for the collaboration we have done together, which I hope will be continued after this thesis is completed.

I express my gratitude to International Science Programme (ISP) through the Eastern African Universities Mathematics Programmes (EAUMP) for the financial support during my master study. In particular, I would like to express gratitude towards Prof. Jean Marie NTAGANDA, Dr. Wellars BANZI, Dr. Denis NDANGUZA, Dr. Froduald MINANI, Mrs. Emelyne UMUNOZA GASANA for their support, their advice and that they inspired my interest in mathematics.

I would like to express my deepest gratitude towards the staff of our department and in particular to all lecturers who contributed to my academic master studies. I would like to forward my deepest gratitude to everyone who contributed to this thesis by prayers and ideas.

Finally, I am deeply grateful to my family and my friends for their support. Thank you to my beloved mother: there are no enough ways I can express my gratitude for the love and support that you have given me since what seems like forever.

Contents

1 Introduction

This chapter includes the theoretical background, the motivation, the objectives of the thesis, the problem statement and the structure of the thesis.

1.1 Theoretical background

The method of separation variables is a common name for a family of related methods of finding particular or general solutions of systems of ordinary and differential equations. It is a vast field of mathematics that develops very rapidly (see for example the review article [47]). In this thesis we focus on separation of variables for systems of ordinary differential equations that can be written as so called Hamiltonian dynamical systems. In the classical formulation, a Hamiltonian system is a system of $2n$ ordinary differential equations of the form

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \quad i = 1, \ldots, n, \tag{1.1}$$

where q_i are coordinates on some smooth manifold $\mathcal{Q}$ of dimension n and p_i are the conjugated momenta in the cotangent bundle $\mathcal{M} = T^*\mathcal{Q}$ so that the coordinates on the cotangent bundle $\mathcal{M} = T^*\mathcal{Q}$ are $(q, p) = (q_1, \ldots, q_n, p_1, \ldots, p_n)$ while $H = H(q, p)$ is a smooth real valued function (Hamiltonian) on $\mathcal{M}$ (for a coordinate-free formulation of Hamiltonian systems, see Chapter 2 of this thesis; a thorough reference in the subject is [1]). Hamiltonian systems were introduced by William Rowan Hamilton in 1835 [34] and since then have been very intensively studied [1, 3, 33, 43, 53]. Solving of (1.1) in the so called quadratures (i.e. a finite sequence of elementary algebraic operations (including calculation of roots), inverting of functions

and integrating of known functions of one variable) is seldom possible. In 1842 (see his lectures published in [36]) C.G. Von Jacobi proposed therefore another approach. Instead of solving (1.1) directly in the (q, p)- coordinates one can attempt to solve the related PDE:

$$H\left(q_1, \ldots, q_n, \frac{\partial W}{\partial q_1}, \cdots, \frac{\partial W}{\partial q_n}\right) = a_1 \tag{1.2}$$

called today the *Hamiltonian-Jacobi (HJ) equation*, for the unknown function $W = W(q, a)$ where $a = (a_1, \ldots, a_n)$ are real parameters. If a solution $W(q, a)$ is found then it is easy to show (this is the content of the famous Jacobi theorem, see Chapter 4 of this thesis) that it is also a generating function for a canonical transformation $(q, p) \to (b, a)$ such that in the new variables (b, a) the flow of the system (1.1) is linear:

$$\frac{db_i}{dt} = \delta_{i1}, \quad \frac{da_i}{dt} = 0, \quad i = 1, \ldots, n$$

(where δ_{ij} is the Kronecker delta) and can therefore be immediately integrated. In reality solving (1.2) can be even more difficult than solving (1.1); however, this method can be successfully applied in the situations when there exists an additively separable solution of the form:

$$W = \sum_{i=1}^{n} W_i(q_i, a), \quad \det\left(\frac{\partial^2 W}{\partial q_i \partial a_j}\right) \neq 0 \ ,$$

where W_i depends on q_i only (and in general on all parameters a_j). In such a case we say that the Hamilton-Jacobi equation (1.2) (or simply the system (1.1)) is separable. The technique of separation of variables is known to be one of the most powerful methods for integration of equations of motion for dynamical systems, see [6, 16, 47] and references therein.

Separable Hamiltonian systems were studied very extensively from the time of their discovery. Among major milestones we can mention the work of Paul Stäckel, who in 1891 [52] initiated the program of classifying the separable systems presenting the conditions for separability of the HJ equations in orthogonal coordinates (this result is now called Stäckel theorem, see Theorem 4.3), the works of Eisenhart [23, 24, 25], Levi-Civita [41] and also the geometric formulation of the theory, with the use of

special conformal Killing tensors, developed by Sarlet and Crampin in [20] and by Sergio Benenti and others [14, 15, 16, 17, 18].

Recently, a new approach, initiated in the famous paper of Sklyanin [49], has arisen. Instead of trying to work out whether a given Hamiltonian systems is separable or not in some coordinate system, which is often a tedious task, we can ask the following question: what is the most natural way of obtaining separable systems? The answer is that the most efficient way to obtain separable systems is to define them directly in their separation coordinates with the help of the so called separation relations, that is n algebraic relations of the form

$$\varphi_i(\lambda_i, \mu_i, a_1, \ldots, a_n) = 0, \quad i = 1, \ldots, n, \tag{1.3}$$

where each function φ_i depends on only one pair (λ_i, μ_i) of canonical coordinates and on (possibly all) parameters $a_1, \ldots, a_n$. Assuming that relations (1.3) can be almost globally solved with respect to the parameter a_i we obtain $a_i = H_i(\lambda, \mu), \ \ i = 1, \ldots, n$, where the right hand sides $H_i(\lambda, \mu)$ can be treated as Hamiltonians on our manifold $\mathcal{M}$. By construction, all the Hamiltonians H_i Poisson-commute and moreover the coordinates (λ, μ) are separation coordinates for all the Hamiltonians H_i [10]. This way we also gain a deep geometric insight into these systems.

One of the most important class of separable systems are the so called Stäckel systems. Stäckel systems belong to the class of integrable and separable Hamiltonian systems with quadratic in momenta constants of motion which makes them especially suitable for describing the physical systems of classical mechanics. Stäckel systems are obtained from separation relations linear in the Hamiltonians H_i and quadratic in momenta μ_i. In this thesis we restrict our attention to Stäckel systems defined using separation relations of the form

$$\sum_{j=1}^{n} S_{ij} H_j = \frac{1}{2} f_i(\lambda_i) \mu_i^2 + \sigma_i(\lambda_i), \quad i = 1, \ldots, n, \tag{1.4}$$

where n represents the number of degrees of freedom of the systems in a $2n$-dimensional phase space (i.e. manifold $\mathcal{M}$), f_i, and σ_i are $2n$ arbitrary, unless otherwise specified, smooth functions of one variable and where S_{ij} is a so called *Stäckel matrix*, that is

a matrix with row i depending only on the variable λ_i. Solving (1.4) with respect to H_j we obtain n Poisson-commuting Stäckel Hamiltonians in separation coordinates $(\lambda, \mu) = (\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n)$.

An important subclass of Stäckel systems are the so called *Benenti systems*, generated by the separation relations of the form

$$\sum_{j=1}^{n} \lambda_i^{n-j} H_j = \frac{1}{2} f_i(\lambda_i)\mu_i^2 + \sigma_i(\lambda_i), \quad i = 1, \ldots, n \tag{1.5}$$

in which the Stäckel matrix S_{ij} is the Vandermonde matrix $S_{ij} = \lambda_i^{n-j}$. We will focus on these systems in the later part of this thesis.

1.2 Motivation of the thesis

In majority of the textbooks treating the theory of Hamiltonian systems separable in the sense of Hamilton-Jacobi theory a dominant approach is the classical one, in which one considers only one Hamiltonian with one corresponding time (evolution parameter) and it is a priori not clear whether this Hamiltonian is separable or not. A more modern mathematical description, starting from the level of separation relations, is thus needed, which motivated writing this thesis. We shall not forget, however, that as many other excellent mathematical theories, also this modern separation theory has its roots in the profound problems of physics, in this case classical mechanics.

1.3 Objectives of the thesis

The objective of the thesis is to present a *modern, comprehensive and self-sustained exposition of theory of classical Hamiltonian systems that are additively separable in the sense of Hamilton-Jacobi theory.* The exposition starts with an elementary background presentation of differential-geometric notions necessary for our purposes. Then I present a general theory of separable Hamiltonian systems and later I shift focus to separable systems of Stäckel- and generalized Stäckel-type. The presented exposition is novel in that it stems from the notion of a Hamiltonian system defined

by separation curves (or more generally: separation relations) rather than directly by Hamiltonian functions, as it is still done in an overwhelming majority of the books and articles on the subject.

1.4 Problem statement

Stäckel systems, written in their separation coordinates (λ, μ), are complicated functions with metric tensors depending rationally on position variables and as such are difficult to work with and difficult to relate with the classical separable systems known in literature. The problem that is considered in thesis is to investigate possible canonical transformations to variables in which these Hamiltonians have a polynomial form.

One such transformation has been previously known (leading to the so called Viète coordinates), another method, transformation to what is called Newton coordinates, has been discovered very recently in [19]. In this thesis we investigate the map between both coordinate systems; as both maps are point transformations on the configurational manifold of the system, the map between both sets of variables must be a point transformation too. Using these new results we will then present a new proof of the results presented in [19].

1.5 Structure of the thesis

This thesis is organized as follows: the introductory chapter contains a basic theoretical background, motivation of the thesis, the objective of the thesis, the problem statement and a short description of the structure of the thesis. Chapter 2 is devoted to description of Hamiltonian systems and their properties. In order to be able to do that in a coordinate-free manner we also describe several basic concepts of differential geometry, necessary for our exposition. We want to stress that this part is only an outline of the necessary concepts; the details can be found in numerous references in the text. Chapter 3 introduces the concept of a completely integrable system and the well-known Liouville-Arnold theorem. Chapter 4 contains modern exposition of

theory of separable systems, including the Hamilton-Jacobi equation, the Levi-Civita condition, orthogonal separation for natural Hamiltonians, Stäckel theorem, separation relations and Jacobi theorem from the point of view of separation relations. Chapter 5 is devoted to Stäckel separable systems and in particular to Stäckel systems of Benenti type. In this chapter we discuss two maps that turn Benenti Hamiltonians into polynomial form. Chapter 6 contains discussion of this work and conclusion.

2 Hamiltonian systems

In this chapter we will present the definition and some of the basic properties of Hamiltonian systems. Hamiltonian systems are systems of ODE's of evolutionary type that exists on Poisson manifolds, that is smooth manifolds equipped with a Poisson structure. Thus, before we introduce the concept of a Hamiltonian system, we give a short review of smooth manifolds and Poisson structures. The exposition in this chapter is only an outline and can *in no way* be treated as a complete introduction into the world of smooth manifolds and Poisson structures. An interested reader should follow the references in order to get a more complete picture of these introductory matter, for example [22, 40, 43, 51].

2.1 Smooth manifolds

2.1.1 Definition of a smooth manifold and some examples

An n-dimensional smooth manifold is a Hausdorff topological space [39] that is locally homeomorphic to an open set in $\mathbb{R}^n$; in a sense it means that a manifold at each point looks locally as a piece of $\mathbb{R}^n$. This concept in a natural way generalizes the concept of a smooth hypersurface in $\mathbb{R}^n$. Unless more structures are introduced on a manifold, every manifold can be realized (embedded) as a smooth surface in $\mathbb{R}^n$, a theorem due to Whittaker [40].

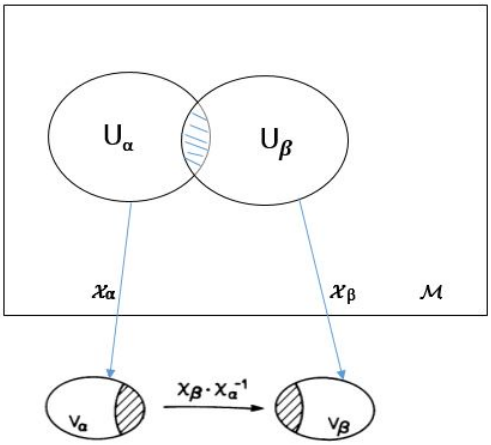
Definition 2.1. *An n-dimensional smooth real manifold is a Hausdorff topological space $\mathcal{M}$, together with a collection $\{U_\alpha\}_{\alpha\in I}$ (where I is an index set) of open subset $U_\alpha \subset \mathcal{M}$ and homeomorphisms $\chi_\alpha : U_\alpha \longrightarrow V_\alpha$ onto open subsets $V_\alpha \subset \mathbb{R}^n$, such that*

the following properties are satisfied:

- The sets U_α cover $\mathcal{M}$, that is $\cup_\alpha U_\alpha = \mathcal{M}$
- If $U_\alpha \cap U_\beta \neq \emptyset$ for a pair of indices $\alpha, \beta \in I$, then the map

$$\chi_\beta \circ \chi_\alpha^{-1} : \chi_\alpha(U_\alpha \cap U_\beta) \longrightarrow \chi_\beta(U_\alpha \cap U_\beta)$$

from $\chi_\alpha(U_\alpha \cap U_\beta) \subset \mathbb{R}^n$ to $\chi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{R}^n$ is smooth i.e. has continuous derivatives of arbitrary order.



See the figure that illustrate this definition. Let us shortly comment on the content and ingredients of this definition. The demand that a topological space $\mathcal{M}$ is a Hausdorff space means that any two distinct points have disjoint neighbourhoods, i.e. that if x and y are two different points of $\mathcal{M}$, then there exists open subsets $V_x \subset \mathcal{M}$ and $V_y \subset \mathcal{M}$ such that $x \in V_x, y \in V_y$ and such that $V_x \cap V_y = \emptyset$. The open sets U_α in the context of the above definition are called *charts*, while the functions χ_α are called *local coordinate maps* (or simply local coordinate system or local coordinates). The functions $\chi_\alpha \circ \chi_\beta^{-1}$ are called *transition functions* and describe coordinate transformations on the manifold $\mathcal{M}$. The collection $\mathbb{K} = \{(U_a, \chi_\alpha)\}$ is called an *atlas* on the manifold $\mathcal{M}$. Every manifold has infinitely many (equivalent) atlases that together define a *differentiable structure* on the manifold. For more expositions and explanations on the definition of a smooth manifold, see [22, 40, 51].

Here we note only two more things. Firstly: manifolds can be in general modelled on Banach spaces, in the sense that the homeomorphisms χ_α can map open subsets of $\mathcal{M}$ into open subsets of a Banach space; in such a case we speak about *Banach manifolds*. In this thesis we will solely consider manifolds modelled on $\mathbb{R}^n$ and in most cases we will restrict ourselves to the simplest manifold that exists, namely $\mathbb{R}^n$ itself. *However, in order to understand that all the ideas of this thesis are geometric we feel obliged to introduce the right geometric environment of the theory, namely the theory of Hamiltonian systems on Poisson manifolds.* Secondly, the transition functions are composed of homeomorphisms but they themselves are diffeomorphisms, i.e. smooth homeomorphisms with smooth inverses. For definitions of a topological space, Hausdorff space, homeomorphism and diffeomorphism, see any textbook on topology, for instance [22, 39].

Below we present some examples of smooth manifolds:

Example 2.2. *The Euclidean space $\mathbb{R}^n$ is an n-dimensional manifold with a single coordinate chart $U = \mathbb{R}^n$, and a local coordinate map defined by the identity map: $\chi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $\chi(x) = x$. Any open subset of $U \subset \mathbb{R}^n$ is also an n- dimensional manifold with a single coordinate chart given by U itself with the identity map as a local coordinate map.*

Example 2.3. *The n-sphere is an n-dimensional manifold that is realized as a surface in $\mathbb{R}^{n+1}$ given by*

$$S^n = \left\{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : \ x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1\right\},$$

with an atlas given by $\mathbb{K}_1 = \{(U_\alpha, \chi_\alpha), (V_\alpha, \psi_\alpha),\ 1 \leq \alpha \leq n+1\}$ for where $U_\alpha = \{(x_1, x_2, \ldots, x_{n+1}) \in S^n : x_\alpha > 0\}$ together with the local coordinate map

$$\chi_\alpha(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_{\alpha-1}, 0, x_{\alpha+1}, \ldots, x_{n+1})$$

and $V_\alpha = \{(x_1, \ldots, x_{n+1}) \in S^n : x_\alpha < 0\}$ together with coordinate maps

$$\psi_\alpha(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_{\alpha-1}, 0, x_{\alpha+1}, \ldots, x_{n+1}).$$

Both χ_α and ψ_α are simply orthogonal projections of U_α respectively V_α onto the hyperplane $x_\alpha = 0$. It is easy to check that all the transition functions $\chi_\alpha \circ \chi_\beta^{-1}$, $\chi_\alpha \circ \psi_\beta^{-1}$ etc. are smooth. Another possible atlas on S^n is given by

$$\mathbb{K}_2 = \left\{ (S^n \backslash \{N\}, \chi_1), (S^n \backslash \{S\}, \chi_2) \right\},$$

where

$$\chi_1 (x_1, x_2, \ldots, x_{n+1}) = \left(\frac{x_1}{1 - x_{n+1}}, \cdots, \frac{x_n}{1 - x_{n+1}} \right)$$
$$\chi_2 (x_1, x_2, \ldots, x_{n+1}) = \left(\frac{x_1}{1 + x_{n+1}}, \cdots, \frac{x_n}{1 + x_{n+1}} \right),$$

are stereographic projections from $N = (0, \ldots, 0, 1)$ and from $S = (0, \ldots, 0, -1)$, i.e. from the north and the south pole, respectively. The only (up to inverse) nontrivial transition function in this atlas $\chi_1 \circ \chi_2^{-1} : \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{R}^n \setminus \{0\}$ is a diffeomorphism given by the formula

$$\chi_2 \circ \chi_1^{-1} (x_1, x_2, \ldots, x_n) = \left(\frac{x_1}{x_1^2 + \cdots + x_n^2}, \cdots, \frac{x_n}{x_1^2 + \cdots + x_n^2} \right) \in S^{n-1}. \qquad (2.1)$$

Let us demonstrate the formula (2.1) i.e., let us calculate $\chi_2 \circ \chi_1^{-1}$. To do this, we need an explicit expression for χ_1^{-1}. Denote $u_i = \frac{x_i}{w}$ with $w = 1 - x_{n+1}$ and $u = (u_1, \ldots, u_n)$. Since $x_{n+1}^2 = 1 - (x_1^2 + \cdots + x_n^2)$ we have $(1 - w)^2 = 1 - w^2 |u|^2$, where $|u|$ denote the norm of vector u. Then $1 - 2w + w^2 = 1 - w^2 |u|^2$ and since $w \neq 0$ (otherwise $x_{n+1} = 1$, but $(0, \ldots, 0, 1) \notin S^n \setminus \{N\}$) $w + w |u|^2 = 2$ so that $w = \frac{2}{1+|u|^2}$. We obtain therefore that

$$x_1 = \frac{2u_1}{1 + |u|^2}, \cdots, x_n = \frac{2u_n}{1 + |u|^2} \text{ and } x_{n+1} = \frac{|u|^2 - 1}{1 + |u|^2}.$$

Thus

$$\chi_1^{-1}(u_1, \ldots, u_n) = \left(\frac{2u_1}{1 + |u|^2}, \cdots, \frac{2u_n}{1 + |u|^2}, \frac{|u|^2 - 1}{1 + |u|^2} \right),$$

and an easy calculation yields

$$\chi_2 \circ \chi_1^{-1} (u_1, u_2, \ldots, u_n) = \chi_2 \left(\frac{2u_1}{1 + |u|^2}, \cdots, \frac{2u_n}{1 + |u|^2}, \frac{|u|^2 - 1}{1 + |u|^2} \right) = \frac{u}{|u|^2}$$

i.e., the formula (2.1). In a similar way we can obtain that

$$\chi_2^{-1}(u_1, \ldots, u_n) = \left(\frac{2u_1}{1 + |u|^2}, \cdots, \frac{2u_n}{1 + |u|^2}, \frac{1 - |u|^2}{1 + |u|^2} \right),$$

and further that

$$\chi_1 \circ \chi_2^{-1}(u_1, u_2, \ldots, u_n) = \chi_1\left(\frac{2u_1}{1+|u|^2}, \cdots, \frac{2u_n}{1+|u|^2}, \frac{|u|^2-1}{1+|u|^2}\right) = \frac{u}{|u|^2}$$

and thus also $\chi_1 \circ \chi_2^{-1}(u_1, u_2, \ldots, u_n) = \dfrac{u}{|u|^2}$. *A moment of reflection is enough to realize that* $\chi_2 \circ \chi_1^{-1}$ *is a smooth (as a rational function) bijection, and since its inverse is given by the same formula, it is smooth as well. Therefore,* $\chi_2 \circ \chi_1^{-1}$ *is a diffeomorphism.*

Example 2.4. *The circle*

$$S^1 = \left\{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\right\}$$

can be endowed with yet another atlas: $\mathbb{K} = \{(U_1, \chi_1), (U_2, \chi_2)\}$ *with* $U_1 = S^1 \setminus (-1, 0)$ *and* $\chi_1 = \arctan\dfrac{y}{x}$ *with* $-\pi < \chi_1 < \pi$ *and* $U_2 = S^1 \setminus (1, 0)$ *and* $\chi_2 = \arctan\dfrac{y}{x}$ *with* $0 < \chi_2 < 2\pi$. *Note that the (only nontrivial in this atlas) transition function is the identity function,* $\chi_1 \circ \chi_2^{-1} = 1$, *therefore obviously a diffeomorphism.*

2.1.2 Vector fields on a manifold

In order to define a vector field on a manifold, let us start by describing the concept of a tangent vector. Assume that a smooth curve γ is given on a manifold $\mathcal{M}$. Suppose that γ is parametrized by a smooth map $\Phi : I \longrightarrow \mathcal{M}$, where I is an interval in $\mathbb{R}$, and assume that we have a local coordinate system $x = (x^1, \ldots, x^n)$ on $\mathcal{M}$ in which the curve γ is given by a smooth function $\Phi(t) = (\Phi^1(t), \ldots, \Phi^n(t))$.

Remark 2.5. *From now we will use the shorthand notation, like* $(x^1, \ldots, x^n)$, *for coordinate systems on* $\mathcal{M}$, *postponing mentioning the chart* U *as well as the detailed form of the coordinate functions* x^i. *We stress that* x^i *are smooth functions on manifold.*

Definition 2.6. *A tangent vector to the curve* γ *at the point* $x = \Phi(t) \in \mathcal{M}$ *is the linear differential operator of the first order given by*

$$v|_x : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M}), \quad v|_x = \frac{d\Phi^1(t)}{dt}\left.\frac{\partial}{\partial x^1}\right|_x + \cdots + \frac{d\Phi^n(t)}{dt}\left.\frac{\partial}{\partial x^n}\right|_x,$$

where $\left(\dfrac{\partial}{\partial x^i}\right)\Big|_x$ *is the usual partial derivative with respect to* x^i *taken at the point* x*.*

This definition is one of many possible equivalent definitions of a tangent vector; it stems from the simple fact that in $\mathbb{R}^n$ a vector at a point x is uniquely determined by the set of directional derivatives of all possible functions along this vector. For further explanations, including the proof why this concept is actually coordinate-independent, see for example [51]. We stress that $v|_x$ does depend on the chosen parametrization of the curve.

Definition 2.7. *Given any* $C^\infty(\mathcal{M})$ *of dimension* n *and let* $x \in \mathcal{M}$*, a tangent vector to the manifold is any equivalence class of* C^1*-curve* γ *thought the point* x *on* $\in \mathcal{M}$ *such that* $\gamma(0) = x$*.*

Example 2.8. *Consider a helix in* $\mathbb{R}^3$ *parametrized by* $\Phi(t) = (\cos t, \sin t, t)$ *with* $t \in I = \mathbb{R}$*. The coordinates in* $\mathbb{R}^3$ *are labeled by* (x_1, x_2, x_3)*. The corresponding tangent vector at any point* $x = \Phi(t) = (\cos t, \sin t, t)$ *of the helix is then given by*

$$v_x = -\sin t \left.\frac{\partial}{\partial x^1}\right|_x + \cos t \left.\frac{\partial}{\partial x^2}\right|_x + \left.\frac{\partial}{\partial x^3}\right|_x$$

Definition 2.9. *The collection of all vectors tangent to all possible curves passing through a point* $x \in \mathcal{M}$ *is called the tangent space to* $\mathcal{M}$ *at* x *and it is denoted by* $T_x\mathcal{M}$*.*

In a given coordinate system $x = (x^1, \ldots, x^n)$ on $\mathcal{M}$ the tangent space $T_x\mathcal{M}$ is spanned by all the vectors $\left.\dfrac{\partial}{\partial x^i}\right|_x$ which constitute a linearly independent set, as it is stated in the following lemma.

Lemma 2.10. *For any point* $x \in \mathcal{M}$ *the tangent space* $T_x\mathcal{M}$ *is an* n*-dimensional vector space. In a given coordinate system* $(x^1, \ldots, x^n)$ *the set of tangent vectors*

$$\left\{ \left.\frac{\partial}{\partial x^1}\right|_x, \ldots, \left.\frac{\partial}{\partial x^n}\right|_x \right\} \tag{2.2}$$

constitutes a basis for $T_x\mathcal{M}$*; it is called the basis of* $T_x\mathcal{M}$ associated *with the coordinate system* $(x^1, \ldots, x^n)$*.*

Finally, we define a vector field on $\mathcal{M}$ as a collection of tangent vectors defined in a smooth way over the whole manifold.

Definition 2.11. *A vector field v on $\mathcal{M}$ is a map that to each point $x \in \mathcal{M}$ assigns a tangent vector $v|_x \in T\mathcal{M}_x$ with $v|_x$ varying smoothly as x varies smoothly over $\mathcal{M}$. Thus, in the local coordinates $(x^1, \dots, x^n)$, any vector field has the form*

$$v = \xi^1 \frac{\partial}{\partial x^1} + \cdots + \xi^n \frac{\partial}{\partial x^n},$$

where $\xi^i = \xi^i(x)$ are smooth real valued functions on $\mathcal{M}$, called components of v in the coordinate system $(x^1, \dots, x^n)$.

Note that any vector field v on $\mathcal{M}$ defines a smooth map $F \longmapsto v(F) : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$ that in any coordinate system $(x^1, \dots, x^n)$ is given by

$$v(F) = \xi^1 \frac{\partial F}{\partial x^1} + \cdots + \xi^n \frac{\partial F}{\partial x^n}, \tag{2.3}$$

where ξ^i are components of v in the coordinate system x (here and throughout the thesis $C^\infty(\mathcal{M})$ denotes the real algebra of smooth real valued functions on $\mathcal{M}$). It is easy to show that this definition actually does not depend on the choice of the coordinate system. The map $F \longmapsto v(F)$ has all properties of a differentiation: for any two smooth functions $F, G \in C^\infty(\mathcal{M})$ and any two real constants α, β we have

$$v(\alpha F + \beta G) = \alpha v(F) + \beta v(G) \text{ (linearity)} \tag{2.4}$$

$$v(FG) = Fv(G) + Gv(F) \text{ (Leibniz rule)}, \tag{2.5}$$

(where FG is the usual point wise multiplication of functions) so in this sense *a vector field on $\mathcal{M}$ is simply a differentiation on $C^\infty(\mathcal{M})$*.

We can therefore define vector field on a manifold $\mathcal{M}$ in an equivalent way.

Definition 2.12. *A vector field on $\mathcal{M}$ is a linear first order differential operator on $C^\infty(\mathcal{M})$ that satisfies (2.4) and (2.5).*

Such definition is obviously invariant; we do not need any coordinate system to formulate it.

Let us conclude this subsection by a very important definition of a tangent bundle.

Definition 2.13. *The tangent bundle $T\mathcal{M}$ to a smooth n-dimensional manifold $\mathcal{M}$ is the $2n$-dimensional smooth manifold that is a disjoint union of all tangent spaces*

$$T\mathcal{M} = \bigcup_{x\in\mathcal{M}} T_x\mathcal{M}$$

with the topological and differential structures such that the projection map $\pi : T\mathcal{M} \to \mathcal{M}$ sending any $v \in T_x^*\mathcal{M}$ to $x \in \mathcal{M}$ is smooth.

We refer the reader to details in [40, 51] or other books on differential geometry. Here we only mention that $T\mathcal{M}$ is the so called *fibre bundle* over $\mathcal{M}$ with fibres $T_x\mathcal{M}$ and with the typical fibre $\mathbb{R}^n$.

One can also say that a vector field on a manifold $\mathcal{M}$ is a *section* of $T\mathcal{M}$, i.e., a smooth mapping v from $\mathcal{M}$ to $T\mathcal{M}$ such that $v(x) \in T_x\mathcal{M}$.

2.1.3 Cotangent bundle and tensor fields on smooth manifolds

Definition 2.14. *Let V be a vector space. The dual space to V is the linear space of all real valued linear functionals $\omega : V \to \mathbb{R}$. This space is denoted by V^*.*

Definition 2.15. *Let $\mathcal{M}$ be a smooth manifold. For each $x \in \mathcal{M}$ we define the cotangent space to x as the dual space to the tangent space $T_x\mathcal{M}$, and denote it by $T_x^*\mathcal{M}$. The elements of $T_x^*\mathcal{M}$ are called* covectors at the point *x.*

Definition 2.16. *The cotangent bundle $T^*\mathcal{M}$ to an n-dimensional smooth manifold $\mathcal{M}$ is the $2n$-dimensional smooth manifold that is a disjoint union of all cotangent spaces*

$$T^*\mathcal{M} = \bigcup_{x\in\mathcal{M}} T_x^*\mathcal{M}$$

such that the projection map $\pi^* : T^*\mathcal{M} \longrightarrow \mathcal{M}$ defined through $\omega \in T_x^*\mathcal{M} \mapsto x \in \mathcal{M}$ is smooth.

Again, see [40, 51] for details. We only mention here that $T^*\mathcal{M}$ is a fiber bundle over $\mathcal{M}$ with fibres $T_x^*\mathcal{M}$ and with the typical fibre $\mathbb{R}^n$.

Definition 2.17. *The map $\langle\cdot,\cdot\rangle_x : T_x^*\mathcal{M} \to T_x\mathcal{M}$ defined through $\langle\omega, v\rangle_x = \omega(v)$ for any $\omega \in T_x^*\mathcal{M}$ and for any $v \in T_x\mathcal{M}$, is called the dual map between the cotangent and the tangent spaces at x.*

If $(x^1, \ldots, x^n)$ is a local coordinate system on $\mathcal{M}$ then the basis of $T_x^*\mathcal{M}$ dual to the basis (2.2) of $T_x\mathcal{M}$ will be denoted by $\{dx^1, \ldots, dx^n\}$. Thus, at any point belonging to this coordinate system we have that

$$\left\langle dx^i, \frac{\partial}{\partial x^j} \right\rangle = dx^i \left(\frac{\partial}{\partial x^j} \right) = \frac{\partial x^i}{\partial x^j} = \delta^i_j.$$

Finally, a *covector field* is a smooth mapping that to each point $x \in \mathcal{M}$ attaches a covector $\omega \in T_x^*\mathcal{M}$. It is therefore a smooth section of the cotangent bundle $T^*\mathcal{M}$.

Example 2.18. *Given $F \in C^\infty(\mathcal{M})$, the* differential *of F is the covector field dF defined as $dF(v) = v(F)$. Thus, due to (2.3), in a coordinate system $(x^1, \ldots, x^n)$ the differential dF is given by*

$$dF = \sum_{i=1}^{n} \frac{\partial F}{\partial x^i} dx^i.$$

Of course only some covector fields are differentials of functions.

In general, once can define (r, s)-tensor fields on every smooth manifold [30]. One way of doing this is the following:

Definition 2.19. *$A_n(r, s)$-tensor field on $\mathcal{M}$ is a mapping that with each coordinate system $x = (x^1, \ldots, x^n)$ associates a collection of n^{r+s} functions $T^{i_1 \ldots i_r}_{j_1 \ldots j_s}(x)$, (where every* contravariant *index i_α and every* covariant *index j_β vary between* 1 *and n) such that in any coordinate system $y = (y^1, \ldots, y^n)$ overlapping with $(x^1, \ldots, x^n)$ the following transformation rule is valid*

$$T^{k_1 \ldots k_r}_{l_1 \ldots l_s}(y) = \frac{\partial y^{k_1}}{\partial x^{i_1}} \cdots \frac{\partial y^{k_r}}{\partial x^{i_r}} \frac{\partial x^{j_1}}{\partial y^{l_1}} \cdots \frac{\partial x^{j_s}}{\partial y^{l_s}} T^{i_1 \cdots i_r}_{j_1 \ldots j_s}(x)$$

(here we use the Einstein summation convention which implies summation over repeated indices of opposite variance) where $y^i = y^i(x)$ are the transition functions between the coordinates $(x^1, \ldots, x^n)$ and $(y^1, \ldots, y^n)$.

This definition is in fact a geometrical one. A vector field is in fact a $(1, 0)$-tensor field while a covector field is in fact a $(0, 1)$-tensor field. So, if $\xi_i(x)$ is a covector field on $\mathcal{M}$ given in coordinates $(x^1, \ldots, x^n)$ then in the coordinates $(y^1, \ldots, y^n)$ given by the transition functions $y^i = y^i(x)$ this tensor field is given by

$$\xi_i(y) = \frac{\partial x^j}{\partial y^i} \xi_j(x)$$

or, in matrix form

$$\xi(y) = \left(J^{-1}\right)^T \xi(x), \tag{2.6}$$

where J is the Jacobi matrix of the transformation from $(x^1, \ldots, x^n)$ to $(y^1, \ldots, y^n)$ so that $J^i_j = \frac{\partial y^i}{\partial x^j}$. Similarly, if $G^{ij}(x)$ is a $(2,0)$-tensor field on $\mathcal{M}$ given in coordinates $(x^1, \ldots, x^n)$ then in the coordinates $(y^1, \ldots, y^n)$ given by the transition functions $y^i = y^i(x)$ this tensor field is given by

$$G^{kl}(y) = \frac{\partial y^k}{\partial x^i}\frac{\partial y^l}{\partial x^j} G^{ij}(x)$$

or, in matrix form

$$G(y) = J\ G(x)\, J^T. \tag{2.7}$$

Likewise, if L is a $(1,1)$-tensor field on $\mathcal{M}$ given in coordinates $(x^1, \ldots, x^n)$ then in the coordinates $(y^1, \ldots, y^n)$ given by the transition functions $y^i = y^i(x)$ this tensor field is given by

$$L^k_l(y) = \frac{\partial y^k}{\partial x^i}\frac{\partial x^j}{\partial y^l} L^i_j(x)$$

or, in matrix form

$$L(y) = J\, L(x)\, J^{-1}. \tag{2.8}$$

Now we need to define a Killing Tensor, conformal Killing tensor and special conformal Killing tensor.

Definition 2.20. *The $(1,1)$ Killing tensor $K(\lambda) = \left[K^i_j(\lambda)\right]$ on the base/configurational manifold Q with coordinates λ and equipped with a Riemann metric g is a Killing tensor (for this metric g) if the function on $M = T^*Q$ given by*

$$F(\lambda) = \sum_{i,j=1}^{n} A^{ij}(\lambda)\mu_i\mu_j$$

(where $A^{ij} = G^{ik}K^j_k$ is the contravariant form of K) is a constant of motion for the geodesic flow defined by g.

Definition 2.21. *We call L-tensor is a conformal Killing two -tensor L^{ij} on the Riemannian manifold (Q, g_{ij}), such that $L = (L^j_i)$, $L^j_i = g_{hi}L^{jh}$ is torsion-less and with real simple eigenvalues [20].*

The conformal Killing tensor whose torsion vanishes and which has functionally independent eigenfunctions must satisfies this necessary condition

$$L^k_{ij} = \frac{1}{2}\left(\alpha_i g_{jk} + \alpha_j g_{ik}\right), \text{ where } \alpha_i = \frac{1}{\det L} L^j_i \frac{\partial}{\partial q^j} \det L,$$

therefore L^k_{ij} is called *special conformal Killing tensor* [20].

2.2 Poisson structures and Hamiltonian vector fields

2.2.1 Poison brackets

A basic concept in the theory of Hamiltonian systems is the notion of a Poisson bracket. A Poisson bracket is a bilinear operation that turns a pair of smooth functions on a manifold into a new smooth function on the same manifold.

Definition 2.22. *A Poisson bracket on $\mathcal{M}$ is a map $\{\cdot,\cdot\} : C^\infty(\mathcal{M}) \times C^\infty(\mathcal{M}) \longrightarrow C^\infty(\mathcal{M})$ with the following properties: for any $F, G, H \in C^\infty(\mathcal{M})$ and for any $a, b \in \mathbb{R}$.*

- *Bilinearity: $\{aF + bG, H\} = a\{F, H\} + b\{G, H\}$ and $\{F, aH + bG\} = a\{F, H\} + b\{G, H\}$ for all $a, b \in \mathbb{R}$.*
- *Skew-symmetry : $\{F, H\} = -\{H, F\}$.*
- *Jacobi identity: $\{\{F, G\}, H\} + \{\{G, H\}, F\} + \{\{H, F\}, G\} = 0$.*
- *Leibniz rule: $\{FG, H\} = F\{G, H\} + G\{F, H\}$.*

In other words, $C^\infty(\mathcal{M})$ equipped with the Poisson bracket $\{\cdot,\cdot\}$ is a Lie algebra with the Lie bracket that satisfies the Leibniz rule.

Definition 2.23. *A manifold $\mathcal{M}$ equipped with a Poisson bracket $\{\cdot,\cdot\}$ is called Poisson manifold. One also says that a Poisson bracket $\{\cdot,\cdot\}$ defines* a Poisson structure *on $\mathcal{M}$.*

The most important example of a Poisson bracket is the so called *canonical Poisson bracket*:

Example 2.24. *Let $\mathcal{M} = \mathbb{R}^{2n}$ with the natural coordinates $(p, q) = (q^1, \ldots, q^n, p^1, \ldots, p^n)$. For any two smooth functions $F(p, q)$and $H(p, q)$ we define their Poisson bracket as*

$$\{F, H\} = \sum_{i=1}^{n} \left(\frac{\partial F}{\partial q^i} \frac{\partial H}{\partial p^i} - \frac{\partial F}{\partial p^i} \frac{\partial H}{\partial q^i} \right). \tag{2.9}$$

The expression $\{F, H\}$ in 2.9 satisfies all the properties of Poisson brackets as required by Definition 2.22.

This bracket is often presented in textbooks on classical mechanics. In the context of classical mechanics the variables q^i are called (generalized) positions while p^i are called (generalized) momenta.

Definition 2.25. *Any smooth real valued function $C \in C^\infty(\mathcal{M})$ such that*

$$\{C, H\} = 0 \text{ for all } H \in C^\infty(\mathcal{M})$$

is called a distinguished function of the Poisson bracket $\{\cdot, \cdot\}$ (or the Casimir function or simply Casimir of $\{\cdot, \cdot\}$). Thus, C is a Casimir function of the Poisson bracket $\{\cdot, \cdot\}$ if the Poisson bracket of C with any other real valued function vanishes identically.

The only Casimirs of the canonical Poisson bracket 2.9 are constant functions.

2.2.2 Hamiltonian vector fields

We are now in position to present the first definition of a fundamental concept of this thesis, *a Hamiltonian vector field on a Poisson manifold*. This definition is coordinate-free i.e. we do not need to use any coordinate system to formulate it.

Definition 2.26. *Let $\mathcal{M}$ be a Poisson manifold and let $H : \mathcal{M} \longrightarrow \mathbb{R}$ be a smooth function on $\mathcal{M}$. The Hamiltonian vector field associated with H is the unique smooth vector field w_H on $\mathcal{M}$ satisfying*

$$w_H(F) = \{F, H\} \tag{2.10}$$

for any smooth function $F : \mathcal{M} \longrightarrow \mathbb{R}$.

Definition 2.27. *The system of first order equations on the Poisson manifold* $\mathcal{M}$

$$\frac{dx}{dt} = w_H \tag{2.11}$$

with $H \in C^\infty(\mathcal{M})$ *and* w_H *given by (2.10), is called Hamiltonian system with the Hamiltonian function* H*.*

The system of ODE's (2.11) is a dynamical system that defines a flow (a one-parameter local group of diffeomorphisms) on $\mathcal{M}$ [44]. While Hamiltonian systems owe its name to Sir William Rowan Hamilton, the qualitative study of dynamical systems has originated in the famous works of Henri Poincaré (see for example a translation of his original work on dynamical systems in [45] or [46]). In a given coordinate system $x = (x^1, \dots, x^n)$ we have $w_H = \sum\limits_{i=1}^{n} \zeta^i(x) \dfrac{\partial F}{\partial x^i}$ where $\zeta^i(x) = w_H(x^i) = \{x^i, H\}$ (as we explained in Remark 2.5, x^i for a fixed i is a local function on the manifold $\mathcal{M}$; this function maps a point in a chart to the value of its i-th coordinate). Thus the Hamiltonian vector field (2.10) generated by the Hamiltonian H can be written as

$$w_H = \sum_{i=1}^{n} \left\{x^i, H\right\} \frac{\partial}{\partial x^i}. \tag{2.12}$$

Example 2.28. *Let us consider* $\mathcal{M} = \mathbb{R}^m$ *for* $m = 2n + l$ *with the coordinates*

$$(q, p, z) = (q^1, \dots, q^n, p^1, \dots, p^n, z^1, \dots, z^l)$$

and the Poisson bracket defined as

$$\{F, H\} = \sum_{i=1}^{n} \left(\frac{\partial F}{\partial q^i} \frac{\partial H}{\partial p^i} - \frac{\partial F}{\partial p^i} \frac{\partial H}{\partial q^i} \right) \tag{2.13}$$

i.e. given formally by the same expression as (2.9). Since the variables z *do not enter (2.13), any function* $C = C(z)$ *depending only on* z^i*-coordinates is a Casimir function of this bracket. Consider also an arbitrary smooth function* $H = H(p, q, z)$ *on* $\mathcal{M}$*. It is easy to calculate that the Hamiltonian vector field associated with* H *is*

$$w_H = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial p^i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p^i} \right).$$

The corresponding Hamiltonian system has the form

$$\frac{dq^i}{dt} = \frac{dH}{dp^i}, \quad \frac{dp^i}{dt} = -\frac{dH}{dq^i}, \quad i = 1, \dots, n, \quad \frac{dz^j}{dt} = 0, \quad j = 1, \dots, l \tag{2.14}$$

and we see that the z-variables (called in this context Casimir variables) are constants of motion.

2.2.3 Structure functions

We will now show how to express a Poisson bracket in local coordinates. Suppose that $x = (x^1, \ldots, x^n)$ is a local coordinate system on a Poisson manifold $\mathcal{M}$. We want to express the function $\{F, H\}$ as a function of $(x^1, \ldots, x^n)$ given that we know the expressions for F and H as functions of $(x^1, \ldots, x^n)$. From (2.12) we know that

$$w_H = \sum_{i=1}^{n} \left\{x^i, H\right\} \frac{\partial}{\partial x^i}$$

is the Hamiltonian vector field corresponding to H. Thus we obtain that $\{F, H\} = w_H(F) = \sum_{i=1}^{n} \{x^i, H\} \dfrac{\partial F}{\partial x^i}$. Moreover, due to skew-symmetry,

$$\left\{x^i, H\right\} = -\left\{H, x^i\right\} = -w_{x^i}(H) = -\sum_{j=1}^{n} \left\{x^j, x^i\right\} \frac{\partial H}{\partial x^j},$$

so finally we obtain

$$\{F, H\} = \sum_{i,j=1}^{n} \left\{x^i, x^j\right\} \frac{\partial F}{\partial x^i} \frac{\partial H}{\partial x^j} \tag{2.15}$$

as the expression for the Poisson bracket $\{F, H\}$ in coordinates x. Thus, the local expression for the Poisson bracket is completely determined by the functions $\pi^{ij} = \{x^i, x^j\}$ that are simply Poisson brackets of coordinate functions. The functions π^{ij} are called the *structure functions* of the Poisson bracket $\{\cdot, \cdot\}$ relative to the given local coordinate system x. The $n \times n$ matrix $\pi = [\pi^{ij}]$ is called *the structure matrix* of the Poisson bracket $\{\cdot, \cdot\}$ and these functions can be shown to be components of a $(2, 0)$-tensor field that is called *Poisson tensor* (see below for details). Thus, we have that

$$\{F, H\} = \langle dF,\, \pi\, dH \rangle\,,$$

where dF and dH are differentials of F and H, respectively, and where $\langle \cdot, \cdot \rangle$ is the dual map between tangent and cotangent spaces (see Definition 2.17).

It turns out that the structure functions (i.e. components of Poisson tensor) uniquely determine the Poisson bracket or Poisson structure, in the following sense.

Proposition 2.29. *Let $\pi(x) = \pi^{ij}(x)$ be an $n \times n$ matrix of functions of $x = (x^1, \ldots, x^n)$ defined on a smooth manifold $\mathcal{M}$. Then $\pi(x)$ is the structure matrix for the Poisson bracket $\{F, H\} = \langle dF, \pi\, dH\rangle$ on $\mathcal{M}$ if and only if it has the following properties:*

1. *The skew-symmetry*

$$\pi^{ij}(x) = -\pi^{ji}(x), \quad i, j = 1, \ldots, n \tag{2.16}$$

2. *Jacobi-Identity*

$$\sum_{l=1}^{n} \left\{ \pi^{il}\partial_l \pi^{jk} + \pi^{kl}\partial_l \pi^{ij} + \pi^{jl}\partial_l \pi^{ki} \right\} = 0, \quad i, j, k = 1, \ldots, n \tag{2.17}$$

for all $x \in \mathcal{M}$. The notation $\partial_l = \partial/\partial x^l$ is used.

The proof of this proposition is elementary, but we omit it here. We refer the reader to [43]. We only note that the property (2.16) obviously implies that the expression $\langle dF, \pi\, dH\rangle$ is antisymmetric: $\langle dF, \pi\, dH\rangle = -\langle dH, \pi\, dF\rangle$.

Example 2.30. *(Example 2.28 continued) For the Poisson structure given in Example 2.28 the structure matrix with respect to the natural coordinates (q, p, z) has the form*

$$\begin{bmatrix} 0 & I & 0 \\ -I & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where I represents the $n \times n$ identity matrix.

From (2.12), we can easily obtain the coordinate expression for the Hamiltonian vector field w_H involving the structure functions π^{ij}:

$$w_H = \sum_{i,j=1}^{n} \pi^{ij}(x) \frac{\partial H}{\partial x^j} \frac{\partial}{\partial x^i}. \tag{2.18}$$

Example 2.31. *An important example of a Poisson structure is the so-called Lie-Poisson structure. Assume we have an n-dimensional Lie algebra η with C_{ij}^k for $i, j, k = 1, \ldots, n$ being the structure constant of η relative to some basis. Let us also*

assume that V is an n-dimensional vector space with coordinates of $x = (x^1, \ldots, x^n)$ relative to some basis. We can now define ***the Lie-Poisson bracket*** *between two smooth functions $F, H : V \longrightarrow \mathbb{R}$ through*

$$\{F, H\} = \sum_{i,j,=1}^{n} \left(\sum_{k=1}^{n} C_{ij}^{k} x^k \right) \frac{\partial F}{\partial x^i} \frac{\partial H}{\partial x^j}.$$

Clearly, the Lie-Poisson bracket above has the form as (2.15) with the structure functions π^{ij} that are linear in x and given by $\pi^{ij}(x) = \sum_{k=1}^{n} C_{ij}^{k}\ x^k$. The algebraic properties of the structure constants C_{ij}^{k} guarantee that the structure functions π^{ij} satisfy both (2.16) and (2.17).

2.2.4 Poisson operator of a Poisson structure

We will now demonstrate that the structure functions π^{ij} of a Poisson bracket are components of a twice contravariant tensor field that is called - depending on the context - *Poisson tensor*, *Hamiltonian operator, Poisson operator* or *Poisson bi-vector*. Throughout the thesis we will use the notion *Poisson operator*.

Proposition 2.32. *Let $\mathcal{M}$ be a Poisson manifold and $x \in \mathcal{M}$. There exists unique map $\pi : T^*\mathcal{M} \longrightarrow T\mathcal{M}$ such that $\pi|_x : T_x^*\mathcal{M}| \longrightarrow T_x\mathcal{M}$ and such that for any smooth real valued function $H : \mathcal{M} \longrightarrow \mathbb{R}$ we have $(\pi dH)\,|_x = w_H|_x$.*

Proof. As we know from a previous section , at every $x \in \mathcal{M}$ the cotangent space $T^*\mathcal{M}|_x$ is spanned by the set $\{dx^1, \ldots, dx^n\}$ of all the differentials dx_i of coordinate functions defining a local coordinate system near x. Assume that π^{ij} are the structure functions for the Poisson bracket. Define the linear operator $\pi|_x : T^*\mathcal{M}|_x \longrightarrow T\mathcal{M}|_x$ through its action on the basis vectors dx^i (dx^i span the cotangent space at each point) as

$$\pi\left(dx^i\right)\Big|_x = \sum_{i=1}^{n} \pi^{ij}(x)\ \left.\frac{\partial}{\partial x^j}\right|_x, \quad i = 1, \ldots, n.$$

Then, since $dH = \sum_{j=1}^{n} \frac{\partial H}{\partial x_j} dx_j$ and due to linearity of $\pi|_x$ we have

$$(\pi dH)\,|_x = \sum_{i=1}^{n} \pi^{ij}(x)\ \frac{\partial H}{\partial x_j} \left.\frac{\partial}{\partial x^j}\right|_x = w_H|_x\,,$$

where the last equality follows from (2.18). Varying x we obtain the map π. □

The Poisson operator π is thus a twice-contravariant (or $(2,0)$) tensor field on $\mathcal{M}$ with the components being exactly the structure functions π^{ij}.

The proposition above shows that there is a one-to-one correspondence between Poisson brackets and Poisson operators. From the view point of Poisson geometry they are the same. Defining a Poisson bracket (i.e. a Poisson structure) automatically defines the corresponding Poisson operator and vice versa.

Example 2.33. *Consider $\mathcal{M} = \mathbb{R}^m$ with $m = 2n+l$ with natural coordinates (q,p,z) and with the Poisson bracket as in Example 2.28. Any $c \in T^*\mathcal{M}$ has then the form*

$$c = \sum_{i=1}^{n} \left(a_i \; dq^i + b_i \; dp^i \; \right) + \sum_{j=1}^{l} \; e_j \; dz^j,$$

where a_i, b_i and e_j are $2n+l$ smooth functions on $\mathcal{M}$. Then

$$\pi(c) = \sum_{i=1}^{n} \; \left(b_i \; \frac{\partial}{\partial q^i} - a_i \; \frac{\partial}{\partial p^i} \right).$$

Note that $dz^1, \ldots, dz^l$ span the kernel of π, $\pi(dz^j) = 0$ for $j = 1, \ldots, l$.

Of a fundamental importance is a theorem due to Jean Gaston Darboux [43] that states that every Poisson structure has locally the form given as in Example 2.28. This is a fact that we will heavily explore in the rest of this thesis

Theorem 2.34. *(Darboux Theorem) Suppose that $\mathcal{M}$ is an m-dimensional Poisson manifold (with $m = 2n + l$) whose Poisson operator π is of a constant rank $2n \leq m$ everywhere (if $2n = m$ then $l = 0$ and π is nondegenerate i.e. $\det[\pi^{ij}(x)] \neq 0$ for all x). For any point $x \in \mathcal{M}$ there exists a neighbourhood $U \ni x$ and a coordinate system*

$$(q,p,z) = (q_1, \ldots, q_n, p_1, \ldots, p_n, z_1, \ldots, z_l)$$

in U in which the Poisson operator π attains the constant form

$$\pi = \begin{bmatrix} 0 & I & 0 \\ -I & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

or, equivalently, in which the Poisson bracket attains the form

$$\{F, H\} = \sum_{i=1}^{n} \left(\frac{\partial F}{\partial q^i} \frac{\partial H}{\partial p^i} - \frac{\partial F}{\partial p^i} \frac{\partial H}{\partial q^i} \right). \tag{2.19}$$

Definition 2.35. *A coordinate system in which π attains the above form is called* canonical coordinates *for π or* Darboux coordinates *for π.*

This theorem is far from trivial and for its proof we refer the reader to a literature (see [43] and references therein). When we study the proof of Darboux theorem we notice two things:

1. Finding canonical (Darboux) coordinates for a given Poisson operator π is often a tedious task, since it basically requires solving the system of corresponding Hamiltonian equations (see below)

2. For a given π there always exists infinitely many canonical coordinate systems around each point. So, Darboux/canonical coordinates are far from unique.

2.2.5 Hamiltonian systems

We can now reformulate Definition 2.26 using the notion of Poisson operator.

Definition 2.36. *Suppose that π is a Poisson operator on $\mathcal{M}$ and $H \in C^\infty(\mathcal{M})$. The system of first order ODE's of the form*

$$\frac{dx}{dt} = \pi\, dH \tag{2.20}$$

is called the Hamiltonian system on $\mathcal{M}$ generated by the Hamiltonian H.

In other words, a Hamiltonian system on $\mathcal{M}$ is any dynamical system given by a Hamiltonian vector field. For a fixed π and H the system (2.20) defines the *Hamiltonian flow* on $\mathcal{M}$.

An important tool to study Hamiltonian systems is their constants of motion. Before we define this concept, let us note that the rate of change of a function $F \in C^\infty(\mathcal{M})$ along the Hamiltonian flow (2.20) is (we calculate it in an arbitrary coordinate system x)

$$\frac{dF}{dt} = \sum_{i=1}^{n} \frac{\partial F}{\partial x^i}\frac{dx^i}{dt} = \sum_{i=1}^{n} \frac{\partial F}{\partial x^i}\pi^{ij}\frac{dH}{dx^j} = \{F, H\}$$

so if $\{F, H\} = 0$ then F does not change along the flow of (2.20). Such a function is called t*he constant of motion (or first integral)* of the Hamiltonian system (2.20) or, if π is implicitly understood, the constant of motion of the Hamiltonian H.

Definition 2.37. *A function $F \in C^\infty(\mathcal{M})$ is a constant of motion for the Hamiltonian flow (2.20) if $\{F, H\} = 0$.*

Knowing constants of motion of a Hamiltonian system significantly simplifies solving it, since any solution of a Hamiltonian system must always lie on the common level surface of all constant of motion. This fact is heavily exploited in the next chapter that deals with *completely integrable* Hamiltonian systems i.e. such systems that have enough functionally independent integrals of motion so that they can be in principle solved by quadratures (which is a content of the famous Liouville theorem).

In classical mechanics one often considers the case when $\mathcal{M}$ is a $2n$-dimensional (i.e. even-dimensional) smooth manifold that is a cotangent bundle to some other smooth manifold, so that $\mathcal{M} = T^*\mathcal{Q}$ ($\mathcal{M}$ is in this context called the *phase space*, while $\mathcal{Q}$ itself is an n-dimensional manifold called the *configuration space*) while π is of maximal rank $2n$. Often π is chosen to be

$$\pi = \sum_{i=1}^{n} \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p^i},$$

where q^i are some coordinates on $\mathcal{Q}$ while p_i are the induce coordinates in $T^*\mathcal{Q}$. In such case the coordinates $(q, p) = (q^1, \ldots, q^n, p^1, \ldots, p^n)$ are canonical (Darboux) coordinates for π and π has the matrix form

$$\pi = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.$$

The coordinates q are called generalized positions while the coordinates p are called generalized momenta. One of the most profound methods of solving of such systems is the *Hamilton-Jacobi method* which we explain in chapter 4.

Sometimes one deals with Hamiltonian systems with Hamiltonians depending explicitly on time t, called non-autonomous Hamiltonian systems, see for example [12, 9]:

$$\frac{dx}{dt} = \pi(x) dH(x, t) \tag{2.21}$$

(the above notation stresses that π depends on x only while H may depend explicitly on the evolution parameter t as well). Then, as it is easy to calculate, the function $P(x,t)$ is a constant of motion for (2.21) if and only if

$$\frac{\partial P}{\partial t} + \{P, H\} = 0$$

is valid for all $x \in \mathcal{M}$ and for all $t \in \mathbb{R}$. Note that if H is time independent then it is automatically the constant of motion of the Hamiltonian system it defines.

Below we present some examples of Hamiltonian systems that stem from classical mechanics. All these examples are given in Darboux coordinates.

1. *The harmonic oscillator* is a mass-spring system with the Hamiltonian:

$$H(x,p) = \frac{p^2}{2m} + \frac{1}{2}kx^2,$$

where x represents the displacement of the mass m from the equilibrium and p is the momentum of the mass (the spring is assumed to be weightless). Since H is a constant of motion, the trajectories of the system constitute a family of ellipses $\frac{1}{2}kx^2 + \frac{p^2}{2m} = E$. The corresponding Hamiltonian system is given by

$$\frac{dx}{dt} = \frac{\partial H}{\partial p} = \frac{p}{m}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial x} = -kx.$$

2. Consider a system of N particles with masses m_i, interacting pairwise with potential forces depending on distances between particles. The Hamiltonian H of such system is given by

$$H(p,q) = \sum_{i=1}^{N} \frac{\|\overrightarrow{p_i}\|^2}{2m_i} + \sum_{i=2}^{N}\sum_{j=1}^{i-1} V\left(\|\overrightarrow{q_i} - \overrightarrow{q_j}\|\right),$$

where $\overrightarrow{q_i}, \overrightarrow{p_i} \in \mathbb{R}^3$ are vectors representing the position and momentum of the i^{th} particle of mass m_i and $V(r_{ij})\,(i > j)$ is the interaction potential between the particle i and the particle j, depending only on the distance $r_{ij} = \|\overrightarrow{q_i} - \overrightarrow{q_j}\|$ between both particles. As usual, $\|\|$ denotes the length of a vector. The Hamiltonian equations of motion become then

$$\frac{d\overrightarrow{q_i}}{dt} = \frac{1}{m_i}\,\overrightarrow{p_i}\,, \quad \frac{d\overrightarrow{p_i}}{dt} = \sum_{j=1}^{N} V(r_{ij})\left(\overrightarrow{q_i} - \overrightarrow{q_j}\right), \; i = 1, \ldots, N.$$

Our solar system is an example of such system, with

$$V(r_{ij}) = -\frac{Gm_i m_j}{r_{ij}},$$

where G is the gravitational constant.

3. *The mathematical pendulum* is a Hamiltonian system defined by the Hamiltonian

$$H(\theta, p) = \frac{p^2}{2ml^2} + mgl\left(1 - \cos(\theta)\right),$$

where m is the mass of the pendulum, l is the length of its string, θ is the distortion angle and p is the conjugate momentum. The corresponding Hamiltonian system is

$$\frac{d\theta}{dt} = \frac{\partial H}{\partial p} = \frac{p}{ml^2}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial \theta} = mgl\sin\theta.$$

3 Completely integrable systems and Liouville-Arnold theorem

In this chapter we introduce the fundamental notion of a *completely integrable (Liouville integrable) Hamiltonian system* and discuss the content of Arnold-Liouville theorem, describing the motion of such system. Roughly speaking, this theorem guarantees that if a Hamiltonian system has enough commuting and independent integrals of motion then there exist so called *action-angle variables* in which the dynamics of this system is particularly simple. Completely integrable systems have been studied for many years and have been applied to many problems of mathematical physics and celestial mechanics (see for example [2] or [48]).

There are several related notions of integrability in literature. On a general level one defines a system of ordinary or partial differential equations with one dependent variable (usually called time) to be an *integrable system* if there exists a (usually algebraic) integral (solution) of the system that satisfies arbitrary chosen initial conditions and that involves a finite number of arbitrary constants and arbitrary functions [31]. In the context of Hamiltonian systems however there is a more specific definition of an integrable system.

Definition 3.1. *The Hamiltonian system*

$$\frac{dx}{dt} = \pi \, dH \tag{3.1}$$

on a $2n$-dimensional Poisson manifold $\mathcal{M}$ is called completely integrable in the sense of Arnold-Liouville (or Liouville integrable) if it has n integrals of motion $f_1, \ldots, f_n$ such that $\{f_i, f_j\} = 0$ for $i, j = 1, \ldots, n$ (i.e. they pairwise Poisson-commute) and

such that their differentials df_i are linearly independent at each point of an open dense subset of $\mathcal{M}$.

The importance of this definition comes from the following theorem that describes the dynamics of Liouville integrable systems [3].

Theorem 3.2. *(Liouville-Arnold theorem) Suppose the Hamiltonian system (3.1) has n Poisson commuting integrals $f_1, ..., f_n$. Denote by $\mathcal{M}_c = \{f_1 = c_1, \ldots, f_n = c_n\}$ the common invariant level set of all the integrals. Then:*

- *If $\mathcal{M}_c$ is regular (i.e. if the differentials of $f_1, \ldots, f_n$ are independent on $\mathcal{M}_c$), compact and connected, then it is diffeomorphic to the n-dimensional torus $T^n = \{(\varphi_1, \ldots, \varphi_n) \mod 2\pi\}$.*
- *In a neighbourhood of the set $\mathcal{M}_c$ there exist canonical variables $(I, \varphi \mod 2\pi)$ (so that $\{\varphi_i, I_j\} = \delta_{ij}$, $\{\varphi_i, \varphi_j\} = 0$, $\{I_i, I_j\} = 0$ for $i, j = 1, \ldots, n$) called the action-angle variables such that: $I_i = I_i(f_1, \ldots, f_n)$ for $i = 1, \ldots, n$ and such that in the variables (I, φ) the Hamiltonian system (3.1) attains the form*

$$\frac{dI_i}{dt} = 0, \ \frac{d\varphi_i}{dt} = \omega_i(I). \tag{3.2}$$

The above theorem clarifies the geometry of solutions of any Liouville integrable system. Of course the tori $\mathcal{M}_c$ can be located in the manifold $\mathcal{M}$ in a very complicated manner. Moreover, finding the action-angle variables for a given Hamiltonian system can be a tedious task. Once found, however, they immediately yield us the solution of the Liouville integrable system (3.1):

$$I_i(t) = I_i(0), \ \varphi_i(t) = \omega_i(I)t + \varphi_i(0),$$

where $(I(0), \varphi(0))$ are the initial conditions. We may also remark that usually one of the f_i, say f_1, is taken as the Hamiltonian itself: $f_1 = H$.

In what follows we will need two important concepts: that of canonical transformation and that of generating function of a canonical transformation.

Definition 3.3. *Suppose that (q, p) are canonical coordinates on a Poisson manifold $(\mathcal{M} = T^*\mathcal{Q}, \pi)$. The map $(q, p) \to (a, b) = (a^1, \ldots, a^n, b_1, \ldots, b_n)$ is a canonical map (canonical transformation) if (a, b) are also canonical coordinates for π.*

A convenient way to define a canonical transformation is through its *generating function*. Suppose that we have a function $W(q, b)$ depending on "*old*" positions q^i and "*new*" momenta b_i. Define the functions $q_i(a, b)$ implicitly through

$$b_i = \frac{\partial W(q, a)}{\partial a_i}, \quad i = 1, \ldots, n \tag{3.3}$$

and then define $p_i(a, b)$ through

$$p_i(a, b) = \left.\frac{\partial W(q, a)}{\partial q_i}\right|_{q=q(a,b)}, \quad i = 1, \ldots, n. \tag{3.4}$$

It turns out then that the inverse map $(q, p) \to (a, b)$ is canonical.

Remark 3.4. *One way of finding action-angle canonical variables (φ, I) described in Liouville-Arnold theorem is to find an appropriate canonical transformation from the original variables to (φ, I). A possible way to do this is to solve a partial differential equation that is called Hamilton-Jacobi equation, associated with the Hamiltonian of the system. This method will be central to this thesis; we discuss it in Chapter 4.*

Below we present some examples of Liouville integrable systems, together with the corresponding action-angle coordinates. Many formulas in this part of the thesis will be presented without proof. An interested reader can find the details in the references within the examples.

1. Every time-independent Hamiltonian system in a two-dimensional phase space is of course always completely integrable. Consider as an example harmonic oscillator with the Hamiltonian $H(q, p) = \frac{1}{2}(p^2 + \omega^2 q^2)$. In this example $n = 1$ and $f_1 = H$ and we denote c_1 (i.e. value of H along a given trajectory) by E. The invariant tori $\mathcal{M}_{c_1} = \mathcal{M}_E$ are given by the ellipses $H(q, p) = E$ that vary as the value of E varies. The action variable I is given by

$$I = \frac{1}{2\pi} \oint_{\mathcal{M}_E} p dq = \frac{1}{2\pi} \int\int_{S_E} dp dq = \frac{E}{\omega}, \tag{3.5}$$

(where $\oint_{\mathcal{M}_E}$ denotes the definite integral along the ellipse $\mathcal{M}_E$). Here we have used Stokes' theorem in order to express the contour integral in terms of the area S_E enclosed by $\mathcal{M}_E$. This result means that I is simply a rescaled value of the energy E. In the action-angle variables (I, φ) the Hamiltonian H becomes $H = \omega I$ and thus (3.2) becomes

$$\frac{dI}{dt} = 0, \quad \frac{d\varphi}{dt} = \frac{\partial H}{\partial I} = \omega,$$

so that I is (obviously) constant during the motion while $\varphi(t) = \omega t + \varphi(0)$. Let us now express the action-angle variables (I, φ) in terms of original variables (q, p). We have of course $I = \dfrac{E}{\omega} = \dfrac{1}{2}(p^2\omega^{-1} + \omega q^2)$. The first half of the canonical map $(p, q) \to (I, \varphi)$ linearizing the flow can be found from the generating function

$$W(q, I) = \int p dq = \int \sqrt{2I\omega - \omega^2 q^2} dq.$$

Explicitly, according with (3.3)

$$\varphi = \frac{\partial W(q, I)}{\partial I} = \int \frac{\omega}{\sqrt{2I\omega - \omega^2 q^2}} dq = \arcsin\left(q\sqrt{\frac{\omega}{2I}}\right) - \varphi(0),$$

from which we obtain that

$$q = \sqrt{\frac{2I}{\omega}} \sin(\varphi + \varphi(0)).$$

The formula (3.4) reads

$$p = \frac{\partial W(q, I)}{\partial q} = \pm\sqrt{2I\omega - \omega^2 q^2}$$

from which we can easily reconstruct (3.5). It is now straightforward to calculate the solution in the original variables (q, p), using the solution in (I, ϕ). We obtain

$$q = \sqrt{\frac{2E}{\omega^2}} \sin(\omega t + \varphi(0)), \quad p = \sqrt{2E} \cos\left(\omega t + \varphi(0)\right),$$

of course, the above solution can be obtained by elementary calculations as well, but we choose this example to illustrate Liouville-Arnold theorem in a simple context. See [21] for details.

2. Let us consider the so called Kepler problem (that is motion of a point particle in a gravitational field generated by a point source that does not move) in polar coordinates (r, ϕ) and consider thus the four dimensional phase space $\mathcal{M}$ parametrized by the canonical coordinates ($q_1 = r$, $q_2 = \phi$, $p_1 = P_r = m\frac{dr}{dt}$, $p_2 = P_\phi = mr^2\frac{d\phi}{dt}$). The Kepler Hamiltonian is given by

$$H = \frac{1}{2}P_r^2 + \frac{1}{2}\frac{P_\phi^2}{r^2} - \frac{\alpha}{r},$$

where $\alpha > 0$ is a constant. This time $n = 2$ and we take $f_1 = H$ and denote c_1 by E. It is easy to check that $\{H, P_\phi\} = 0$ so that $f_2 = P_\phi$ is the second integral of motion, clearly independent from H, thus making the Kepler system integrable in the Liouville sense. Let us find action-angle variables for this system. Denote c_2 by μ. The level set $\mathcal{M}_c$ is given thus by $H = E$ and $P_\phi = \mu$ which also gives

$$P_r = \pm\sqrt{2E - \frac{\mu^2}{r^2} + \frac{2\alpha}{r}}.$$

It is possible to obtain the generating function $W(r, \phi, I_r, I_\phi)$ of the canonical map $(r, \phi, P_r, P_\phi) \to (\varphi_1, \varphi_2, I_1, I_2)$ to action-angle variables; the result is

$$W(r, \phi, I_r, I_\phi) = \frac{I_\phi \phi}{2\pi} + \int \sqrt{\frac{-2m\alpha^2}{(I_r + I_\phi)^2} - \frac{I_\phi^2}{4\pi^2 r^2}}dr. \tag{3.6}$$

From the equation (3.6) we can calculate (up to a quadrature) angle variables using the formulas (3.3)-(3.4), i.e.

$$\varphi_1 = \frac{\partial W(r, \phi, I_r, I_\phi)}{\partial I_r}, \quad \varphi_2 = \frac{\partial W(r, \phi, I_r, I_\phi)}{\partial I_\phi}$$

(the reader should distinguish the angle coordinates φ_i from the coordinate ϕ) while the Hamiltonian H in action-angle variables becomes

$$H = -\frac{\alpha^2}{2(I_1 + I_2)^2}$$

and thus the equations of motion become:

$$\frac{dI_1}{dt} = \frac{dI_2}{dt} = 0, \ \frac{d\varphi_1}{dt} = \frac{\partial H}{\partial I_1} = \frac{\alpha^2}{(I_1 + I_2)^3} \text{ and } \frac{d\varphi_2}{dt} = \frac{\partial H}{\partial I_2} = \frac{\alpha^2}{(I_1 + I_2)^3}.$$

See again [21] for details.

3. Let us examine the motion of a particle with mass m in space with the spherical coordinates (r, φ, θ) in the presence of the central force coming from the central potential $V(r) = -\frac{\mu}{r}$, where $\mu = GMm$ is called gravitational mass. Due to $V(r)$ that depends only on one of the 3 variable coordinates, i.e that Hamiltonian Jacobi equations is completely separable. The Hamiltonian function of this problem is:

$$H = \frac{1}{2m}\left(P_r^2 + \frac{P_\theta^2}{r} + \frac{P_\varphi^2}{r^2 \sin^2\theta}\right) - \frac{\mu}{r}, \tag{3.7}$$

where the momenta P_r, P_θ and P_φ are given by $P_r = m\frac{dr}{dt}$, $P_\theta = mr^2\frac{d\theta}{dt}$ and $P_\varphi = mr^2 \sin^2\theta \frac{d\varphi}{dt}$, respectively. In this example we will demonstrate that the Hamiltonian system given by (3.7) is completely integrable. Indeed, consider the following three functions on the phase space of this system:

$$f_1 = H, \;\; f_2 = P_\varphi, \;\; f_3 = P_\theta^2 + \frac{P_\varphi^2}{\sin^2\theta}.$$

Obviously, f_i are independent. It is also clear that $\{f_1, f_2\} = \{f_1, f_3\} = 0$. It remains to show that $\{f_2, f_3\} = 0$. This is done by direct calculation, using the formula (2.19). We obtain

$$\{f_2, f_3\} = -\frac{\partial}{\partial\theta}\left(P_\theta^2 + \frac{P_\varphi^2}{\sin^2\theta}\right)\frac{\partial P_\varphi}{\partial P_\theta} = 0$$

since the functions P_φ and P_θ belong to the same coordinate system. Thus, all f_i are mutually in involution and the system (3.7) is Liouville integrable. For construction of action-angle coordinates, we refer the reader to literature, for example to [32] or [48].

Other examples of Liouville integrable systems include the motion of particle confined to a sphere and under the force coming from a harmonic potential, the geodesic motion on a surface of revolution, the Toda lattice, the Hénon-Heiles system and many more.

4 Separable systems

From this chapter on we will focus solely on the situation $\mathcal{M} = T^*\mathcal{Q}$ and we will write q_i instead of q^i to denote coordinates on $\mathcal{Q}$, partially to simplify the notation but also to stress that in $\mathcal{M}$ the coordinates p and q have now equal status, as we will consider canonical maps that mix up p's and q's to new "*positions*" and "*momenta*".

4.1 Hamilton-Jacobi equation

The action-angle variables of a Liouville integrable system, if found, yield immediately its general solution. However, in practice, there is no good way of finding these variables. Beginning from the works of C. G. Jacobi [36] another approach to solving Hamiltonian systems was developed. This approach was based on the method of separation of variables in *the Hamilton-Jacobi (HJ) equation*

$$H\left(q_1,\dots,q_n,\frac{\partial W(q,a)}{\partial q_1},\cdots,\frac{\partial W(q,a)}{\partial q_n}\right)=a_1 \tag{4.1}$$

associated with a given Hamiltonian $H(q_1,\dots,q_n,p_1,\dots,p_n)$ written in canonical coordinates $(q,p)=(q_1,\dots,q_n,p_1,\dots,p_n)$ (we remind that it means that $\{q_i,q_j\}=\{p_i,p_j\}=0$ for any $i,j=1,\dots,n$ while $\{q_i,p_j\}=\delta_{ij}$); here $a=(a_1,\dots,a_n)$ is a set of n real parameters.

According to the Jacobi theorem (see below) if we are able to find a so called complete integral $W(q,a)$ of (4.1) then the function $W(q,a)$ is a generating function for a canonical change of coordinates to new coordinates $(b,a)=(b_1,\dots,b_n,a_1,\dots,a_n)$ and in the coordinates (b,a) the Hamiltonian system defined by H is trivial as it takes the form

$$\frac{da_i}{dt}=0,\frac{db_i}{dt}=\delta_{i1},$$

i.e. in the variables (b, a) the Hamiltonian flow is linearized, or the corresponding Hamiltonian vector field w_H is straightened. This is actually obvious as the Hamiltonian H in the new variables (b, a) is simply $H(b, a) = a_1$.

The Hamilton-Jacobi method is beautiful but it has a major drawback: the Hamilton-Jacobi equation (4.1) is a partial differential equation that is usually much harder to solve than the original Hamiltonian equations associated with the Hamiltonian H. The idea of Jacobi was then to look for the so called *additively separable* solutions of (4.1), i.e. complete integrals $W(q, a)$ of the form

$$W(q, a) = \sum_{i=1}^{n} W_i(q_i, a), \tag{4.2}$$

where each function W_i depends on q_i only (and on all a). In such a case the Hamilton-Jacobi equation (a PDE) can often be decoupled into a system of decoupled ODE's (that are still coupled through the parameters a_i) that can be solved by quadratures.

As a simple illustration of the above idea, let us consider the so called natural Hamiltonian:

$$H(q, p) = \frac{1}{2} \sum_{k=1}^{n} p_k^2 + V(q_1, \ldots, q_n).$$

The corresponding Hamilton-Jacobi equation has the form

$$\frac{1}{2} \sum_{k} \left(\frac{\partial W}{\partial q_k} \right)^2 + V(q_1, \ldots, q_n) = E \tag{4.3}$$

and is usually (that is for a randomly chosen potential V) impossible to solve analytically. Assume thus, additionally, that we are lucky and in the chosen coordinate system $(q_1, \ldots, q_n)$ the potential V is a sum of the form: $V(q) = \sum\limits_{k} V_k(q_k)$ where each V_k is an arbitrary differentiable function of one variable. Let us now look for an additively separable solution (4.2) to (4.3). We get

$$\frac{1}{2} \sum_{k} \left(\frac{\partial W_k(q_k, a)}{\partial q_k} \right)^2 + \sum_{k} V_k(q_k) = E. \tag{4.4}$$

The PDE (4.4) splits into n decoupled ODE's

$$\frac{1}{2} \left(\frac{\partial W_k}{\partial q_k} \right)^2 + V_k(q_k) = a_k, \quad k = 1, \ldots, n,$$

where $\sum_k a_k = E$, which can be immediately solved yielding the generating function of the sought canonical transformation $(q,p) \to (b,a)$:

$$W(q,a) = \sum_k \int^{q_k} \sqrt{2a_k - V_k(x)}dx.$$

Thus, the canonical transformation $(q,p) \to (b,a)$ is given by formulas (3.3) and (3.4), or

$$b_k = \frac{\partial W(q,a)}{\partial a_k} = \int^{q_k} (2a_k - V_k(x))^{-1/2}\, dx, \;\; p_k = \left.\frac{\partial W(q,a)}{\partial q_k}\right|_{q_k = q_k(a,b)},$$

where the first equation can be used to find $q_k = q_k(a,b)$ which is used in the substitution in the second formula. Since, due to the Jacobi theorem (see below), $b_1 = t + const$, while all remaining b_k and all a_k are constants, the above formulas also yield the time evolution of the system in the original variables (q,p).

The procedure of splitting the HJ equation into set of decoupled ODE's is called separation of variables. The constants a_k are in this context called separation constants [36]. The method of separation of variables has been successfully applied to many problems in analytical mechanics and in celestial mechanics, but many other systems, such as Manakov top, showed to be intractable by this method. Finding separation coordinates for a given system has proven to be a tedious task.

4.2 Levi-Civita condition

Given a Hamiltonian $H = H(q,p)$ in an a priori given canonical coordinate system (q,p) it is not at all obvious whether these coordinates are separable coordinates for H (that is, whether the HJ equation for H admits an additively separable solution) and if not, whether there exists a canonical transformation to some separable coordinates and finally, and if this is the case, how to find them. The simpler question whether a given Hamiltonian $H(q,p)$ is separable in a priori given canonical (Darboux) coordinate system (q,p) is answered in the famous theorem of Levi-Civita [41].

Theorem 4.1. ***(Levi-Civita, 1904)****: A necessary and sufficient conditions for a coordinate system (q,p) to be separable for a Hamiltonian H is that H satisfies the*

following condition (Levi-Civita condition):

$$\partial_i\partial_j H\partial^i H\partial^j H - \partial_j\partial^i H\partial^j H\partial_i H + \partial^i\partial^j H\partial_i H\partial_j H - \partial^j\partial_i H\partial_j H\partial^i H = 0, \qquad (4.5)$$

for all $i \neq j, i, j = 1, \ldots, n$, *where* $\partial_i = \frac{\partial}{\partial q_i}$ *while* $\partial^i = \frac{\partial}{\partial p_i}$ *(in (4.5) there is no summation over the repeated indices).*

For the proof of this theorem, see for example [54]. This theorem is a consequence of a more fundamental theorem in the theory of PDE's called *Frobenius theorem*. Thus, checking if H is separable in a given coordinate system (q, p) is a straightforward, albeit sometimes tedious, task. Finding out whether there *is* a separable coordinate system for a given Hamiltonian H is far from trivial and is a subject of vigorous research. Waksjö and Rauch [54] solved this problem for natural Hamiltonians (see below) of the form: $H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + V(q)$ but for an arbitrary Hamiltonian the problem is still open. And even for the case solved by Waksjö and Rauch, the procedure is tedious and requires a computer.

4.3 Orthogonal separation for natural Hamiltonians and Stäckel theorem

In this section we will focus on the so called natural Hamiltonians. A natural Hamiltonian is a Hamiltonian of the form

$$H(q, p) = \frac{1}{2}\sum_{i=1}^{n} \frac{p_i^2}{H_i^2(q)} + V(q) \qquad (4.6)$$

which describes the motion of a particle of mass $m = 1$ in the potential $V(q)$. The functions $H_i^2(q)$ can be interpreted as components of a diagonal metric tensor

$$g = \begin{pmatrix} H_1^2 & & \\ & \ddots & \\ & & H_n^2 \end{pmatrix}, \qquad (4.7)$$

so it also means that the coordinates q are orthogonal with respect to g i.e. the coordinate curves are at each point pairwise orthogonal in the metric g (the reader should

be careful here not to mix up the functions H_i with the Hamiltonian H itself). In this context the Hamiltonian (4.6) describes the motion of a particle on a Riemannian manifold.

Theory of separability of such Hamiltonian systems have been heavily studied, see for example [37, 38] and references therein. Here we mention only one result, due to P. Stäckel. Again, this result is valid only in a given coordinate system and says nothing about possibility of separating the considered Hamiltonian in other coordinate systems. Before we formulate this result, we need to formulate some definitions.

Definition 4.2. *A **Stäckel matrix** is a non-singular matrix $S(q)$ (so that $\det S(q) \neq 0$) such that its row i depends on the coordinate q_i only:*

$$S(q) = \begin{pmatrix} S_{11}(q_1) & \dots & S_{1n}(q_1) \\ \dots & \dots & \dots \\ S_{n1}(q_n) & \dots & S_{nn}(q_n) \end{pmatrix}.$$

A metric g of the form (4.7) is in the Stäckel form (in a given coordinate system q), if there exist a Stäckel matrix $S(q)$ such that

$$\left(H_1^{-2}, \dots, H_n^{-2}\right) S(q) = (1, 0, \dots, 0),$$

(which means that $\left(H_1^{-2}, \dots, H_n^{-2}\right)$ is the first row of $S^{-1}(q)$). Finally, a real valued function V is called a *Stäckel multiplier* in the metric g if V can be factorized to the form

$$V = \sum_{i=1}^{n} H_i^{-2} f_i(q_i),$$

where f_i are n smooth functions of one variable.

Theorem 4.3. ***(Stäckel,1891)**: The Hamiltonian-Jacobi equation*

$$\frac{1}{2} \sum_{i=1}^{n} H_i^{-2}(q) \left(\frac{\partial W}{\partial q_i}\right)^2 + V(q) = E$$

associated with the Hamiltonian (4.6) admits a complete separated solution in the coordinate system q orthogonal with respect to the metric g given by (4.7) if and only if:

- *The metric g is in the Stäckel form in q*
- *The potential V is a Stäckel multiplier in the metric g.*

For the proof of this theorem, see for example [54].

4.4 Separation relations and Jacobi theorem

In the previous sections we introduced the concept of additive variable separation for Hamilton-Jacobi equations and briefly mentioned some results that help us check whether a given Hamiltonian is separable in a given coordinate system. However, as we also mentioned above, finding out whether a given Hamiltonian is separable in some, unknown from the beginning, coordinate system is far from trivial. In this section we present therefore another approach - a modern theory of Liouville separable systems that has its origin in the work of Sklyanin [49]. Roughly speaking, the idea is to look for a "*machinery*" producing separable systems rather than finding our whether a given system is separable or not. In this way we can gain deeper geometric insights about separable systems.

We begin our study of separable Liouville systems in this new spirit by defining *separation relations*, a concept introduced by Sklyanin in [49].

Definition 4.4. *Consider a $2n$-dimensional manifold $\mathcal{M}$ equipped with a Poisson bracket π. Suppose also that $(\lambda, \mu) = (\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n)$ are Darboux coordinates on $\mathcal{M}$ (which means that $\{\lambda_i, \lambda_j\} = \{\mu_i, \mu_j\} = 0$ for all $i, j = 1, \ldots, n$ while $\{\lambda_i, \mu_j\} = \delta_{ij}$). A set of algebraic equations of the form*

$$\varphi_i(\lambda_i, \mu_i, a_1, \ldots, a_n) = 0,\ i = 1, \ldots, n \tag{4.8}$$

(where (λ_i, μ_i) are pairs of canonically conjugate Darboux coordinates) is called separation relations if it is globally solvable (except possibly for a union of lower dimensional submanifolds) with respect to the parameters $a_j \in \mathbb{R}$.

Solving (4.8) with respect to a_j yields

$$a_j = H_j(\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n), \quad j = 1, \ldots, n, \tag{4.9}$$

where the functions $H_j : \mathcal{M} \to \mathbb{R}$ are easily proven to be in involution with respect to the Poisson bracket defined by π. Actually, the following theorem is valid [42]:

Theorem 4.5. *Suppose that* $\{\varphi_i, \varphi_j\} = 0$ *for* $i, j = 1, \ldots, p$ *(and for all values of the parameters* a_r*) where* $\varphi_i = \varphi_i(x, a_1, \ldots, a_p)$ *with* $x \in \mathcal{M}$*. Then the functions* $H_1, \ldots, H_p$ *defined implicitly through*

$$\varphi_i(x, H_1, \ldots, H_p) = 0, \;\; i = 1, \ldots, p,$$

also Poisson commute: $\{H_i, H_j\} = 0$ *for* $i, j = 1, \ldots, p$.

Proof. Differentiating the identity above with respect to x_j, we obtain

$$\frac{\partial \varphi_i}{\partial x_j} + \sum_{k=1}^{p} \frac{\partial \varphi_i}{\partial a_k} \frac{\partial H_k}{\partial x_j} = 0 \Rightarrow \frac{\partial H_k}{\partial x_j} = -\sum_{s=1}^{p} A_{ks} \frac{\partial \varphi_s}{\partial x_j},$$

where $A = \left[\frac{\partial \varphi_i}{\partial a_k}\right]^{-1}$ is the inverse of the $p \times p$ matrix $\left[\frac{\partial \varphi_i}{\partial a_k}\right]$. Thus

$$\{H_i, H_j\} = \langle dH_i, \pi dH_j \rangle = \sum_{s,r=1}^{p} A_{is} A_{jr} \{\varphi_s, \varphi_r\} = 0.$$

□

Therefore, since every function φ_i in (4.8) depends only on λ_i and μ_i (and not on the remaining λ_j and μ_j), the functions φ_i in (4.8) are in involution and thus, due to the above theorem, the functions H_i defined in (4.9) are also in involution. That also means that the set of n Hamiltonian systems

$$\frac{dx}{dt_i} = \pi dH_i, \; i = 1, \ldots, n \tag{4.10}$$

constitute a Liouville integrable system *as soon as* H_i are independent. We will now demonstrate that each Hamiltonian system in (4.10) is also separable in the sense of Hamilton-Jacobi theory, which explains why the relations (4.8) are called separation relations. Before we do that, we formulate and prove the main theorem of Hamilton-Jacobi theory:

Theorem 4.6. *(Jacobi) Assume that n Hamiltonians H_i constitute a completely integrable system and that a function $W(\lambda_1, \ldots, \lambda_n, a_1, \ldots, a_n)$ simultaneously solves all the corresponding HJ equations*

$$H_i\left(\lambda_1, \ldots, \lambda_n, \frac{\partial W(\lambda, a)}{\partial \lambda_1}, \cdots, \frac{\partial W(\lambda, a)}{\partial \lambda_n}\right) = a_i,\ i = 1, \ldots, n. \tag{4.11}$$

Then the functions

$$b_i = \frac{\partial W(\lambda, a)}{\partial a_i}, \quad i = 1, \ldots n \tag{4.12}$$

satisfy $b_i = t_i + const_i$.

Note that (4.12) is simply (3.3) and as such it can be used to obtain the functions $\lambda_i = \lambda_i(a, b)$ i.e. the first half of the map $(\lambda, \mu) \to (b, a)$ between the Darboux coordinates and the coordinates (b, a). The second half of this map is obtained through (3.3), i.e.

$$\mu_i(a, b) = \left.\frac{\partial W(\lambda, a)}{\partial \lambda_i}\right|_{\lambda=\lambda(a,b)}.$$

Thus, the canonical transformation $(\lambda, \mu) \overset{W(\lambda,a)}{\to} (b, a)$ linearize all the flows (4.10) at once: in the (b, a)-coordinates the equations (4.10) become:

$$\frac{db_i}{dt_j} = \delta_{ij},\ \ \frac{da_i}{dt_j} = 0 \Rightarrow b_i = t_i + c_i, \quad a_i = const_i, \tag{4.13}$$

so they are all simultaneously linearized by the canonical transformation $(\lambda, \mu) \overset{W(\lambda,a)}{\to} (b, a)$. Finally, reversing this canonical map we can obtain the (usually nonlinear) time evolution of the system in the original variables (λ, μ). Note that the equations (4.13) follow easily from the fact that the Hamiltonian H_i is given by $H_i(b, a) = a_i$.

Proof. Differentiation of (4.11) w.r.t. a_k yields

$$\sum_{s=1}^{n} \frac{\partial H_i}{\partial \mu_s} \frac{\partial^2 W(\lambda, a)}{\partial a_k \partial \lambda_s} = \delta_{ik} \text{ for all } i, k = 1, \ldots, n, \tag{4.14}$$

while differentiating b_k in (4.12) w.r. t. t_i yields

$$\frac{\partial b_k}{\partial t_i} = \sum_{s=1}^{n} \frac{\partial^2 W(\lambda, a)}{\partial a_k \partial \lambda_s} \frac{\partial \lambda_s}{\partial t_i} \ \text{ for all } i, k = 1, \ldots, n. \tag{4.15}$$

But, due to Hamilton equations of motion

$$\frac{\partial \lambda_s}{\partial t_i} = \frac{\partial H_i}{\partial \mu_s}.$$

Thus, comparing (4.14) and (4.15) we conclude, that

$$\frac{\partial b_k}{\partial t_i} = \delta_{ik} \text{ for all } i, k = 1, \ldots, n.$$

□

Now, if we substitute $\mu_i = \frac{\partial W_i(\lambda_i, a)}{\partial \lambda_i}$ into the separation relations (4.8), we obtain n decoupled ODE's

$$\varphi_i \left(\lambda_i, \tfrac{\partial W_i(\lambda_i, a)}{\partial \lambda_i}, a_1, \ldots, a_n \right) = 0, \;\; i = 1, \ldots, n. \tag{4.16}$$

If we now find for each of these equations a solution (i.e. a function $W_i(\lambda_i, a)$ that solves equation i in (4.16)) then

$$W(\lambda, a) = \textstyle\sum_{i=1}^{n} W_i(\lambda_i, a)$$

becomes an additively separable solution of each of the Hamilton-Jacobi equations (4.11),

$$H_i \left(\lambda_1, \ldots, \lambda_n, \mu_1 = \frac{\partial W(\lambda, a)}{\partial \lambda_1}, \cdots, \mu_n = \frac{\partial W(\lambda, a)}{\partial \lambda_n} \right) = a_i,$$

simply because solving the separation relations (4.8) to (4.9) is a purely algebraic operation, with no derivations involved. Thus, all the Hamiltonian systems in (4.10) are separable. The Darboux coordinates (λ, μ) are in this context called *separation coordinates* as in these coordinates all the Hamilton-Jacobi equations (4.11) admit a common additively separable solution (i.e. simultaneously separate). Of course, there is still the issue of actually solving each of the ODE's in (4.16), but it is usually much simpler to solve an ODE than to solve a (usually highly nonlinear) PDE.

In the case that all the functions φ_i are identical, $\varphi_i = \varphi$, we can encode all the separation relations (4.8) into one geometric object called *separation curve*

$$\varphi(\lambda, \mu, a_1, \ldots, a_n) = 0$$

in the sense that taking n copies of this curve at n distinct points (λ_i, μ_i) reproduces the separation relations (4.8) with all $\varphi_i = \varphi$.

To illustrate the above ideas, consider the separation curve of the form

$$H_1\lambda + H_2 = \frac{1}{2}\lambda\mu^2 + \lambda^4$$

(so that $n = 2$), see [8]. This curve yields the following separation relations

$$H_1\lambda_1 + H_2 = \frac{1}{2}\lambda_1\mu^2 + \lambda_1^4, \;\; H_1\lambda_2 + H_2 = \frac{1}{2}\lambda_2\mu^2 + \lambda_2^4.$$

Solving this linear system with respect to H_1 and H_2 we get a Liouville integrable system in the four-dimensional space parametrized by the coordinates (λ, μ). Explicitly,

$$H_1 = \frac{1}{2}\frac{\lambda_1\mu_1^2 - \lambda_2\mu_2^2 + 2\lambda_1^4 - 2\lambda_2^4}{\lambda_1 - \lambda_2}, \;\; H_2 = \frac{1}{2}\frac{\lambda_1\lambda_2\left(\mu_1^2 - \mu_2^2 + 2\lambda_1^3 - 2\lambda_2^3\right)}{\lambda_2 - \lambda_1}.$$

Consider the following transformation between the position coordinates

$$q_1 = \lambda_1 + \lambda_2, \;\; q_2 = 2\sqrt{-\lambda_1\lambda_2}$$

and the transformation of conjugate momenta being the differential consequence of the above transformation. That is, we consider a so called point transformation in the phase space which means that if the Jacobian of the transformation from λ_i to q_i is J then the map between new and old momenta in $T^*\mathcal{Q}$ is linear at each point of $\mathcal{Q}$ and given by

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \left(J^{-1}\right)^T \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}.$$

As it is easy to check, a point transformation is always a canonical transformation, so the new coordinates (q, p) are also canonical. Since in our example the Jacobian J is given explicitly by

$$J = \begin{pmatrix} 1 & 1 \\ \\ \dfrac{-\lambda_2}{\sqrt{-\lambda_1\lambda_2}} & \dfrac{-\lambda_1}{\sqrt{-\lambda_1\lambda_2}} \end{pmatrix},$$

and since

$$J^{-T} = \begin{pmatrix} \dfrac{\lambda_1}{\lambda_1 - \lambda_2} & -\dfrac{\lambda_2}{\lambda_2 - \lambda_2} \\ \\ \dfrac{\sqrt{-\lambda_1\lambda_2}}{\lambda_1 - \lambda_2} & -\dfrac{\sqrt{-\lambda_1\lambda_2}}{\lambda_2 - \lambda_1} \end{pmatrix},$$

we obtain

$$p_1 = \frac{\lambda_1\mu_1 - \lambda_2\mu_2}{\lambda_1 - \lambda_2},\ p_2 = \frac{\sqrt{-\lambda_1\lambda_2}\,(\mu_1 - \mu_2)}{\lambda_1 - \lambda_2},$$

and an explicit calculation shows that the Hamiltonians H_i in the new variables (q, p) attain the form

$$H_1 = \frac{1}{2}\left(p_1^2 + p_2^2\right) + q_1^3 + \frac{1}{2}q_1q_2^2,\ H_2 = \frac{1}{2}q_2p_1p_2 - \frac{1}{2}q_1p_2^2 + \frac{1}{16}q_2^4 + \frac{1}{4}q_1^2q_2^2.$$

The function $H_1(q, p)$ turns out to be the Hamiltonian function of one of the integrable cases of Hénon-Heiles system while the $H_2(q, p)$ is an additional integral of the system, in involution with H_1. This system has been first studied by Michel Hénon and Carl Heiles, who worked on the problem of nonlinear motion of a star around a galactic centre with motion restricted to the plane [35].

5 Stäckel separable systems

5.1 Generalized Stäckel systems

Among all possible separations relations (4.8) a natural subclass consists of the separation relations that are linear in the Hamiltonians H_k:

$$\sum_{k=1}^{n} S_{ik}(\lambda_i, \mu_i) H_k = \psi_i(\lambda_i, \mu_i), \qquad i = 1, \ldots, n. \tag{5.1}$$

Here S_{ik} and ψ_i are arbitrary smooth functions of two arguments (we use here H_k rather than a_k since a_k become anyway H_k after solving (4.8) to (4.9)). The relations (5.1) are called the *generalized Stäckel separation relations* and the related dynamical systems, obtained by solving (5.1) with respect to H_k, are called the *generalized Stäckel separable systems*. The matrix $S = [S_{ik}(\lambda_i, \mu_i)]$ is called a *generalized Stäckel matrix* (cf. Definition 4.2). The system (5.1) can of course be cast into the matrix form

$$S\mathbf{H} = \mathbf{U},$$

where $\mathbf{H} = (H_1, \ldots, H_n)^T$ and $\mathbf{U} = (\psi_1, \ldots, \psi_n)^T$, with the obvious formal solution

$$\mathbf{H} = S^{-1}\mathbf{U},$$

(note that the explicit algebraic form of S^{-1} can be very complicated). Although the restriction to separation relations linear in H_k seems to be very strong, it appears that an overwhelming majority of all separable systems considered in the literature falls into various subclasses of this class. For example, the class of separation relations with momenta entering exponentially the functions ψ_i

$$\sum_{j=1}^{n} H_j {\lambda_i}^{n-j} = \exp(a\mu_i) + \exp(-b\mu_i) + \varphi_i(\lambda_i), \quad i = 1, \ldots, n, \quad a, b \in \mathbb{R}_+$$

includes such systems as the periodic Toda lattice [28], the KdV dressing chain [5], the Ruijsenaars-Schneider system [4] and others. Another example is the class where ψ_i depends cubically on the momenta μ:

$$\mu_i \sum_{j=1}^{n_1} H_j^{(1)} {\lambda_i}^{n_1-j} + \sum_{j=1}^{n_2} H_j^{(2)} {\lambda_i}^{n_2-j} = \mu_i^3 + \mu_i \varphi_1(\lambda_i) + \varphi_2(\lambda_i), \qquad i = 1, \dots, n.$$

This class contains for instance the stationary flows of the so called Boussinesq hierarchy (for $n_1 = 1$ and $n_2 = n - 1$) [4, 27], while dynamical system on the loop algebra $\hat{\mathfrak{sl}}(3)$ belongs to this class with $n_1 = 2$ and $n_2 = 4$ [29]. However, by far the most important class of systems in (5.1) is the class of *classical Stäckel systems*, that is systems with the matrix S is a Stäckel matrix (see Definition 4.2) (so that $S_{ik} = S_{ik}(\lambda_i)$) and with ψ_i being *quadratic* in momenta μ:

$$S_{ik}(\lambda_i, \mu_i) = S_{ik}(\lambda_i), \qquad \psi_i(\lambda_i, \mu_i) = \tfrac{1}{2} f_i(\lambda_i) \mu_i^2 - \varphi_i(\lambda_i).$$

so that the separation relations (5.1) attain the form

$$\varphi_i(\lambda_i) + \sum_{k=1}^{n} S_{ik}(\lambda_i) H_k = \tfrac{1}{2} f_i(\lambda_i) \mu_i^2, \qquad i = 1, \dots, n. \tag{5.2}$$

The relations (5.2) are called *Stäckel separation relations*. A particular Stäckel system is thus defined by a choice of the Stäckel matrix $S_{ik}(\lambda_i)$ and by a choice of $2n$ functions f_i and φ_i. Solving the relations (5.2) with respect to H_k we obtain n quadratic in momenta functions (Hamiltonians) on $\mathcal{M}$

$$H_r = \frac{1}{2} \mu^T A_r \mu + V_r(\lambda), \qquad r = 1, \dots, n, \tag{5.3}$$

where A_r are $n \times n$ matrices given by

$$A_r = \operatorname{diag}\left(f_1(\lambda_1) \left(S^{-1}\right)_{r1}, \dots, f_n(\lambda_n) \left(S^{-1}\right)_{rn} \right), \qquad r = 1, \dots, n.$$

As the Hamiltonians (5.3) are defined through separation relations, they are in involution with respect to the canonical Poisson bracket on $\mathcal{M}$ and they are also separable in the variables (λ, μ), see Chapter 4.

There is a beautiful geometric interpretation of Stäckel systems given by (5.3). If we factorize A_r as $A_r = K_r G$, where

$$G = A_1 = \operatorname{diag}\left(f_1(\lambda_1) \left(S^{-1}\right)_{11}, \cdots, f_n(\lambda_n) \left(S^{-1}\right)_{1n} \right)$$

and

$$K_r = \mathrm{diag}\left(\frac{(S^{-1})_{r1}}{(S^{-1})_{11}}, \cdots, \frac{(S^{-1})_{rn}}{(S^{-1})_{1n}}\right), \qquad r = 1, \ldots, n$$

(so that $K_1 = I$) then we can treat (interpret) the matrix G as a contravariant form of a metric tensor on a manifold $\mathcal{Q}$ such that $\mathcal{M} = T^*\mathcal{Q}$ is the cotangent bundle to $\mathcal{Q}$. The corresponding covariant metric tensor will be denoted by g so that $gG = I$. It can be shown that the matrices K_r are then $(1,1)$-Killing tensors of the metric G. For a fixed Stäckel matrix S we have thus the whole family of metrics G parametrized by n arbitrary functions f_i of one variable. The tensors K_r are then Killing tensors for any metric from this family. Thus, the Stäckel Hamiltonians H_r in (5.3) are geodesic Hamiltonians of a Liouville integrable system in the Riemannian space $(\mathcal{M}, g)$. Finally, the functions $V_r(\lambda)$ are functions on $\mathcal{Q}$ defined by the separation relations

$$\sum_{k=1}^{n} S_{ik}(\lambda_i)V_k = -\varphi_i(\lambda_i), \qquad i = 1, \ldots, n,$$

and are called in literature separable potentials on $\mathcal{Q}$.

5.2 Stäckel systems of Benenti type

In the remaining part of this dissertation we will focus on a particular class of Stäckel systems that is called in literature *Benenti systems*, to honour the Italian mathematician Sergio Benenti who made major contributions to theory of separation of variables in various types of PDE's [14]- [18]. Benenti systems have been studied much in literature recently, see for example [8, 9] and references therein. These systems are defined by the separation relations of the form

$$\varphi_i(\lambda) + \sum_{j=1}^{n} \lambda_i^{n-j} H_j = \frac{1}{2} f_i(\lambda_i)\mu_i^2, \qquad i = 1, \ldots, n. \tag{5.4}$$

Thus, we assume from now on that the Stäckel matrix S in (5.2) is of the very particular form $S_{ij} = \lambda_i^{n-j}$ or explicitly:

$$S = \begin{pmatrix} \lambda_1^{n-1} & \lambda_1^{n-2} & \ldots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_n^{n-1} & \lambda_n^{n-2} & \ldots & 1 \end{pmatrix}$$

thus being a Vandermonde matrix. The inverse of this matrix is given by the following lemma.

Lemma 5.1. *If S is the $n \times n$ Vandermonde matrix given by $S_{ij} = \lambda_i^{n-j}$ then*

$$\left[S^{-1}\right]_{ij} = -\frac{1}{\Delta_j}\frac{\partial \rho_i}{\partial \lambda_j},$$

where

$$\rho_i = (-1)^i \sigma_i(\lambda), \qquad \Delta_j = \prod_{k \neq j} (\lambda_j - \lambda_k)$$

and where $\sigma_r(\lambda)$ are elementary symmetric polynomials.

By definition

$$\sigma_i(\lambda) = \sum_{1 \leq j_1 < \ldots < j_i \leq n} \lambda_{j_1} \ldots \lambda_{j_i}, \quad i = 1, \ldots, n,$$

so that

$$\sigma_0 = 1, \;\; \sigma_1 = \sum_{i=1}^{n} \lambda_i, \;\; \sigma_2 = \sum_{1 \leq i < j \leq n} \lambda_i \lambda_j, \;\; \ldots, \;\; \sigma_n = \prod_{i=1}^{n} \lambda_i.$$

Lemma 5.1 can be proved by a direct calculation. By this lemma, solving (5.4) with respect to H_r yields n functions (Hamiltonians) H_r on $\mathcal{M}$

$$H_r = -\frac{1}{2}\sum_{i=1}^{n} \frac{\partial \rho_r}{\partial \lambda_i}\frac{f_i(\lambda_i)\mu_i^2}{\Delta_i} + V_r(\lambda) \equiv \frac{1}{2}\mu^T K_r G \mu + V_r(\lambda), \qquad r = 1, \ldots, n \tag{5.5}$$

called *Benenti Hamiltonians*. Thus, for Benenti Hamiltonians the metric tensor G is given by

$$G = \operatorname{diag}\left(\frac{f_1(\lambda_1)}{\Delta_1}, \cdots, \frac{f_n(\lambda_n)}{\Delta_n}\right)$$

while the Killing tensors K_r are given by

$$K_r = -\operatorname{diag}\left(\frac{\partial \rho_r}{\partial \lambda_1}, \cdots, \frac{\partial \rho_r}{\partial \lambda_n}\right) \quad r = 1, \ldots, n. \tag{5.6}$$

From now on we will assume that all f_i are equal, and likewise all φ_i:

$$f_i = f, \quad \varphi_i = \varphi$$

so that all the Hamiltonians (5.5) are generated by the single separation curve:

$$\varphi(\lambda) + \sum_{j=1}^{n} \lambda^{n-j} H_j = \frac{1}{2} f(\lambda)\mu^2 \tag{5.7}$$

and are given explicitly by:

$$H_r = -\frac{1}{2}\sum_{i=1}^{n}\frac{\partial\rho_r}{\partial\lambda_i}\frac{f(\lambda_i)\mu_i^2}{\Delta_i} + V_r(\lambda) \equiv \frac{1}{2}\mu^T K_r G \mu + V_r(\lambda), \qquad r = 1,\dots,n \tag{5.8}$$

and thus the metric tensor G is now given by

$$G = \operatorname{diag}\left(\frac{f(\lambda_1)}{\Delta_1},\cdots,\frac{f(\lambda_n)}{\Delta_n}\right)$$

of particular interest is the case $f(\lambda_i) = \lambda_i^m$ with $m \in \mathbb{Z}$. In such a case the metric tensor G will be denoted by G_m:

$$G_m = \operatorname{diag}\left(\frac{\lambda_1^m}{\Delta_1},\cdots,\frac{\lambda_n^m}{\Delta_n}\right),\ m \in \mathbb{Z}.$$

Of course, if f is a Laurent polynomial (polynomial)

$$f(\lambda) = \sum_{\alpha\in A} a_\alpha \lambda_i^\alpha,$$

where $A \subset \mathbb{Z}$ is a finite set, then

$$G = \sum_{\alpha\in A} a_\alpha G_\alpha. \tag{5.9}$$

It can be shown that the metric G_m is flat for $m \in \{0,\dots,n\}$ and of constant curvature for $m = n+1$ (by linearity of (5.7) the same is true for f being a polynomial in λ of order m). Moreover

$$G_m = L^m G_0,\ \ G_0 = \operatorname{diag}\left(\frac{1}{\Delta_1},\cdots,\frac{1}{\Delta_n}\right), \tag{5.10}$$

where

$$L = \operatorname{diag}(\lambda_1,\dots,\lambda_n)$$

is a $(1,1)$-tensor called *special conformal Killing tensor* [20]. So, if it can be shown that all the Killing tensors K_r can be calculated from the formula

$$K_1 = I, \quad K_r = \sum_{k=0}^{r-1}\rho_k L^{r-1-k}, \quad r = 2,\dots,n. \tag{5.11}$$

In order to illustrate the form of separable potentials $V_r(\lambda)$ in the Benenti case, let us further assume that φ is a Laurent sum, that is

$$\varphi(\lambda) = \sum_{\alpha\in A} c_\alpha \lambda_i^\alpha, \tag{5.12}$$

where $A \subset \mathbb{Z}$ is finite and where c_α are some real constants. The Benenti separation relations (5.4) specify then to

$$\sum_{\alpha\in A} c_\alpha \lambda_i^\alpha + \sum_{j=1}^{n} \lambda_i^{n-j} H_j = \frac{1}{2} f(\lambda_i)\mu_i^2, \quad i = 1, \ldots, n \tag{5.13}$$

and due to their linearity we have

$$V_r = \sum_{\alpha\in A} c_\alpha V_r^{(\alpha)},$$

where $V_r^{(\alpha)}$ are so called basic separable potentials. By linearity of (5.13), the potentials $V_r^{(\alpha)}$ satisfy the relations

$$\lambda_i^\alpha + \sum_{r=1}^{n} V_r^{(\alpha)} \lambda_i^{n-r} = 0, \quad i = 1, \ldots, n$$

and by Lemma 5.1, they are thus given by

$$V_r^{(\alpha)} = \sum_{i=1}^{n} \frac{\partial \rho_r}{\partial \lambda_i} \frac{\lambda_i^\alpha}{\Delta_i}, \qquad r = 1, \ldots, n.$$

The basic separable potentials $V_r^{(\alpha)}$ can be explicitly constructed by the following formula (see [13]):

$$V^{(\alpha)} = R^\alpha V^{(0)}, \quad V^{(\alpha)} = (V_1^{(\alpha)}, \ldots, V_n^{(\alpha)})^T, \tag{5.14}$$

where

$$R = \begin{pmatrix} -\rho_1 & 1 & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ \vdots & 0 & 0 & 1 \\ -\rho_n & 0 & 0 & 0 \end{pmatrix} \tag{5.15}$$

and $V^{(0)} = (0, \ldots, 0, -1)^T$. The first n basic potentials are trivial

$$V_k^{(\alpha)} = -\delta_{k,n-\alpha}, \quad \alpha = 0, \ldots, n-1.$$

The first nontrivial positive potential is

$$V^{(n)} = (\rho_1, \ldots, \rho_n)^T$$

and higher potentials are more complicated polynomials in q_i. The first negative potential is

$$V^{(-1)} = \left(\frac{1}{\rho_n}, \cdots, \frac{\rho_{n-1}}{\rho_n}\right)^T$$

and the higher negative potentials are more complicated rational functions of all ρ_i. Note also that the recursion formulas (5.14)-(5.15) are not tensorial; they look the same in any coordinate system.

5.3 Polynomial form of Benenti systems

As we saw in the previous section, even the relatively simple Benenti Hamiltonians are complicated rational functions when expressed in separation variables (λ, μ). In this section we will demonstrate two canonical maps that under certain conditions transform Benenti Hamiltonians (5.8) to a polynomial form.

5.3.1 Benenti systems in Viète coordinates

Suppose that we change the position coordinates on the base manifold $\mathcal{Q}$ through the map

$$q_i = \rho_i(\lambda) \quad i = 1, ..., n. \tag{5.16}$$

According with (2.6), this map induces the map (point transformation) on $T^*\mathcal{Q}$:

$$p = \left(J_V^{-1}\right)^T \mu, \tag{5.17}$$

where J_V is the Jacobian of the map (5.16):

$$(J_V)_{ij} = \frac{\partial \rho_i}{\partial \lambda_j}. \tag{5.18}$$

Let us find an explicit form of (5.17). To do this we need a simple lemma from linear algebra.

Lemma 5.2. *Denote by k_i the i-th column of an $n \times n$ nondegenerate matrix A:*

$$A = (k_1|k_2|\ldots|k_n)$$

and by r_j the j-th row of its inverse

$$A^{-1} = \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{pmatrix}.$$

Then, if $\alpha_i \in \mathbb{R}$ for $i = 1, \ldots, n$

$$(\alpha_1 k_1 | \alpha_2 k_2 | \ldots | \alpha_n k_n)^{-1} = \begin{pmatrix} r_1/a_1 \\ r_2/\alpha_2 \\ \vdots \\ r_n/\alpha_n \end{pmatrix}.$$

This elementary lemma follows from the fact that $r_i k_j = \delta_{ij}$. An analogous lemma is of course true if we consider rows of A instead of its columns. Combining lemmas 5.1 and 5.2 we obtain that

$$\left(J_V^{-1}\right)_{ij} = -\frac{\lambda_i^{n-j}}{\Delta_i} \tag{5.19}$$

and thus the map (5.17) can be written as

$$p_i = -\sum_{k=1}^{n} \frac{\lambda_k^{n-i}\mu_k}{\Delta_k}, \quad i = 1, ..., n. \tag{5.20}$$

The coordinates (q, p) defined by (5.16) and (5.20) are called Viète coordinates. To summarize, the map $(\lambda, \mu) \to (q, p)$ from separation coordinates to Viète coordinates is given by

$$q_i = \rho_i(\lambda), \quad p_i = -\sum_{k=1}^{n} \frac{\lambda_k^{n-i}\mu_k}{\Delta_k}, \quad i = 1, ..., n. \tag{5.21}$$

Being a point transformation, the map (5.21) is a canonical map which means that Viète coordinates are Darboux (canonical) coordinates as well:

$$\{q_i, q_j\} = \{p_i, p_j\} = 0, \quad \{q_i, p_j\} = \delta_{ij}.$$

Let us now investigate the structure of Benenti Hamiltonians (5.8) in Viète coordinates (q, p). The Hamiltonians (5.8) are of course written in tensorial form so that in Viète coordinates

$$H_r(q, p) = \frac{1}{2} p^T K_r(q) G(q) p + V_r(q), \qquad r = 1, \ldots, n \tag{5.22}$$

where, by transformation laws for tensors, see (2.8) and (2.7),

$$K_r(q) = J_V K_r \left(J_V\right)^{-1}, \;\; G(q) = J_V G \left(J_V\right)^T . \tag{5.23}$$

The first formula in (5.23) yields, after some calculation

$$\left(K_r(q)\right)^i_j = \begin{cases} q_{i-j+r-1}, & i \leq j \text{ and } r \leq j \\ -q_{i-j+r-1}, & i > j \text{ and } r > j \\ 0 \quad \text{otherwise} \end{cases} . \tag{5.24}$$

Here and throughout the whole section we use the convention that $q_0 = 1$ and $q_k = 0$ for $k < 0$ and for $k > n$. Thus, all the $K_r(q)$ are linear in q-variables. Further, for the monomial case $f(\lambda_i) = \lambda_i^m$ with $m \in \{0, \ldots, n+1\}$ we can obtain from the second formula in (5.23) that

$$G^{ij}_m(q) \;\; = \begin{cases} q_{i+j+m-n-1}, & i,j = 1, \ldots, n-m \\ -q_{i+j+m-n-1}, & i,j = n-m+1, \ldots, n \qquad\qquad m = 0, \ldots, n \\ 0 \quad \text{otherwise} \end{cases} \tag{5.25}$$

$$G^{ij}_m(q) \;\; = q_i q_j - q_{i+j}, \quad i,j = 1, \ldots, n, \qquad m = n+1.$$

The formulas (5.24) and (5.25) can alternatively be obtained with the help of the special conformal Killing tensor L by using the formulas (5.11) and (5.10), respectively, and the fact that the tensor L can be easily calculated in Viète coordinates, due to the tensor transformation law (2.8) i.e. $L(q) = J_V L \left(J_V\right)^{-1}$. We obtain

$$L^i_j(q) = -\delta^1_j q_i + \delta^{i+1}_j$$

that is

$$L(q) = \begin{pmatrix} -q_1 & 1 & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ \vdots & 0 & 0 & 1 \\ -q_n & 0 & 0 & 0 \end{pmatrix} . \tag{5.26}$$

Note therefore that L happens to have the same form in q-coordinates as the recursion matrix (5.15). This is a pure coincidence without any deeper meaning; we stress again that R in (5.15) is not a tensor.

In any case, due to the fact that all the entries in L are linear in q_i we see that all the entries in G_m are linear in q_i for $m = 0, \ldots, n+1$, quadratic in q_i for $m = n+1$ and higher order polynomials for higher m. Moreover, by (5.24), all entries in $K_r(q)$ are linear in q_i. Using all these facts and the formula (5.9) we obtain the following important corollary:

Corollary 5.3. *If f is a polynomial then the geodesic parts of Benenti Hamiltonians (5.22) have a polynomial form. Moreover, if the right hand side of (5.12) is a pure polynomial, then by the recursive relations (5.14)-(5.15) also the potentials V_r in the Benenti Hamiltonians (5.8) are in this case polynomials in q_i. Thus, in such a case, the whole Hamiltonians $H_r(q,p)$ (and not just their geodesic parts) are polynomials.*

The reader should remind that we have assumed $f = f_i(\lambda_i)$ being smooth function and the generated single separation relation help us to derive the equation (5.8)-(5.22).

Example 5.4. *Consider the case $n = 2$, $f(\lambda) = 1$ (this is then a purely monomial situation with $m = 0$ in (5.25), so that $G = G_0$) and $\varphi(\lambda) = \lambda^3$. Then the separation curve (5.13) becomes*

$$\lambda^3 + \lambda H_1 + H_2 = \frac{1}{2}\mu^2$$

and yields the Hamiltonians H_i in the explicit form

$$H_1 = \frac{1}{2(\lambda_1 - \lambda_2)}(\mu_1^2 - \mu_2^2) - (\lambda_1^2 + \lambda_1\lambda_2 + \lambda_2^2)$$
$$H_2 = \frac{1}{2(\lambda_1 - \lambda_2)}(\lambda_1\mu_2^2 - \lambda_2\mu_1^2) + \lambda_1\lambda_2(\lambda_1 + \lambda_2)$$

so both Hamiltonians are rational functions of separation coordinates (λ, μ). The above Hamiltonians have exactly the form (5.8) with the metric

$$G = G_0 = diag\left(\frac{1}{\Delta_1}, \frac{1}{\Delta_2}\right),$$

and with the Killing tensors (5.6) given explicitly by:

$$K_1 = I, \;\; K_2 = -diag(\lambda_2, \lambda_1).$$

The map (5.21) to Viète coordinates has the explicit form:

$$q_1 = -(\lambda_1 + \lambda_2), \quad q_2 = \lambda_1\lambda_2,$$
$$p_1 = \frac{1}{\lambda_2 - \lambda_1}(\lambda_1\mu_1 - \lambda_2\mu_2), \quad p_2 = \frac{1}{\lambda_2 - \lambda_1}(\mu_1 - \mu_2).$$

An elementary calculation shows that H_i in these variables attain the simple form

$$H_1(q,p) = \frac{1}{2}q_1p_2^2 + p_1p_2 - q_1^2 + q_2$$
$$H_2(q,p) = \frac{1}{2}p_1^2 + q_1p_1p_2 + \frac{1}{2}q_1^2p_2^2 - \frac{1}{2}p_2^2q_2 - q_1q_2$$

which is in agreement with (5.25) and (5.24). Explicitly:

$$G_0(q) = \begin{pmatrix} 0 & 1 \\ 1 & q_1 \end{pmatrix}, \; K_1(q) = I, \;\; K_2(q) = \begin{pmatrix} 0 & 1 \\ -q_2 & q_1 \end{pmatrix}.$$

Thus, the Hamiltonians H_r become polynomial in Viète coordinates (p, q).

Finally, let us demonstrate a three-dimensional example.

Example 5.5. *Consider the case $n = 3$, $f(\lambda) = \lambda$ (so that $m = 1$ in (5.25) and thus $G = G_1$) and $\varphi(\lambda) = \lambda^5$. Then the separation curve (5.13) becomes*

$$\lambda^5 + \lambda^2H_1 + \lambda H_2 + H_3 = \frac{1}{2}\lambda\mu^2.$$

Solving the corresponding separation coordinates yields the Benenti Hamiltonians (5.8) with the metric $G_1 = LG_0$ with

$$L = diag\,(\lambda_1, \lambda_2, \lambda_3)$$

so that

$$G_1 = LG_0 = diag\left(\frac{\lambda_1}{\Delta_1}, \frac{\lambda_2}{\Delta_2}, \frac{\lambda_3}{\Delta_3}\right)$$

and with the Killing tensors (5.6) given explicitly by

$$K_1 = I, \quad K_2 = diag\,(\lambda_2 + \lambda_3, \lambda_1 + \lambda_3, \lambda_1 + \lambda_2)\,,$$
$$K_3 = -diag\,(\lambda_2\lambda_3, \lambda_1\lambda_3, \lambda_1\lambda_2)\,,$$

while the potentials $V_r = V_r^{(5)}$ *have the form*

$$\begin{aligned}
V_1^{(5)} &= \lambda_1^3 + \lambda_2^3 + \lambda_3^3 + \lambda_1^2\lambda_2 + \lambda_1^2\lambda_3 + \lambda_1\lambda_2^2 + \lambda_1\lambda_3^2 + \lambda_2^2\lambda_3 + \lambda_2\lambda_3^2 + \lambda_1\lambda_2\lambda_3,\\
V_2^{(5)} &= \lambda_1^3\lambda_2 + \lambda_1^3\lambda_3 + \lambda_1^2\lambda_2^2 + 2\lambda_1^2\lambda_2\lambda_3 + \lambda_1^2\lambda_3^2 + \lambda_1\lambda_2^3 + 2\lambda_1\lambda_2^2\lambda_3,\\
&\quad + 2\lambda_1\lambda_2\lambda_3^2 + \lambda_1\lambda_3^3 + \lambda_2^3\lambda_3 + \lambda_2^2\lambda_3^2 + \lambda_2\lambda_3^3,\\
V_3^{(5)} &= \lambda_1\lambda_2\lambda_3\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3\right).
\end{aligned}$$

The map (5.21) to Viète coordinates has now the form

$$q_1 = -(\lambda_1 + \lambda_2 + \lambda_3), \quad q_2 = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3, \quad q_3 = -\lambda_1\lambda_2\lambda_3$$

and

$$\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \left(J_V^{-1}\right)^T \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix}$$

with J_V *and* J_V^{-1} *given by (5.18) and (5.19) respectively. Explicitly*

$$J_V = \begin{pmatrix} -1 & -1 & -1 \\ \lambda_2+\lambda_3 & \lambda_1+\lambda_3 & \lambda_1+\lambda_2 \\ -\lambda_2\lambda_3 & -\lambda_1\lambda_3 & -\lambda_1\lambda_2 \end{pmatrix}$$

and

$$J_V^{-1} = -\begin{pmatrix} \frac{\lambda_1^2}{\Delta_1} & \frac{\lambda_1}{\Delta_1} & \frac{1}{\Delta_1} \\ \frac{\lambda_2^2}{\Delta_2} & \frac{\lambda_2}{\Delta_2} & \frac{1}{\Delta_2} \\ \frac{\lambda_2^2}{\Delta_3} & \frac{\lambda_2}{\Delta_3} & \frac{1}{\Delta_3} \end{pmatrix}.$$

An elementary calculation shows that H_i *in these variables attain the form*

$$\begin{aligned}
H_1(q,p) &= \frac{1}{2}q_1p_2^2 + p_1p_2 - \frac{1}{2}q_3p_3^2 + q_1^3 - 2q_1q_2 + q_3,\\
H_2(q,p) &= \frac{1}{2}p_1^2 - \frac{1}{2}q_2p_2^2 - \frac{1}{2}q_1q_3p_3^2 + q_1p_1p_2 - q_3p_2p_3 + q_1^2q_2 - q_1q_3 - q_2^2,\\
H_3(q,p) &= -\frac{1}{2}q_3p_2^2 - \frac{1}{2}q_2q_3p_3^2 - q_3p_1p_3 - q_1q_3p_2p_3 + q_1^2q_3 - q_2q_3,
\end{aligned}$$

which is in agreement with (5.25) and (5.24). Explicitly:

$$G_0(q) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & q_1 \\ 1 & q_1 & q_2 \end{pmatrix},\ K_1(q) = I,\ K_2(q) = \begin{pmatrix} 0 & 1 & 0 \\ -q_2 & q_1 & 1 \\ -q_3 & 0 & q_1 \end{pmatrix},$$

$$K_3(q) = \begin{pmatrix} 0 & 0 & 1 \\ -q_3 & 0 & q_1 \\ 0 & -q_3 & q_2 \end{pmatrix},$$

while the tensor L attains the form as in (5.26):

$$L(q) = \begin{pmatrix} -q_1 & 1 & 0 \\ -q_2 & 0 & 1 \\ -q_3 & 0 & 0 \end{pmatrix}.$$

Note again that the Hamiltonians H_r become polynomial in Viète coordinates (p, q).

5.3.2 Benenti systems in Newton coordinates

There is another way of turning Benenti Hamiltonians (5.8) into a polynomial form, by passing to so called Newton coordinates. This method has been discovered recently by V. M. Buchstaber and A. V. Mikhailov [19]. In this section we present our own proof of this result, independent from the work [19]. We also investigate in detail the structure of Benenti Hamiltonians (5.8) in Newton coordinates. Consider the following map (consisting of a sequence of Newton polynomials) on the base manifold $\mathcal{Q}$:

$$Q_i = \frac{1}{i}\sum_{s=1}^{n} \lambda_s^i. \tag{5.27}$$

By (2.6), this map induces the map on $T^*\mathcal{Q}$:

$$P = \left(J_N^{-1}\right)^T \mu, \tag{5.28}$$

where $P = (P_1, \ldots, P_n)^T$ and J_N is the Jacobian of the map (5.27),

$$(J_N)_{ij} = \frac{\partial Q_i}{\partial \lambda_j} = \lambda_j^{i-1}.$$

Thus, $J_N = V^T$, where V is the Vandermonde matrix, but different from S:

$$V = \begin{pmatrix} 1 & \lambda_1 & \dots & \lambda_1^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_n & & \lambda_n^{n-1} \end{pmatrix}. \tag{5.29}$$

That also means that (5.28) reads as $P = V^{-1}\mu$.

Lemma 5.6. *In the above notation*

$$\left(V^{-1}\right)_{ij} = -\frac{1}{\Delta_j}\frac{\partial \rho_{n-i+1}}{\partial \lambda_j}.$$

The reader should compare this lemma 5.6 with Lemma 5.1. Thus, the map (5.27) induces the following map on $T^*\mathcal{Q}$

$$Q_i = \frac{1}{i}\sum_{s=1}^{n}\lambda_s^i, \;\; P_i = -\sum_{j=1}^{n}\frac{1}{\Delta_j}\frac{\partial \rho_{n-i+1}}{\partial \lambda_j}\mu_j, \;\; i = 1, \dots, n \tag{5.30}$$

and we will call the coordinates (Q, P) *Newton coordinates*. The reader should compare this map with (5.21). Again, since the map $(\lambda, \mu) \to (Q, P)$ is a point map on $T^*\mathcal{Q}$, the Newton coordinates (Q, P) are Darboux (canonical), that is

$$\{Q_i, Q_j\} = \{P_i, P_j\} = 0, \quad \{Q_i, P_j\} = \delta_{ij}.$$

Let us now investigate the structure of Benenti Hamiltonians (5.8) in (Q, P)-coordinates. The Hamiltonians (5.8) are written in tensorial form and thus

$$H_r(Q, P) = \frac{1}{2}P^T K_r(Q) G(Q) P + V_r(Q), \qquad r = 1, \dots, n. \tag{5.31}$$

In the monomial case, i.e., when $f(\lambda) = \lambda^m$ we have

$$H_r(Q, P) = \frac{1}{2}P^T K_r(Q) L^m(Q) G_0(Q) P + V_r(Q), \qquad r = 1, \dots, n. \tag{5.32}$$

Let us now investigate the structure of (5.31), and in particular (5.32), in Newton coordinates. Of course, due to the tensor transformation law (2.8)

$$L(Q) = J_N L \left(J_N\right)^{-1}, \quad K_r(Q) = J_N K_r \left(J_N\right)^{-1} \tag{5.33}$$

and due to the tensor transformation law (2.7)

$$G(Q) = J_N G \left(J_N\right)^T, \tag{5.34}$$

but in order to express explicitly the right hand sides of (5.33)-(5.34) we need to invert the map $\lambda \to Q$ given by (5.27), which is in general not algebraically invertible. Let us thus consider the map $q \to Q$ between the Viète coordinates, considered in the previous subsection, and the Newton coordinates. In a very recent paper [11] the authors show that this map is given by

$$Q_r = -\frac{1}{r}\sum_{k=1}^{\gamma} V_k^{(n+r-k)}(q), \quad r = 1, \ldots, n, \tag{5.35}$$

where $V_k^{(\alpha)}(q)$ are the basic separable potentials as given by (5.14)-(5.15). Below we present a theorem in which we considerably extend the understanding of the above formula.

Theorem 5.7. *The map $q \to Q$ as given by (5.35) has the following structure:*

$$Q_r = \sum_{\substack{\alpha_1,\alpha_2,i,j=0 \\ \alpha_1 i+\alpha_2 j=r,\ i+j\leq r,\ i\neq j}}^{r} \frac{(-1)^{\alpha_1+\alpha_2}}{\max(\alpha_1,\alpha_2)}\, {q_i}^{\alpha_1} {q_j}^{\alpha_2}, \quad r = 1, \ldots, n, \tag{5.36}$$

where we denote as usual $q_0 = 1$. In consequence

$$Q_r(q) = -q_r + \tau_r^{(r-1)}(q_1, \ldots, q_{r-1}), \quad r = 1, \ldots, n, \tag{5.37}$$

where $\tau_r^{(\alpha)}$ denotes a polynomial of order α and where $\tau_1^{(0)} = 0$. Therefore, the map $q \to Q$ is algebraically invertible, with the inverse map

$$q_r = \sum_{\substack{\alpha_1,\alpha_2,i,j=0 \\ \alpha_1 i+\alpha_2 j=r,\ i+j\leq r,\ i\neq j}}^{r} \frac{(-1)^{\alpha_1+\alpha_2}}{\max(\alpha_1,\alpha_2)!}\, {Q_i}^{\alpha_1} {Q_j}^{\alpha_2}, \quad r = 1, \ldots, n, \tag{5.38}$$

(where we denote $Q_0 = 1$) which implies that

$$q_r(Q) = -Q_r + \eta_r^{(r-1)}(Q_1, \ldots, Q_{r-1}), \quad r = 1, \ldots, n, \tag{5.39}$$

where $\eta_r^{(\alpha)}$ denotes a polynomial of order α with $\eta_1^{(0)} = 0$. Moreover, neither $\tau_r^{(\alpha)}$ nor $\eta_r^{(\alpha)}$ depends on n.

One proves this theorem 5.7 by direct calculations, using the properties of basic separable potentials $V_k^{(\alpha)}$. This theorem means that both the map $q \to Q$ and its inverse $Q \to q$ are polynomial maps and moreover that the transformation between

the first n variables, i.e. between $q_1, \ldots, q_n$ and $Q_1, \ldots, Q_n$, does not change after increasing n to $n+1$. It is now possible to calculate the tensor L in the Newton variables Q. After some calculations we obtain:

$$L(Q) = J_N L \left(J_N\right)^{-1} = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ -q_n(Q) & -q_{n-1}(Q) & -q_{n-2}(Q) & \ldots & -q_1(Q) \end{pmatrix}, \quad (5.40)$$

or, equivalently

$$L(Q)^i_j = -q_{n-j+1}(Q)\delta^i_n + \delta^i_{j-1}, \quad i,j = 1, \ldots, n,$$

where the functions $q_i(Q)$ are given by (5.36) or by (5.37). Thus, the entries of $L(Q)$ are polynomial, and so of course will be for any positive power of L, i.e. for $L^m(Q)$ for any $m \in \mathbb{N}$.

Let us now calculate the Killing tensors K_r in Newton coordinates Q. We will do it by transforming $K_r(q)$, as given by (5.24), to Q variables, by the formula (2.8), which in this case reads $K_r(Q) = J_{VN} K_r(q) \left(J_{VN}\right)^{-1}$. First we find that

$$(J_{VN})_{i,j} = \sum_{s=0}^{n} q_s (J_{VN})_{i-s,j} - q_1 q_{i-s} + q_i, \quad i,j = 1, \ldots, n,$$

with $(J_{VN})_{1,j} = -1$, $(J_{VN})_{2,j} = q_1$, $(J_{VN})_{3,j} = -q_1^2 + q_2$ for any fixed j. This also yields, perhaps surprisingly, that

$$(J_{VN})^{-1}_{i,j} = -q_{i-j}, \quad i,j = 1, \ldots, n.$$

Note that this last result also means that the map (5.36) or (5.37) can now be extended to the whole manifold $\mathcal{M} = T^*\mathcal{Q}$ by completing it with the map between the canonical momenta:

$$P_i = \left[(J_{VN})^{-1}\right]^T_{ij} p_j = -\sum_{j=1}^{n} q_{j-i} p_j, \quad i = 1, ..., n. \quad (5.41)$$

After some calculations we obtain that

$$(K_r(Q))^i_j = \begin{cases} q_{i-j+r-1}(Q), & i-j \leq 0 \text{ and } r \leq n-i+1 \\ -q_{i-j+r-1}(Q), & i-j > 0 \text{ and } r > n-i+1 \\ 0 \quad \text{otherwise} \end{cases}, \tag{5.42}$$

cf. (5.24). Thus, since all $q_i(Q)$ by (5.37) are polynomials, all the entries of $K_r(Q)$ are polynomials (in Q_i) as well. Finally, let us consider $G_0(Q)$, i.e. the metric G_0 in Newton coordinates, by transforming $G_0(q)$, as given by (5.25), to Q variables, by the formula (2.7), that reads now as $G_0(Q) = J_{VN} G_0(q) \left(J_{VN}\right)^T$. The result is the following lemma.

Lemma 5.8. *The metric G_0 in Newton coordinates (5.27) attains the form of lower-triangular Hankel matrix given by the recurrence formulas*

$$G_0(Q)_{i,j} = \begin{cases} -\sum_{s=1}^{k} q_s(Q) \left(G_0\right)_{i-s,j} + q_1(Q) q_{i-1}(Q) - q_i(Q), & i \geq j \\ 0, & i < j \end{cases} \quad \text{for } i,j = 3,\ldots,n, \tag{5.43}$$

with $G_0(Q)_{1,j} = 1, G_0(Q)_{2,j} = -q_1$, and $G_0(Q)_{3,j} = q_1^2 - q_2$ for arbitrary fixed j.

Taking into account the formulas (5.40), (5.42) and Lemma 5.8 we obtain a corollary that is the main aim for this subsection. It is an analogue of Corollary 5.3.

Corollary 5.9. *If f is a polynomial then the geodesic parts of Benenti Hamiltonians $H_r(Q,P)$ in (5.31) have in Newton coordinates (5.30) a polynomial form. Moreover, if the right hand side of (5.12) is a pure polynomial, then the potentials $V_r(Q)$ in the Benenti Hamiltonians (5.31) are in this case also polynomials. Thus, in such a case, all the Hamiltonians $H_r(Q,P)$ (and not just their geodesic parts) are polynomials.*

The reader should remind that we have assumed $f = f_i(\lambda_i)$ being smooth function and the generated single separation relation help us to derive the equation (5.31).

We finish this chapter by presenting some examples.

Example 5.10. *We will now consider the same setting as in Example 5.4, i.e.* $n = 2$, $f(\lambda) = 1$ *(so that* $m = 0$*) and* $\varphi(\lambda) = \lambda^3$*, but in Newton coordinates. The map (5.37)-(5.41) reads now*

$$Q_1 = -q_1, \quad Q_2 = \frac{1}{2}q_1^2 - q_2,$$
$$P_1 = -p_1 - q_1 p_2, \quad P_2 = -p_2$$

and it transforms the Hamiltonians from Example 5.21 to the form

$$H_1(Q,P) = \frac{1}{2}P_2^2 Q_1 + P_1 P_2 - Q_2 - \frac{1}{2}Q_1^2,$$
$$H_2(Q,P) = -\frac{1}{4}P_2^2 Q_1^2 + \frac{1}{2}P_2^2 Q_2 + \frac{1}{2}P_1^2 + \frac{1}{2}Q_1^3 - Q_1 Q_2,$$

which is in agreement with (5.43) and (5.42). Explicitly:

$$G_0(Q) = \begin{pmatrix} 0 & 1 \\ 1 & Q_1 \end{pmatrix}, \; K_1(Q) = I, \; K_2(Q) = \begin{pmatrix} -Q_1 & 1 \\ Q_2 - \frac{1}{2}Q_1^2 & 0 \end{pmatrix}.$$

Moreover, L *becomes*

$$L(Q) = \begin{pmatrix} 0 & 1 \\ Q_2 - \frac{1}{2}Q_1^2 & Q_1 \end{pmatrix}.$$

Example 5.11. *We will now consider Example 5.5 in Newton coordinates i.e. the case* $n = 3$, $m = 1$ *and* $\varphi(\lambda) = \lambda^5$*. As* $n = 3$ *the map (5.37) is now*

$$Q_1 = -q_1, \quad Q_2 = \frac{1}{2}q_1^2 - q_2, \quad Q_3 = -\frac{1}{3}q_1^3 + q_1 q_2 - q_3, \tag{5.44}$$

and its inverse (5.39) is

$$q_1 = -Q_1, \quad q_2 = \frac{1}{2}Q_1^2 - Q_2, \quad q_3 = -\frac{1}{6}Q_1^3 + Q_1 Q_2 - Q_3.$$

The map (5.41) between momenta is

$$\begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix} = \begin{pmatrix} -1 & -q_1 & -q_2 \\ 0 & -1 & -q_1 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}, \tag{5.45}$$

with the inverse

$$\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} -1 & -Q_1 & -\frac{1}{2}Q_1^2 - Q_2 \\ 0 & -1 & -Q_1 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix}.$$

The map (5.44)-(5.45) transforms the Hamiltonians $H_r(q,p)$ in Example 5.5 to the form

$$H_1(Q,P) = \frac{1}{2}P^T \begin{pmatrix} 0 & 1 & Q_1 \\ 1 & Q_1 & \frac{1}{2}Q_1^2 + Q_2 \\ Q_1 & \frac{1}{2}Q_1^2 + Q_2 & Q_3 + Q_1Q_2 + \frac{1}{6}Q_1^3 \end{pmatrix} P + V_1^{(5)}(Q)$$

$$H_2(Q,P) = \frac{1}{2}P^T \begin{pmatrix} 1 & 0 & Q_2 - \frac{1}{2}Q_1^2 \\ 0 & Q_2 - \frac{1}{2}Q_1^2 & Q_3 - \frac{1}{3}Q_1^3 \\ Q_2 - \frac{1}{2}Q_1^2 & Q_3 - \frac{1}{3}Q_1^3 & -\frac{1}{12}Q_1^4 - Q_1^2Q_2 + Q_3Q_1 + Q_2^2 \end{pmatrix} P + V_2^{(5)}(Q)$$

$$H_3(Q,P) = \frac{1}{2}P^T \begin{pmatrix} 0 & 0 & \frac{1}{6}Q_1^3 - Q_2Q_1 + Q_3 \\ 0 & \frac{1}{6}Q_1^3 - Q_2Q_1 + Q_3 & \frac{1}{6}Q_1^4 - Q_2Q_1^2 + Q_3Q_1 \\ \frac{1}{6}Q_1^3 - Q_2Q_1 + Q_3 & \frac{1}{6}Q_1^4 - Q_2Q_1^2 + Q_3Q_1 & \frac{1}{12}Q_1^5 - \frac{1}{3}Q_1^3Q_2 + \frac{1}{2}Q_3Q_1^2 \\ & & -Q_1Q_2^2 + Q_3Q_2 \end{pmatrix} P$$

$$+ V_3^{(5)}(Q),$$

where

$$V_1^{(5)}(Q) = -\frac{1}{6}Q_1^3 - Q_1Q_2 - Q_3,$$
$$V_2^{(5)}(Q) = Q_1^2Q_2 - Q_1Q_3 - Q_2^2 + \frac{1}{12}Q_1^4,$$
$$V_3^{(5)}(Q) = -\frac{1}{12}Q_1^5 + \frac{1}{3}Q_1^3Q - \frac{1}{2}Q_1^2Q_3 + Q_1Q_2^2 - Q_2Q_3,$$

which is again in agreement with (5.43) and (5.42). Explicitly:

$$G_0(Q) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & Q_1 \\ 1 & Q_1 & \frac{1}{2}Q_1^2 + Q_2 \end{pmatrix},\ K_1(Q) = I,$$

$$K_2(Q) = \begin{pmatrix} -Q_1 & 1 & 0 \\ 0 & -Q_1 & 1 \\ \frac{1}{6}Q_1^3 - Q_1Q_2 + Q_3 & Q_2 - \frac{1}{2}Q_1^2 & 0 \end{pmatrix},$$

$$K_3(Q) = \begin{pmatrix} \frac{1}{2}Q_1^2 - Q_2 & -Q_1 & 1 \\ \frac{1}{6}Q_1^3 - Q_1Q_2 + Q_3 & 0 & 0 \\ 0 & \frac{1}{6}Q_1^3 - Q_1Q_2 + Q_3 & 0 \end{pmatrix},$$

and

$$L(Q) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{6}Q_1^3 - Q_1Q_2 + Q_3 & Q_2 - \frac{1}{2}Q_1^2 & Q_1 \end{pmatrix}.$$

6 Discussion and Conclusion

In this thesis we presented an outline of the theory of separable (in the sense of Hamilton-Jacobi theory) Hamiltonian systems from the point of view of separation relations and separation curves rather than from the standard viewpoint, that still dominates the majority of textbooks and publications on the subject. The concept of separation relations, introduced by Sklyanin in [49], allows for a systematic analysis of separable systems. In this thesis we applied this concept to a wide class of separable systems of Stäckel type and then we focused on a particular subclass of Stäckel systems that is called in literature Benenti systems. Hamiltonians of Benenti systems turn out to be rational when expressed in their separation coordinates. A particular problem that we posed in this work was thus to investigate possible canonical transformations that turn Benenti Hamiltonians into polynomials. One such transformation - to the so called Viète coordinates (5.21) - was known for some time, another transformation of this kind - to what we call Newton coordinates - has been recently presented by Buchstaber and Mikhailov in [19]. In this thesis we presented also a new proof of Buchstaber and Mikhailov result and we also investigated in detail the structure of Benenti Hamiltonians in Newton coordinates, presenting formulas for all the geometric ingredients of these Hamiltonians (the conformal Killing tensor, $(1,1)$-Killing tensors and the metric) in Newton variables.

Future investigations continuing along this line may include:

1. Investigating possible ways of turning Benenti Hamiltonians H_i to polynomials in case that H_i are not generated by a single separation curve but by the more general separation relations (5.4) i.e. with different f_i and φ_i.

2. The same question can be posed for general Stäckel Hamiltonians, generated by the separation relations (5.2).

3. Investigation of separable systems generated by separation relations linear in H_i but cubic and quadratic in momenta (see the beginning of chapter 5).

4. Investigate other separation relations than linear in H_i. What systems do they lead to? Can we find a geometric interpretation for these systems? What would be the properties of these systems?

Bibliography

[1] Abraham R., Marsden J. E., *Foundations of Mechanics*, London: Benjamin-Cummings, ISBN 978-0-8053-0102-1 (1978).

[2] Arnold V. I., Kozlov V. V., Neĭshtadt A. I., *Mathematical aspects of classical and celestial mechanics*, Springer–Verlag, **3** (1988).

[3] Arnol'd V.I., *Mathematical Methods of Classical Mechanics*, Springer–Verlag, ISBN 978–0–387–96890–2 (1989).

[4] Błaszak M., *Degenerate Poisson Pencils on Curves: New Separability Theory*, *J. Nonl. Math.Phys*, **7** (2) (2000), 213–243.

[5] Błaszak M., *From bi-Hamiltonian geometry to separation of variables: stationary Harry-Dym and the KdV dressing chain*, *J. Nonl. Math. Phys.*, **9** (2001), 1–13.

[6] Błaszak M., *Systematic construction of separable systems with quadratic in momenta first integrals*, arXiv preprint 0309044 (2003).

[7] Błaszak M., *Separable systems with quadratic in momenta first integrals*, Journal of Physics A: Mathematical and General, **38** (8) (2005), 1667–1685.

[8] Błaszak M., Marciniak K., *From Stäckel systems to integrable hierarchies of PDEs: Benenti class of separation relations*, Journal of Mathematical Physics, **47** (3) (2006).

[9] Blaszak M., Marciniak K., *Stäckel transform of Lax equations*, arXiv:1901.02368 (2019).

[10] Błaszak M., Marciniak K., Domanski Z, *Separable quantizations of Stäckel systems*, arXiv:1501.00576, **371** (2016), 460–477.

[11] Blaszak M., Marciniak K., Domanski Z., *Systematic construction of non-autonomous Hamiltonian equations of Painlevé-type. I. Frobenius integrability.* To be published.

[12] Błaszak M., Marciniak K., Sergyeyev A., *From deformations of Lie algebras to Frobenius integrable non-autonomous Hamiltonian systems*, arXiv:1712.08155 (2017).

[13] Błaszak M., Sergyeyev A., *Generalized Stäckel systems*, *Math. Phys*, **375** (27) (2011), 2617–2623.

[14] Benenti S., *Separation of variables in the geodesic Hamilton–Jacobi equation.* Progr. Math., **9** (1991),1–36.

[15] Benenti S., *Orthogonal separation of variables on manifolds with constant curvature.* Differential Geometry and its Applications, **2** (1992), 351–367.

[16] Benenti S., *Orthogonal separable dynamical systems. Differential geometry and its applications*, Math. Publ., **1** (1992), 163–184.

[17] Benenti S., *Intrinsic characterization of the variable separation in the Hamilton-Jacobi equation.* J. Math. Phys., **38** (12) (1997), 6578–6602.

[18] Benenti S., Chanu C., Rastelli G., *Variable separation for natural Hamiltonians with scalar and vector potentials on Riemannian manifolds*, Mathematical Physics, **42** (5) (2001), 2065–2091.

[19] Bukhshtaber V. M., Mikhailov A. V. *Polynomial Hamiltonian integrable systems on symmetric powers of plane curves*, Russian Mathematical Surveys, Turpion Ltd, **73** (6) (2018), 1122–1124.

[20] Crampin M., W. Sarlet W., *A class of nonconservative Lagrangian systems on Riemannian manifolds*, J. Math. Phys. **42** (9) (2001), 4313–4326.

[21] Dunajski M., *Integrable systems*, Lecture Notes, (2012).

[22] Dubrovin B. A., Fomenko A. T., Novikov S. P., *Modern geometry methods and applications: Part II: The geometry and topology of manifolds*, Springer Science and Business Media, **104** (2012).

[23] Eisenhart L. P., *Separable systems of Stäckel.* Ann. of Math., **35** (2) (1934), 284–305.

[24] Eisenhart L. P., *Stäckel systems in conformal Euclidean space.* Ann. of Math., **36** (2) (1935), 57–70.

[25] Eisenhart L. P., *Riemannian Geometry*, Second ed., Princeton Univ. Press, Princeton, N. J., (1949).

[26] Elvang H., Hadjiantonis M., *A practical approach to the Hamilton-Jacobi formulation of holographic renormalization*, Journal of High Energy Physics, **2016** (6) (2016), 46.

[27] Falqui G., Magri F., and Tondo G., *Reduction of bi-Hamiltonian systems and separation of variables: an example from the Boussinesq hierarchy*, *Theor. Math. Phys.*, **122** (2) (2000), 176–192.

[28] Falqui G., Magri F., Pedroni M., *Bi-Hamiltonian geometry and separation of variables for Toda lattices*, *J. Nonlinear Math. Phys.*, **8** (2001), 118–127.

[29] Falqui G., Pedroni M., *Separation of variables for bi-Hamiltonian systems*, *Math. Phys. Anal. Geom.*, **6** (2) (2003), 139–179.

[30] Fomenko A. T., *Differential geometry and topology*, New York, London, (1987).

[31] Green G. M., *On the Linear Dependence of Functions of Several Variables, and Certain Completely Integrable Systems of Partial Differential Equations*, **2** (4) (1916), 209.

[32] Goldstein H., Poole C., Safko J., *Classical mechanics*, (2002).

[33] Hand L. N., Finch J. D., *Analytical Mechanics*, Cambridge University Press, (1998).

[34] Hamilton W.R., *On a General Method in Dynamics*, Phil. Trans. Roy. Soc. London, (1834)-(1835), 247–308.

[35] Hénon M., Heiles C., *The applicability of the third integral of motion: Some numerical experiments*, The Astronomical Journal, **69** (1964), 73–79.

[36] Jacobi C. G., Vorlesungen uber Dynamik, Georg Reimer, Berlin. *Jacobi's lectures on dynamics given in Königsberg*, published by A. Clebsch, (1866), 1842–1843.

[37] Kalnins E. G., Miller Jr W., and Reid G. J., *Separation of variables for complex Riemannian spaces of constant curvature-I. Orthogonal separable coordinates for* $S_{n\mathbf{C}}$ and $E_{n\mathbf{C}}$. Proc. Roy. Soc. London Ser., **394** (1806) (1984), 183–206.

[38] Kalnins E. G., Miller Jr W., *Separation of variables on n–dimensional Riemannian manifolds. I. The n–sphere* S^n *and Euclidean n–space* $\mathbb{R}^n$, Journal of Mathematical Physics, **27** (7) (1986), 1721–1736.

[39] Kuratowski K., *Topology*, PWN and academic press Inc. London Ltd, (1968).

[40] Lee J.M., *Introduction to smooth manifolds*. Second ed., Graduate Texts in Mathematics, Springer, New York, **218** (2013).

[41] Levi-Civita T., *Sulla integrazione della equazione di Hamilton-Jacobi per separazione di variabili.* Math. Ann., **59** (3) (1904), 383–397.

[42] Marciniak K., *Presentation on Stäckel Separable systems*, University of Rwanda, Rwanda, (2018).

[43] Olver P. J., *Applications of Lie groups to differential equations*, Springer-Verlag, New York, (1993).

[44] Perko L., *Differential equations and dynamical systems*, Springer Science and Business Media, **7** (2013).

[45] Poincaré H., *New methods of celestial mechanics*, Springer Science and Business Media, **13** (1992).

[46] Poincaré, Henri J., *The three-body problem and the equations of dynamics: Poincaré's foundational work on dynamical systems theory*, Springer International publishing, **443** (2017).

[47] Rauch-Wojciechowski S., Marciniak K., *Separation of variables for differential equations*, (2004).

[48] Romano A., *Classical mechanics with mathematica*, Springer, (2012).

[49] Sklyanin E. K., *Separation of variables– new trends*, Progr. Theoret. Phys. Suppl., **118** (1995), 35–60.

[50] Sklyanin E. K., *Separation of variables in the classical integrable magnetic chain*, Communications in Mathematical Physics, **150** (3) (1992), 181–191.

[51] Spivak M. D., *A comprehensive introduction to differential geometry*, (1970).

[52] Stäckel P., *Die Integration der Hamilton-Jacobischen Differentialgleichung mittelst Separation der Variablen, Habilitationsschrift*, Halle, (1891).

[53] Vinogradov A. M., Kupershmidt B. A., *The structure of Hamiltonian mechanics*, London, Math. Soc. Lect. Notes Ser., London: Cambridge Univ. Press., **60** (1981).

[54] Waksjö C., *Stäckel multipliers in Euclidean space.* Licenciate dissertation, Linköping universitet, Sweden, (2000), ISSN 0280–7971; 833, p. 77

[55] Waksjö C., Rauch-Wojciechowski S., *How to find separation coordinates for the Hamilton Jacobi equation: a criterion of separability for natural Hamiltonian systems*, Mathematical Physics, Analysis and Geometry, **6** (2003), pp.301–348.