Stockholm University

Statistical Inference of Tangency Portfolio in Small and Large Dimension

Stanislas Muhinyuza

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Statistical Inference of Tangency Portfolio in Small and Large Dimension

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Abstract

This thesis considers statistical test theory in portfolio theory. It analyses the asymptotic behavior of the considered tests in the high-dimensional setting, meaning $k/n \to c \in (0, \infty)$ as $n \to \infty$, where k and n are portfolio size and sample size, respectively. It also considers the high-dimensional asymptotic of the product of components involved in the computation of the optimal portfolio. The thesis comprises four manuscripts:

Paper I is concerned with the test on the location of the tangency portfolio on the set of feasible portfolios. Considering the independent and normally multivariate asset returns, we propose a finite-sample test on the mean-variance efficiency of the tangency portfolio (TP). We derive the distribution of the proposed test statistic under both the null and alternative hypotheses, using which we assess the power of the test and construct a confidence interval. The out-of-sample performance of the portfolio determined by the proposed test is conducted and through an extensive simulation study, we show the robustness of the developed test towards the violation of the normality assumptions. We also apply the developed test to real data in the empirical study.

Paper II extends the results of paper I. It is concerned with the study of the asymptotic distributions of the test on the existence of efficient frontier (EF) and the efficiency of the tangency portfolio in the mean-variance space in the high-dimension setting under both the null and alternative hypotheses. Finite-sample performance and robustness of the proposed tests are studied through an extensive simulation study.

In paper III, we study the distributional properties of the TP weights under the assumption of normally distributed logarithmic returns. The distribution of the weights of the TP is given under the form of a stochastic representation (SR). Using the derived SR we deliver the asymptotic distribution of the TP weights under a high-dimensional asymptotic regime. Besides, we consider tests about the elements of the TP weights and derive the asymptotic distribution of the test statistic under the null and alternative hypotheses. In a simulation study, we compare the power function of the high-dimensional asymptotic and the exact tests. Moreover, in an empirical study, we apply the developed theory in analysing the TP weights in a portfolio made of stocks from the S&P 500 index.

In paper IV, we derive a stochastic representation of the product of a singular Wishart matrix and a singular Gaussian vector. We then use the derived SR in the obtention of the characteristic function of that product and in proving the asymptotic normality under the double asymptotic regime. The performance of the obtained asymptotic is shown in the simulation study.

Keywords: *Tangency portfolio, Mean-variance portfolio, High-dimensional asymptotics, Test theory.*

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List of Papers

The following papers, referred to in the text by their Roman numerals, are included in this thesis.

PAPER I: Stanislas Muhinyuza,Taras Bodnar and Mathias Lindholm, **A test on the location of the tangency portfolio on the set of feasible portfolios**,
(Under revision in *Applied Mathematics and Computation*).

PAPER II: Stanislas Muhinyuza, **A test on mean-variance efficiency of the tangency portfolio in high-dimensional setting**,
(To appear in *Theory of Probability and Mathematical Statistics*).

PAPER III: Sune Karlson, Stepan Mazur and Stanislas Muhinyuza, **Statistical inference for the tangency portfolio in high-dimension**,
(Submitted for publication).

PAPER IV: Taras Bodnar, Stepan Mazur, Stanislas Muhinyuza and Nestor Parolya, **On the product of a singular Wishart matrix and a singular Gaussian vector in high dimension**,
Theory of Probability and Mathematical Statistics, 99(2):39-52.
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Author's contribution

PAPER I: Stanislas Muhinyuza contributed in deriving the theoretical results in collaboration with Taras Bodnar and Mathias Lindholm, carried out all computational results in collaboration with Taras Bodnar and wrote the large part of the manuscript with the assistance from Taras Bodnar and Mathias Lindholm.

PAPER II: Stanislas Muhinyuza is the only author of the paper.

PAPER III: Stepan Mazur brought up the idea. I derived the theoretical results in collaboration with Stepan Mazur and Sune Karlson. Based on his suggestions, I conducted all numerical and empirical computations. Stepan improved those sections. We all contributed to the writing of the manuscript.

PAPER IV: Stanislas Muhinyuza contributed to proving the main theorems in collaboration with Stepan Mazur and Nestor Parolya, conducted the simulation study in cooperation with Stepan Mazur and wrote the majority of the manuscript assisted by Taras Bodnar.

General comment: Paper I, and an earlier version of Paper IV, as well as some parts of the introduction, were included in the Licentiate thesis of Stanislas Muhinyuza (Muhinyuza (2018)).

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Contents

1. Introduction

The choice of investment allocation is of great importance for both individuals, e.g., retirement savings, as well as for banks and other institutional investors. The way this choice is done depends on the investor's view on risk about the return. A prominent theory in dealing with that problem is the mean-variance analysis introduced by Markowitz (1952) and later extended by Tobin (1958). It plays an important role in finance and investment for both practitioners and researchers in that area (Merton (1972)). It has been observed that, in practice, mean-variance efficient portfolios are of paramount importance in portfolio management applications, whereas research-wise, it may be used in a larger number of asset pricing theories as well as in empirical tests that involve those theories (Britten-Jones (1999)). It is also known as the first area of finance in which multivariate distribution concepts have been applied (Jajuga (2008)). Unfortunately, its implementation faces several challenges.

In recent years, estimating the covariance matrix by the sample covariance matrix becomes a serious problem if the sample size n is comparable with the number of assets k in a portfolio (see, e.g, Bodnar et al. (2019); Glombek (2014)). Despite substantial advancements in sophisticated mathematical and statistical methods dealing with portfolios managements, still, a lot has to be done. The 2008 financial crisis reminds us that novel and efficient mathematical and statistical models and methods in that area need to be developed.

Indeed, the mean-variance optimization techniques serve as a quantitative tool that considers the trade-off between the risk of the portfolio and its expected return. It also helps the investors in the construction of an optimal portfolio by minimizing the risk for a given level of the expected return or by maximizing the portfolio return for a given level of the portfolio risk (Li et al. (2015)). If there is a possibility of investing in risk-free assets, the tangency portfolio (TP) is constructed and is composed of both risk-free assets and risky assets. There has been significant interest in understanding the statistical properties of the TP. Also, the TP has a far-reaching role in financial literacy and is usually used as a market portfolio in the capital asset pricing model. Having a thorough understanding of the properties of the TP becomes crucial for any financial actors. However, numerous challenges are encountered in the estimation of TP and in studying its statistical properties(Bodnar et al. (2019); Glombek (2014)).

The specific goal of this thesis is to study the statistical properties of TP by developing new statistical models. In particular, it suggests an exact test for the location of the TP on the set of feasible portfolios and provides the distribution of the test statistic under both hypotheses. It delivers the high-dimensional asymptotic distributions of the test statistics for testing both the existence of the efficient frontier and the efficiency of the TP. It also delivers the high-dimensional asymptotic distribution of the estimated TP weights as well as the asymptotic distribution of the statistical test about the elements of the TP. Lastly, it gives the distribution properties of the components involved in the construction of optimal portfolios(mean vector and covariance matrix) together with its asymptotic distribution under the double asymptotic regime.

In the following chapters of this thesis, we give a brief introduction of portfolio management theory. We also focus on multivariate methods used in it. The main emphasis goes on definitions of a multivariate normal distribution, a matrix-variate normal distribution, and a Wishart distribution and provides some of their respective properties, followed by an introduction to high-dimensional asymptotics. At the end, we provide an overview/summary of the papers included in this thesis.

2. Portfolio Management Theory

In this chapter, we define and give the basics of portfolio theory. We start with the Markowitz selection problem and all its possible solutions. We put much emphasis on the construction of optimal portfolios under parameter uncertainty.

2.1 Portfolio selection problem and its solution

Investment strategies and its corresponding statistical challenges have become an eminent research area in finance since its inception by Markowitz (1952). Let us consider an investor that wants to invest in k assets. For now, we exclude the possibility to invest into the risk-free asset. Let $\mathbf{x} = (x_1, \ldots, x_k)'$ be the returns on k assets. We also assume that the vector of asset returns $\mathbf{x}$ has a k-dimensional distribution with mean vector $\boldsymbol{\mu} \in \mathbb{R}^k$ and positive definite covariance matrix $\Sigma \in \mathbb{R}^{k \times k}$.

Definition 2.1.1. A **portfolio** P is a linear combination of the k assets. The symbol $\mathbf{w} = (w_1, \ldots, w_k)' \in \mathbb{R}^k$ denotes a vector of weights (allocation vector) of that linear combination and it fulfils the budget constraint $\mathbf{w}'\mathbf{1} = 1$, where $\mathbf{1} = (1, \ldots, 1)' \in \mathbb{R}^k$.

Allowing short-selling means that the vector of weights $\mathbf{w}$ may contain negative components.

Definition 2.1.2. For a given portfolio P, its expected return and variance are respectively given by

$$\begin{aligned} R &= \mathbb{E}\left(\mathbf{w}'\mathbf{x}\right) = \mathbf{w}'\boldsymbol{\mu}, \\ V &= \mathbb{V}\left(\mathbf{w}'\mathbf{x}\right) = \mathbf{w}'\Sigma\mathbf{w}. \end{aligned}$$

A well-known strategy for an investor to optimally allocate his/her investment is to maximize the portfolio return given the level of portfolio risk or to minimize the portfolio risk given the level of portfolio return (Li et al. (2015)). The latter expression is formulated as an optimization problem by

$$\min \mathbf{w}'\Sigma\mathbf{w} \text{ subject to } \mathbf{w}'\boldsymbol{\mu} = \mu_P, \mathbf{w}'\mathbf{1} = 1, \tag{2.1.1}$$

where μ_P is the given level of portfolio return.

Alternatively, Ingersoll (1987) showed that the mean-variance analysis is fully consistent with the expected utility maximization under special circumstances. The corresponding optimization problem is given by

$$\max \mathbf{w}'\boldsymbol{\mu} - \frac{\alpha}{2}\mathbf{w}'\Sigma\mathbf{w} \text{ subject to } \mathbf{w}'\mathbf{1} = 1, \tag{2.1.2}$$

where α stands for the risk aversion.

The solutions to optimization problems (2.1.1) and (2.1.2) are optimal portfolios, which is called efficient frontier. Merton (1972) showed that the efficient frontier is the upper limb of a parabola in the mean-variance space. A limit of the optimal solutions of the maximization problem (2.1.2) when the risk aversion α tends to infinity gives a global minimum variance portfolio (GMVP). The GMVP is the smallest portfolio with the smallest variance, characterized by its weights, expected return and variance expressed as

$$\mathbf{w}_{GMV} = \frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}'\Sigma^{-1}\mathbf{1}}, R_{GMV} = \frac{\boldsymbol{\mu}'\Sigma^{-1}\mathbf{1}}{\mathbf{1}'\Sigma^{-1}\mathbf{1}} \text{ and } V_{GMV} = \frac{1}{\mathbf{1}'\Sigma^{-1}\mathbf{1}}. \tag{2.1.3}$$

Based on the characteristics of the GMVP, Bodnar and Schmid (2008) proposed a new parameterization of the efficient frontier,

$$(R - R_{GMV})^2 = s(V - V_{GMV}) \text{ where } s = \boldsymbol{\mu}'\mathbf{R}\boldsymbol{\mu} \text{ with } \mathbf{R} = \Sigma^{-1} - \frac{\Sigma\mathbf{1}\mathbf{1}'\Sigma^{-1}}{\mathbf{1}'\Sigma^{-1}\mathbf{1}}. \tag{2.1.4}$$

Here s is the slope of the efficient frontier.

If there is a possibility to invest into the risk-free asset, the choice of allocating a part of the wealth in it will reduce the portfolio risk. In that case, a new quadratic utility function needs to be maximized to obtain optimal portfolio compositions, that is

$$\max r_f + \mathbf{w}'(\boldsymbol{\mu} - r_f\mathbf{1}) - \frac{\alpha}{2}\mathbf{w}'\Sigma\mathbf{w} \tag{2.1.5}$$

An alternative investment strategy in this case for an investor is to maximize the performance of his investment, which is the Sharpe ratio, measured by relating the portfolio expected return to its risk.

Definition 2.1.3. The Sharpe ratio is defined as,

$$SR(\mathbf{w}) = \frac{\mathbf{w}'\boldsymbol{\mu} - r_f\mathbf{1}}{\sqrt{\mathbf{w}'\Sigma\mathbf{w}}}. \tag{2.1.6}$$

The optimization problem based on the Sharpe ratio is given by

$$\max SR(\mathbf{w}) \text{ with respect to } \mathbf{w}. \tag{2.1.7}$$

The solutions to optimization problems (2.1.5) and (2.1.7) are the optimal portfolios in case the investment into the risk-free asset is available. The portfolio that uniquely minimizes the problems (2.1.5) and (2.1.7) is called tangency portfolio (TP)and characterized by its weights expressed as

$$\mathbf{w}_{TP} = \frac{\Sigma^{-1}(\boldsymbol{\mu} - r_f\mathbf{1})}{\mathbf{1}'\Sigma^{-1}(\boldsymbol{\mu} - r_f\mathbf{1})} \tag{2.1.8}$$

provided that $\mathbf{1}'\Sigma^{-1}(\boldsymbol{\mu} - r_f\mathbf{1}) \neq 0$ (Glombek (2012)). The expected return and variance of the TP are

$$R_T = r_f + \frac{(\boldsymbol{\mu} - r_f\mathbf{1})'\Sigma^{-1}(\boldsymbol{\mu} - r_f\mathbf{1})}{\mathbf{1}'\Sigma^{-1}(\boldsymbol{\mu} - r_f\mathbf{1})} \text{ and } V_T = \frac{(\boldsymbol{\mu} - r_f\mathbf{1})'\Sigma^{-1}(\boldsymbol{\mu} - r_f\mathbf{1})}{(\mathbf{1}'\Sigma^{-1}(\boldsymbol{\mu} - r_f\mathbf{1}))^2}. \tag{2.1.9}$$

Again, the equation of the efficient frontier in case with a risk-free asset is given by

$$(R - r_f)^2 = (\boldsymbol{\mu} - r_f\mathbf{1})'\Sigma^{-1}(\boldsymbol{\mu} - r_f\mathbf{1})V. \tag{2.1.10}$$

2.2 Portfolio selection under parameter uncertainty

Practical implementation of the above solutions requires the knowledge of the parameters $\boldsymbol{\mu}$ and Σ. Unfortunately, these parameters are not known and need to be estimated. This problem is known in portfolio selection theory as parameter uncertainty (see, e.g., Kan and Zhou (2007)). For any investor, it is unavoidable to determine his investment policy without estimating the parameters $\boldsymbol{\mu}$ and Σ. Using the random sample of asset returns $\mathbf{x}_1, \ldots, \mathbf{x}_n$, we estimate the parameters $\boldsymbol{\mu}$ and Σ by their empirical counterparts. We have

$$\hat{\boldsymbol{\mu}} = \frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i$$

and

$$\hat{\Sigma} = \frac{1}{n-1}\sum_{i=1}^{n}(\mathbf{x}_i - \hat{\boldsymbol{\mu}})(\mathbf{x}_i - \hat{\boldsymbol{\mu}})'.$$

If $\mathbf{x}_i, i = 1,\dots,n$ are independent and identically distributed with $\mathbf{x}_i \sim \mathcal{N}_k(\boldsymbol{\mu},\Sigma)$, we know from Muirhead (1982) Theorem 3.1.2 that $\hat{\boldsymbol{\mu}} \sim \mathcal{N}_k(\boldsymbol{\mu},\Sigma), (n-1)\hat{\Sigma} \sim \mathcal{W}_k(n-1,\Sigma)$. Moreover, $\hat{\boldsymbol{\mu}}$ and $\hat{\Sigma}$ are independent. The estimated portfolio weights and the equation of the sample efficient frontier are obtained by replacing the parameters $\boldsymbol{\mu}$ and Σ by their respective sample estimators $\hat{\boldsymbol{\mu}}$ and $\hat{\Sigma}$. Hence in case without a possibility to invest into the risk-free asset, the sample efficient frontier becomes

$$(R - \hat{R}_{GMV})^2 = \hat{s}(V - \hat{V}_{GMV}) \text{ with } \hat{s} = \hat{\boldsymbol{\mu}}'\hat{\mathbf{R}}\hat{\boldsymbol{\mu}},$$

where

$$\hat{R}_{GMV} = \frac{\hat{\boldsymbol{\mu}}'\hat{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}'\hat{\Sigma}^{-1}\mathbf{1}} \text{ and } \hat{V}_{GMV} = \frac{1}{\mathbf{1}'\hat{\Sigma}^{-1}\mathbf{1}}$$

are the estimators of the expected return and variance of the global minimum variance portfolio respectively, with estimated weights

$$\hat{\mathbf{w}}_{GMV} = \frac{\hat{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}'\hat{\Sigma}^{-1}\mathbf{1}}.$$

The properties of the characteristics of the sample efficient frontier are detailed in Bodnar and Schmid (2008, 2009); Kan and Zhou (2008). While the properties of the portfolio with a smallest risk among the efficient portfolios are highlighted in Bodnar et al. (2017a,b); Frahm (2010); Glombek (2014); Okhrin and Schmid (2006).

Similarly, if there is a possibility to invest into the risk-free asset, the estimators of the optimal portfolio weights, the expected return and the variance of the TP are obtained by replacing the parameters $\boldsymbol{\mu}$ and Σ with their corresponding sample mean vector and sample covariance matrix $\hat{\boldsymbol{\mu}}$ and $\hat{\Sigma}$, respectively. The sample efficient frontier becomes

$$(R - r_f)^2 = (\hat{\boldsymbol{\mu}} - r_f\mathbf{1})'\hat{\Sigma}^{-1}(\hat{\boldsymbol{\mu}} - r_f\mathbf{1})V, \tag{2.2.1}$$

with the optimal allocation vector given by

$$\hat{\mathbf{w}}_{TP} = \frac{\hat{\Sigma}^{-1}(\hat{\boldsymbol{\mu}} - r_f\mathbf{1})}{\mathbf{1}'\hat{\Sigma}^{-1}(\hat{\boldsymbol{\mu}} - r_f\mathbf{1})}. \tag{2.2.2}$$

The properties and distribution properties of the estimated weights, sample expected return and sample variance of the TP are detailed in Bauder et al. (2018); Bodnar et al. (2019); Bodnar and Zabolotskyy (2017); Britten-Jones (1999); Lo (2002); Okhrin and Schmid (2006); Schmid and Zabolotskyy (2008) and Javed et al. (2020).

3. Multivariate distribution

In this chapter, we review important distributions that are very useful in easy reading of the rest of this thesis.

3.1 Univariate normal distribution and related univariate distributions

This subsection reviews relevant univariate distributions later used in the construction of the stochastic representation.

Definition 3.1.1 (Normal distribution)**.** A random variable x with the probability density function

$$(2\pi\sigma^2)^{-1/2}\exp\{-\frac{1}{2\sigma^2}(x-\mu)^2\}, x \in \mathbb{R} \text{ where } \mu \in \mathbb{R}, \sigma > 0 \tag{3.1.1}$$

is said to have a normal distribution with mean μ and variance σ^2. It is denoted by $x \sim \mathcal{N}(\mu, \sigma^2)$. When $\mu = 0$ and $\sigma = 1$, it is said to have the standard normal distribution and is denoted by $z \sim \mathcal{N}(0,1)$. A univariate normal random variable can also be characterised by its stochastic representation, given by

$$x \overset{d}{=} \mu + \sigma z, \mu \in \mathbb{R}, \sigma > 0.$$

The symbol $\overset{d}{=}$ stands for the equality in distribution.

Definition 3.1.2 (chi-square distribution)**.** The **central chi-square** distribution with n degrees of freedom is defined as the sum of n squared independent standard normal distributions, that is

$$\xi \overset{d}{=} \sum_{i=1}^{n} z_i^2 = \mathbf{z}'\mathbf{z} \sim \chi^2(n), \text{ where } \mathbf{z} = (z_1, \ldots, z_n)' \text{ with } z_i \sim \mathcal{N}(0,1). \tag{3.1.2}$$

Alternatively, if $x_1, \ldots, x_n$ are indepedent and normally distributed with $x_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$, then

$$\xi \overset{d}{=} \sum_{i=1}^{n} \left(\frac{x_i - \mu_i}{\sigma_i} \right)^2 \sim \chi^2(n).$$

Note that $\mathbb{E}(\xi) = n$ and $\mathbb{V}(\xi) = 2n$.
If $x_i = \mu_i + z_i$, where $\mu_i \neq 0$, then

$$\xi \overset{d}{=} \sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} (\mu_i + z_i)^2 \sim \chi^2(n, \lambda). \tag{3.1.3}$$

This is called the **non-central chi-square distribution** with n degrees of freedom and non-centrality parameter $\lambda = \sum_{i=1}^{n} \mu_i^2$, its mean and variance are given by $n + \lambda$ and $2n + 4\lambda$, respectively.

Definition 3.1.3 (t-distribution)**.** The **central t-distribution** with n degrees of freedom is defined as the ratio of independent standard normal distribution as a numerator and the square root of a central chi-square random variable divided by its degrees of freedom,

$$t \overset{d}{=} \frac{z}{\sqrt{\xi / n}} \tag{3.1.4}$$

where $z \sim \mathcal{N}(0,1), \xi \sim \chi^2(n)$ and z and ξ are independent.

On the other side, one can define a **non-central t-distribution** in the following way

$$t \overset{d}{=} \frac{z + \mu}{\sqrt{\xi / n}}, \tag{3.1.5}$$

where $z \sim \mathcal{N}(0,1), \xi \sim \chi^2(n)$ and z and ξ are independent.

Definition 3.1.4 (F-distribution)**.** The **central F-distribution** with k and n degrees of freedom is defined as the ratio of two independent central chi-square random variables divided by their respective degrees of freedom,

$$\eta \overset{d}{=} \frac{\xi_1 / k}{\xi_2 / n} \tag{3.1.6}$$

where $\xi_1 \sim \chi^2(k)$, $\xi_2 \sim \chi^2(n)$ and ξ_1 and ξ_2 are independent.

In similar way, one can define the **non-central F-distribution** as the ratio of two independent chi-square random variables, each divided by its degrees of freedom, where the numerator has a non-central chi-square distribution and the denominator has a central chi-square distribution,

$$\eta \stackrel{d}{=} \frac{\xi_1/k}{\xi_2/n} \tag{3.1.7}$$

where $\xi_1 \sim \chi^2(k,\lambda)$, $\xi_2 \sim \chi^2(n)$ and ξ_1 and ξ_2 are independent.

3.2 Multivariate normal distribution

Definition 3.2.1. A random vector $\mathbf{x} \in \mathbb{R}^p$ is multivariate normally distributed with mean vector $\boldsymbol{\mu} \in \mathbb{R}^p$ and covariance matrix $\Sigma > 0, \Sigma \in \mathbb{R}^{p\times p}$ if its density is given by

$$(2\pi)^{-p/2}|\Sigma|^{-1/2} etr\{-\frac{1}{2}\Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu})(\mathbf{x}-\boldsymbol{\mu})'\}, \tag{3.2.1}$$

where $etr(\cdot) = \exp(tr(\cdot))$; $|\cdot|$ and tr denote the determinant and the trace of a square matrix, respectively. The multivariate normal distribution is usually denoted by $\mathbf{x} \sim \mathcal{N}_p(\boldsymbol{\mu}, \Sigma)$. Its stochastic representation is given by

$$\mathbf{x} \stackrel{d}{=} \boldsymbol{\mu} + \Sigma^{1/2}\mathbf{z},$$

where $\mathbf{z} \sim \mathcal{N}_p(\mathbf{0}, \mathbf{I})$ is a standard multivariate normally distributed vector, i.e. with zero mean vector and identity covariance matrix.

Remark 3.2.2. A random vector $\mathbf{x}$ is said to have a singular normal vector if its covariance matrix Σ is singular.

Definition 3.2.3. A random matrix $\mathbf{X} \in \mathbb{R}^{p\times n}$ is matrix normally distributed with mean matrix $\mathbf{M} \in \mathbb{R}^{p\times n}$ and covariance matrices $\Sigma > 0, \Sigma \in \mathbb{R}^{p\times p}$ and $\boldsymbol{\Psi} > 0, \boldsymbol{\Psi} \in \mathbb{R}^{n\times n}$ if its density is

$$(2\pi)^{-np/2}|\Sigma|^{-n/2}|\boldsymbol{\Psi}|^{-p/2} etr\{-\frac{1}{2}\Sigma^{-1}(\mathbf{X}-\mathbf{M})\boldsymbol{\Psi}^{-1}(\mathbf{X}-\mathbf{M})'\}. \tag{3.2.2}$$

It is denoted by $\mathbf{X} \sim \mathcal{N}_{p,n}(\mathbf{M}, \Sigma \otimes \boldsymbol{\Psi})$.

The following theorem gives some important properties of normal distributions (details and proofs of these results can be found, for example, in Gupta and Nagar (2000); Mathai and Provost (1992); Muirhead (1982)).

Theorem 3.2.4. *Let*

$$\mathbf{x} = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} \sim \mathcal{N}_p(\boldsymbol{\mu}, \Sigma), \boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{pmatrix} \text{ and } \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix},$$

where $\mathbf{x} \in \mathbb{R}^p$, $\mathbf{x}_1, \boldsymbol{\mu}_1 \in \mathbb{R}^r$, $\Sigma_{11} \in \mathbb{R}^{r\times r}$, $\mathbf{x}_2, \boldsymbol{\mu}_2 \in \mathbb{R}^{p-r}$ *and* $\Sigma_{22} \in \mathbb{R}^{(p-r)\times(p-r)}$.

a) *If* $\mathbf{A} \in \mathbb{R}^{q\times p}$, *and* $\mathbf{b} \in \mathbb{R}^q$, *then* $\mathbf{Ax}+\mathbf{b} \sim \mathcal{N}_q(\mathbf{A}\boldsymbol{\mu}+\mathbf{b}, \mathbf{A}\Sigma\mathbf{A}')$. *In particular, if* $\mathbf{a} \in \mathbb{R}^p$ *and* $b \in \mathbb{R}$, *then* $\mathbf{a}'\mathbf{x}+b \sim \mathcal{N}(\mathbf{a}'\boldsymbol{\mu}+b, \mathbf{a}'\Sigma\mathbf{a})$.

b) $\mathbf{x}_1 \sim \mathcal{N}_r(\boldsymbol{\mu}_1, \Sigma_{11})$ *and* $\mathbf{x}_2 \sim \mathcal{N}_{p-r}(\boldsymbol{\mu}_2, \Sigma_{22})$;

c) $\mathbf{x}_1|\mathbf{x}_2 \sim \mathcal{N}_r(\boldsymbol{\mu}_1 + \Sigma_{12}\Sigma_{22}^{-1}(\mathbf{x}_2-\boldsymbol{\mu}_2), \Sigma_{11}-\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})$. *The matrix* $\Sigma_{12}\Sigma_{22}^{-1}$ *is called the matrix of regression coefficient. If* $\Sigma_{12} = \mathbf{0}$, *then the subvectors* $\mathbf{x}_1$ *and* $\mathbf{x}_2$ *are independent;*

d) *Let* $\Sigma = \mathbf{T}'\mathbf{T}$ *be the Cholesky decomposition of* Σ, *where* $\mathbf{T}$ *is the upper triangular matrix.Then* $\mathbf{z} = (\mathbf{T}')^{-1}(\mathbf{x}-\boldsymbol{\mu}) \sim \mathcal{N}_p(\mathbf{0}, \mathbf{I})$, *and* $(\mathbf{x}-\boldsymbol{\mu})'\Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu}) = \mathbf{z}'\mathbf{z} \sim \chi^2_p$;

e) *If* $\mathbf{A} = \mathbf{A}'$, *and* $\Sigma > 0$, *then* $\mathbf{x}'\mathbf{Ax} \sim \chi^2_r(\delta^2), \delta^2 = \boldsymbol{\mu}'\mathbf{A}\boldsymbol{\mu}$ *if and only if* $\mathbf{A}\Sigma\mathbf{A} = \mathbf{A}$ *and* $tr(\mathbf{A}\Sigma) = r$. *If* $\boldsymbol{\mu} = \mathbf{0}$ *it implies that* $\delta^2 = 0$;

f) *Let* $\mathbf{L}_1 = \mathbf{x}'\mathbf{Ax}+\mathbf{a}_1'\mathbf{x}+\mathbf{b}_1$ *such that* $\mathbf{A} = \mathbf{A}'$ *and* $\mathbf{L}_2 = \mathbf{a}_2'\mathbf{x}+\mathbf{b}_2$. *Then the necessary and sufficient conditions of* $\mathbf{L}_1$ *and* $\mathbf{L}_2$ *to be independent are the following: (i)* $\Sigma\mathbf{A}\Sigma\mathbf{a}_2 = \mathbf{0}$ *and (ii)* $(\mathbf{a}_1+2\mathbf{A}\boldsymbol{\mu})'\Sigma\mathbf{a}_2 = \mathbf{0}$;

g) *Let* $\mathbf{x}_1, \ldots, \mathbf{x}_n \sim \mathcal{N}_p(\boldsymbol{\mu}, \Sigma)$ *be independent. Then* $\bar{\mathbf{x}} \sim \mathcal{N}_p(\boldsymbol{\mu}, \frac{1}{n}\Sigma)$;

h) *If* $\mathbf{X} \sim \mathcal{N}_{p,n}(\mathbf{M}, \Sigma\otimes\boldsymbol{\Psi})$, *for any* $\mathbf{A} \in \mathbb{R}^{q\times p}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$ *and* $\mathbf{C} \in \mathbb{R}^{q\times m}$, *then* $\mathbf{AXB}+\mathbf{C} \sim \mathcal{N}_{q,m}(\mathbf{AMB}+\mathbf{C}, \mathbf{A}\Sigma\mathbf{A}'\otimes\mathbf{B}'\boldsymbol{\Psi}\mathbf{B})$;

i) *Let* $\mathbf{X} = \begin{pmatrix} \mathbf{X}_1 & \mathbf{X}_2 \end{pmatrix} \sim \mathcal{N}_{p,n}(\mathbf{M}, \Sigma\otimes\boldsymbol{\Psi})$, $\mathbf{M} = \begin{pmatrix} \mathbf{M}_1 & \mathbf{M}_2 \end{pmatrix}$, $\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$ *and* $\boldsymbol{\Psi} = \begin{pmatrix} \boldsymbol{\Psi}_{11} & \boldsymbol{\Psi}_{12} \\ \boldsymbol{\Psi}_{21} & \boldsymbol{\Psi}_{22} \end{pmatrix}$ *with* $\mathbf{X}_1, \mathbf{M}_1 \in \mathbb{R}^{p\times m}$, *and* $\boldsymbol{\Psi}_{11} \in \mathbb{R}^{m\times m}$, *then*

$$\mathbf{X}_1|\mathbf{X}_2 \sim \mathcal{N}_{p,m}(\mathbf{M}_1 + (\mathbf{X}_2-\mathbf{M}_2)\boldsymbol{\Psi}_{22}^{-1}\boldsymbol{\Psi}_{21}, \Sigma\otimes\boldsymbol{\Psi}_{1.22})$$

where $\boldsymbol{\Psi}_{1.22} = \boldsymbol{\Psi}_{11}-\boldsymbol{\Psi}_{12}\boldsymbol{\Psi}_{22}^{-1}\boldsymbol{\Psi}_{21}$;

j) Let $\mathbf{S} = \mathbf{XAX}'$ *and* $\mathbf{V} = \mathbf{XL}, \mathbf{A} \in \mathbb{R}^{n\times n}, \mathbf{L} \in \mathbb{R}^{n\times q}$*, and* $\mathbf{X} \sim \mathcal{N}_{p,n}(\mathbf{M}, \Sigma \otimes \mathbf{I})$*. The necessary and sufficient condition for* $\mathbf{S}$ *and* $\mathbf{V}$ *to be stochastically independent is that* $\mathbf{AL} = \mathbf{0}$*. Moreover, if* $\mathbf{X} \sim \mathcal{N}_{p,n}(\mathbf{M}, \Sigma \otimes \mathbf{\Psi})$*, then* $\mathbf{S}$ *and* $\mathbf{V}$ *are stochastically independent if and only if* $\mathbf{A\Psi L} = \mathbf{0}$.

3.3 Wishart distributions

The Wishart distribution, which is the subject of study in this section was first derived in Wishart (1928) and belongs to the family of matrix distributions. It is regarded as a multivariate analog of a chi-square distribution in the univariate case. It has been applied in various fields of applied and theoretical statistics, for instance, the inference procedures based on the sample covariance matrix of Gaussian observations.

However, it may be needed to work with the number of observations less than the dimension or the inverse sample covariance matrix. This leads to two other useful versions of Wishart distributions, namely, singular Wishart and inverse Wishart distributions.

3.3.1. Wishart distribution. In the following, we define the Wishart distribution as in Gupta and Nagar (2000); Kollo and von Rosen (2006); Muirhead (1982).

Definition 3.3.1. Let $\mathbf{X} \sim \mathcal{N}_{p,n}(\mathbf{M}, \Sigma \otimes \mathbf{I})$ be $p \times n$ matrix and $\Sigma > \mathbf{0}$. The matrix $\mathbf{S}$ of size $p \times p$ is said to be Wishart distributed if and only if $\mathbf{S} = \mathbf{XX}'$.
If $\mathbf{M} = \mathbf{0}$ we have a central Wishart distribution denoted by $\mathbf{S} \sim \mathcal{W}_p(n, \Sigma)$, otherwise, we have a non central Wishart distribution denoted by $\mathbf{S} \sim \mathcal{W}_p(n, \Sigma, \Omega)$ where $\Omega = \Sigma^{-1}\mathbf{MM}'$.

For a central Wishart distributed matrix with $n \geq p$, the density function is given by

$$\frac{|\mathbf{S}|^{(n-p-1)/2}}{2^{pn/2}\Gamma_p(n/2)|\Sigma|^{n/2}} etr\{-\frac{1}{2}\Sigma^{-1}\mathbf{S}\} \tag{3.3.1}$$

where $\Gamma_p(\cdot)$ denotes the multivariate gamma function. In the following theorem, we summarize the basic properties of a Wishart distribution.

Theorem 3.3.2. *Muirhead (1982, Theorems 3.2.10 & 3.2.11). Let* $\mathbf{X} \sim \mathcal{N}_{p,n}(\mathbf{M}, \Sigma \otimes \mathbf{I})$ *and* $\mathbf{S} \sim \mathcal{W}_p(n, \Sigma)$ *and considering the following partition of* $\mathbf{S}$ *and* Σ*:*

$$\mathbf{S} = \begin{pmatrix} \mathbf{S}_{11} & \mathbf{S}_{12} \\ \mathbf{S}_{21} & \mathbf{S}_{22} \end{pmatrix}, \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} \tag{3.3.2}$$

and put

$$\mathbf{S}_{11.2} = \mathbf{S}_{11} - \mathbf{S}_{12}\mathbf{S}_{22}^{-1}\mathbf{S}_{21}, \Sigma_{11.2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} \tag{3.3.3}$$

with $\dim(\mathbf{S}_{11}) = \dim(\Sigma_{11}) = r \times r, r < p \leq n$

a) If $\mathbf{A} : n \times n$ *symmetric and idempotent, so that* $\mathbf{MA} = \mathbf{0}$*. Then* $\mathbf{XAX}' \sim \mathcal{W}_p(rank(A), \Sigma)$*;*

b) Suppose the partition in (3.3.2)*, then*

- $\mathbf{S}_{11.2} \sim \mathcal{W}_r(n - r + k, \Sigma_{11.2})$ *and is independent of* $\mathbf{S}_{12}$ *and* $\mathbf{S}_{22}$*;*
- $\mathbf{S}_{12}|\mathbf{S}_{22} \sim \mathcal{N}(\Sigma_{12}\Sigma_{22}^{-1}\mathbf{S}_{22}, \Sigma_{11.2} \otimes \mathbf{S}_{22})$*;*
- $\mathbf{S}_{22} \sim \mathcal{W}_{p-r}(n, \Sigma_{22})$*;*

c) Suppose $\mathbf{A} \in \mathbb{R}^{p \times p}$*, then* $\mathbf{ASA}' \sim \mathcal{W}_p(n, \mathbf{A}\Sigma\mathbf{A}')$*;*

d) Suppose $\mathbf{A} \in \mathbb{R}^{k \times p}$ *of rank k, then* $(\mathbf{AS}^{-1}\mathbf{A}')^{-1} \sim \mathcal{W}_k(n - p + k, (\mathbf{A}\Sigma^{-1}\mathbf{A}')^{-1})$.

3.3.2. Singular Wishart distribution. In many applications, the dimension may exceed the number of observations. We make use of singular Wishart distribution to deal with complications raised in that case. Its definition and basic properties are delivered below.

Definition 3.3.3. Let $\mathbf{X} = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$ be independent identically normally distributed, with $\mathbf{x}_i \sim \mathcal{N}_p(\mathbf{0}, \Sigma)$ and $\Sigma > 0$. Let $p > n$. Then the matrix of size $p \times p$, $\mathbf{S} = \mathbf{XX}'$ is said to have a singular Wishart distribution.

Theorem 3.3.4 summarizes the properties of the singular Wishart distribution.

Theorem 3.3.4. *Bodnar and Okhrin (2008, Lemma 1,Theorem 1). Let* $\mathbf{S} \sim \mathcal{W}_p(n, \Sigma)$ *and consider the following partition of* $\mathbf{S}$ *and* Σ*:*

$$\mathbf{S} = \begin{pmatrix} \mathbf{S}_{11} & \mathbf{S}_{12} \\ \mathbf{S}_{21} & \mathbf{S}_{22} \end{pmatrix}, \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} \tag{3.3.4}$$

with $\dim(\mathbf{S}_{11}) = \dim(\Sigma_{11}) = r \times r, r < n < p$. *Then it holds that*

a) $\mathbf{S}_{11} \sim \mathcal{W}_r(n, \Sigma_{11})$*;*

b) $\mathbf{S}_{21}|\mathbf{S}_{11} \sim \mathcal{N}(\Sigma_{21}\Sigma_{11}^{-1}\mathbf{S}_{11}, \Sigma_{22.1} \otimes \mathbf{S}_{11})$*, with* $\Sigma_{22.1} = \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}$*;*

c) the density of $\mathbf{S}_{21}\mathbf{S}_{11}^{-1}$ *is given by*

$$\begin{aligned} f_{\mathbf{S}_{21}\mathbf{S}_{11}^{-1}}(\mathbf{X}) &= \frac{|\Sigma_{11}|^{\frac{n-r}{2}}\Gamma_r(\frac{n+p-r}{2})}{|\Sigma_{22.1}|^{\frac{r}{2}}\pi^{\frac{(p-r)r}{2}}\Gamma_r(\frac{n}{2})} \\ &\times \left|\mathbf{I} + \Sigma_{11}\left(\mathbf{X} - \Sigma_{21}\Sigma_{11}^{-1}\right)'\Sigma_{22.1}^{-1}\left(\mathbf{X} - \Sigma_{21}\Sigma_{11}^{-1}\right)\right|^{-\frac{1}{2}(n+p-r)} \end{aligned}$$

3.3.3. Inverse Wishart distribution. Even though Wishart distribution has numerous applications in statistics, it is not directly applicable in the portfolio theory, which is the subject of this thesis. Rather we use the inverse Wishart distribution denoted by $\mathcal{W}^{-1}(n, \Sigma)$. We define the inverse Wishart distribution as in Gupta and Nagar (2000).

Definition 3.3.5. A $p \times p$ random matrix $\mathbf{S}$ is said to have an inverse Wishart with n degrees of freedom and $p \times p$ parameter matrix Σ, if its density is given by

$$\frac{2^{-\frac{1}{2}(n-p-1)}|\Sigma|^{\frac{(n-p-1)}{2}}}{\Gamma_p[\frac{1}{2}(n-p-1)]|\mathbf{S}|^{\frac{1}{2}n}} \exp\{-\frac{1}{2}\mathbf{S}^{-1}\Sigma\}, \mathbf{S} > 0, \Sigma > 0, n > 2p. \tag{3.3.5}$$

The property of the inverse Wishart are summarized below

Theorem 3.3.6. *Bodnar and Okhrin (2008, Theorem 3). Suppose* $\mathbf{S} \sim \mathcal{W}^{-1}(n, \Sigma)$ *and consider the partition as in* (3.3.2)*, with* $\dim(\mathbf{S}_{11}) = \dim(\Sigma_{11}) = r \times r, r < p < n$. *Then*

a) $\mathbf{S}_{11.2} \sim \mathcal{W}_r^{-1}(n - p + r, \Sigma_{11.2})$ *and is independent of* $\mathbf{S}_{22}$*;*

b) $\mathbf{S}_{12}|\mathbf{S}_{22}, \mathbf{S}_{11.2} \sim \mathcal{N}(\Sigma_{12}\Sigma_{22}^{-1}\mathbf{S}_{22}, \mathbf{S}_{11.2} \otimes \mathbf{S}_{22}\Sigma_{22}^{-1}\mathbf{S}_{22})$*, with* $\mathbf{S}_{11.2}$ *defined as in* (3.3.3)*;*

c) $\mathbf{S}_{22} \sim \mathcal{W}_{p-r}^{-1}(n - 2r, \Sigma_{22})$*;*

d) $\mathbf{S}_{12}\mathbf{S}_{22}^{-1}$ *is independent of* $\mathbf{S}_{22}$*, with the density given by*

$$
\begin{aligned}
f_{\mathbf{S}_{12}\mathbf{S}_{22}^{-1}}(\mathbf{X}) \quad &= \quad \frac{|\Sigma_{11.2}|^{-\frac{p-r}{2}}|\Sigma_{22}|^{\frac{r}{2}}\Gamma_r(\frac{n-r-1}{2})}{\pi^{\frac{(p-r)r}{2}}\Gamma_r(\frac{n-p-1}{2})} \\
&\times \quad \left|\mathbf{I}+\Sigma_{11.2}^{-1}\left(\mathbf{X}-\Sigma_{12}\Sigma_{22}^{-1}\right)\Sigma_{22}\left(\mathbf{X}-\Sigma_{12}\Sigma_{22}^{-1}\right)'\right|^{-\frac{1}{2}(n-p-1)}
\end{aligned}
$$

e) $\mathbf{S}_{22}$ *is independent of* $\mathbf{S}_{12}\mathbf{S}_{22}^{-1}$ *and* $\mathbf{S}_{11.2}$*;*

f) $\mathbf{S}_{11.2}|(\mathbf{S}_{12}\mathbf{S}_{22}^{-1}=\mathbf{X}) \sim \mathcal{W}_r^{-1}\left(n, \Sigma_{11.2}+\left(\mathbf{X}-\Sigma_{12}\Sigma_{22}^{-1}\right)\Sigma_{22}\left(\mathbf{X}-\Sigma_{12}\Sigma_{22}^{-1}\right)'\right)$

3.4 High-dimensional asymptotics

Nowadays, we live in a world where data storage and computing resources allow the production, processing, and storage of an exponentially growing volume of data. Data has become omnipresent in almost every part of human activities, namely, science, medicine, business, and finance to name just a few. Most of the modern data are characterized by the fact that they record several features on each object or individual. Technically, we say that the dimension p is comparable to the sample size n. Such data are said to be high-dimensional (Giraud (2014)).

This type of data renders most of the usual statistical methods obsolete. For example, under the normality assumption of the asset returns, the inverse of their sample covariance matrix is a biased estimator of the precision matrix. The bias tends to zero only when the portfolio dimension is considerably smaller than the sample size. When the portfolio dimension is comparable to the sample size, an improved estimator of the precision matrix is needed. Due to the presence of the precision matrix in the formulas of optimal portfolio weights and their characteristics, the problem becomes very important in portfolio theory.

For a fixed dimension p and a growing sample size n, the standard asymptotics holds (Yang and Le Cam (2000)). On the other hand, if the dimension p is comparable or larger than the sample size n, the following scenarios may be considered

- p, n are both large;

- $p/n \to c > 0, p \to \infty$ and $n \to \infty$;
- $(p, n) \to (\infty, \infty)$ this means
  - first $p \to \infty$ then $n \to \infty$;
  - first $n \to \infty$ then $p \to \infty$;
  - $p \to \infty$ and $n \to \infty$ simultaneously.

Here we are dealing with high-dimensional asymptotics or Kolmogorov asymptotics (Bühlmann and Van De Geer (2011)). In this situation, the classical limit theorems are no longer suitable.

4. Summary

The understanding of the behaviour of the TP and the components involved in its computation in both traditional and high-dimensional asymptotics is of great importance for financial actors. Because of its considerable interest, a number of works in connection to it have been produced (e.g., Bauder et al. (2018); Bodnar (2009); Bodnar et al. (2019); Javed et al. (2020)). Most of them focus on the properties and distributional properties of the TP weights.

In **paper I**, we focus on the determination of the existence of the TP. Specifying the location of the TP on the set of feasible portfolios is a challenging task due to parameter uncertainty. By assuming that the returns are independent and multivariate normally distributed, we propose a finite sample test on the mean-variance efficiency of the TP, and we derive the distribution of the proposed test statistic under both the null and alternative hypotheses. Particularly, we use the derived distribution of the test statistic in assessing the power of the test and in the construction of a confidence set. Moreover, We conduct the out-of-sample performance. We noticed that the performance of the proposed test is better compared to the naive way of keeping the optimal portfolio at hand. We also show through an extensive simulation study that our test is robust towards the violation of the normality assumption and can be used for heavy-tailed stochastic models. At the end, the derived results are illustrated using actual stock returns. We notice the following, when the sample size is relatively large and a stable period is observed on the market, then the mean-variance efficiency of the TP can be justified. We also note that using the developed test statistic, we can draw the decision about the inefficiency of the TP at the end of 2008, which signals the financial crisis period. During this period we are not able to accept the efficiency of the TP.

Paper II extends the results of **paper I**. Firstly, we propose a test on the existence of the efficient frontier based on the slope parameter. Basing on the derivation in Bodnar and Schmid (2008) we deliver the test statistic and provide its distributions under both the null and alternative hypotheses. Moreover, we derive the distribution of the proposed test in the high-dimensional setting and studied the behavior of its power function compared to the empirical one. We found that the high-dimensional expression of the power provides a reliable approximation of the true power function. Secondly, due to the failure in providing good results when the portfolio size k and sample size n is comparable, i.e, $k/n \to c \in (0, \infty)$ as $n \to \infty$, we extend the results in Muhinyuza et al. (2017) by deriving a distribution of the test on the location of the TP on the efficient frontier in high-dimensions. We also deliver the power function of that test in a high-dimensional setting. A good performance of asymptotic power is noticed. Furthermore, through a simulation study, we analyze the performance of the two tests. We found that both tests are robust to the violation of the normality assumption. We also observe that when the slope of the efficient frontier s is small and the Sharpe ratio of the GMVP S_{GMV} is large, then the test based on the slope of the efficient frontier T_λ performs better. On the other hand, if the slope of the efficient frontier s is large and Sharpe ratio of the GMVP S_{GMV} is small, then the test on the location of the TP on the efficient frontier T is preferable.

In **PaperIII**, we assume the existence of the TP and study the distributional properties of the TP weights assuming a normal distribution of the logarithmic returns. We derive a stochastic representation (SR) of the TP that fully describes its distribution. Using the SR, we provide the asymptotic distribution of the TP weights under the high-dimensional asymptotic regime. Furthermore, we consider a test about the elements of the TP weights and derive the asymptotic distribution of the test statistic under the null and alternative hypotheses. The comparison study between the asymptotic distribution of the TP weights and exact finite sample density is conducted and we observe that the asymptotic distribution serves as a good approximation of the exact finite sample distribution. Comparing the power function of the asymptotic test to the power obtained for the exact test we find that both powers are indistinguishable. In an empirical study, we analyse the TP weights in portfolios containing stocks from the S& P 500 index by studying the dynamic behaviour of the p-values obtained from the exact and asymptotic tests. We first note that the p-values obtained from both tests are almost identical, which is a sign of a good performance of high-dimensional asymptotic.

In the last **paper**, we study properties of components involved in the computation of the TP, namely, the properties of the product of a singular Wishart matrix and a singular Gaussian vector. We first derive the distribution of that product in the form of a stochastic representation (SR). The SR provides a fast and efficient way of how the elements the product should be simulated. We then use the derived SR in the obtention of the characteristic function of that product, which is later used in proving the asymptotic normality of that product under the double asymptotic regime. A good performance of the obtained asymptotic distribution is documented in a simulation study even for the case where $c > 1$.

5. Sammanfattning

Att förstå beteendet av den tangerande portföljen (TP) och dess komponenter i beräknandet av den traditionella och högdimensionella asymptotiken är oerhört viktigt för finansiella aktörer. Då det är av så pass stort intresse har mycket forskning bedrivits på den(e.g., Bauder et al. (2018); Bodnar (2009); Bodnar et al. (2019); Javed et al. (2020)). De flesta fokuserar på beteendet och fördelningens beteende av TP portfäljens vikter.

I första manuskriptet fokuserar vi på att bestämma om TP existerar eller inte. Att specificera vart denna ligger på mängden av möjliga portföljer är svårt på grund av parameter osäkerheten från våra skattningar. Genom att anta att våra tillgångslag är oberoende och likafördelade multivariata normalt fördelade föreslår vi ett statistiskt test baserat på ändlig mängd information. Testet ämnar att testa om TP portföljen är effektiv, i markovitz anda. Vi härleder statistikans fördelning under både null och alternativ hypotesen. Genom detta kan vi också uppskatta testets styrka och konstruera konfidensintervall. Utöver detta undersöker vi dess prestanda på ny data, sådant som parametrarna inte känner till. Där ser vi att testet är bättre än att naivt äga denna specifika optimala portfölj. Vi visar också genom en omfattande simuleringsstudie att testet är robust mot brott mot fördelningsantagandet och kan användas för processer med tunga svansar. För att komplettera simuleringsstudien applicerar vi testet på riktig akitedata. Vi ser då att när stickprovsstorleken är relativt stort och marknaden har befunnit sig i en stabil period så kan TP bedömas vara effektiv. Vi ser också att med hjälp av detta test kan vi dra slutsatsen att TP inte är effektiv i slutet av 2008, då finanskrisen uppdagades. Under den följande perioden kan inte TP bedömas vara effektiv alls.

Det andra manuskriptet utökar det första. Först föreslår vi ett test baserat på lutningen av den effektiva fronten. Från teorin presenterad i Bodnar and Schmid (2008) levererar vi en test staitistka och dess fördelning under både null och alternativhypotesen. Vi härleder också testets högdimensionella fördelning och studerar dess styrka i jämförelse med den empiriska. Vi fann att den högdimensionella styrke funktionen approximerar den sanna styrke funktionen bäst. Då vi inte lyckas med att ge ett bra resultat där portföljstorleken k och stickprovsstorleken är jämförbara i storlek, dvs. $k/n \to c \in (0,\infty)$ då $n \to \infty$, utökar vi resultaten frran Muhinyuza et al. (2017) genom att härleda en fördelning för testet för TP plats i högre dimensioner. Vi härleder också dess styrke funktion i högre dimensioner. Denna visar sig ha god asymptotisk styrka. Genom en simuleringsstudie kan vi analysera de två olika testen. de två olika testen är robusta mot brott av fördelningsantagandet. Vi ser också att då lutningen av den effektiva fronten s är liten och Sharpe ration S_{GMV} är stor, så är testet baserat på lutningen bäst. Om det omvånda gäller så är platsen för TP bättre.

I det tredje manuskriptet antar vi att TP existerar och att våra tillgångslag följer en logaritmisk normal fördelning. Vi härleder den stokastiska representationen för TP som beskriver hela dess fördelning. Med hjälp av den stokastiska representationen härleder vi vikterna för TP i den högdimensionella asymptotiska ramverket. Vi tar fram ett test om vikterna i portföljen och härleder dess fördelning under både null och alternativhypotesen i höga dimensioner. Därefter jämför vi TP fördelningen med dess asymptotiska fördelning. Vi ser att den asymptotiska fördelningen är en bra approximation av den exakta fördelningen. I en jämförelse mellan styrke funktionerna för testets asymptotiska fördelning och den som är baserad på dess exakta fördelning finner vi att de går ej att se någon skillnad mellan dem. I en empirisk studie analyserar vi vikterna i TP portföljen med aktier från S&P500 index:et. Här studerar vi det dynamiska beteendet av p-värden från den exakta och asymptotiska testen. Första noterar vi att dessa p-värden är närpå identiska, vilket talar för testen i högre dimensioner.

I sista manuskriptet studerar vi de olika komponenter som är involverade i att konstruera TP vikterna, nämligen produkten mellan en singulär Wishart- och en singulär multivariat fördelning. Först härleder vi dess stokastiska representation. Denna ger oss ett sätt att snabbt simulera denna produkt. Vi använder senare den stokastiska representationen för att härleda den karakteristiska funktionen för produkten vilken senare används för att härleda den högdimensionella fördelningen. Vi ser god prestanda i den asymptotiska fördelningen genom en simuleringsstudie, även i fallet då $c > 1$.

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Part II

I

A test on the location of the tangency portfolio on the set of feasible portfolios

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Abstract

Due to the problem of parameter uncertainty, specifying the location of the tangency portfolio(TP) on the set of feasible portfolios becomes a challenging task. The set of feasible portfolios is a parabola in the mean-variance space with optimal portfolios lying on its upper part. Using statistical test theory, we want to decide if the tangency portfolio is mean-variance efficient, i.e. if it belongs to the upper limb of the efficient frontier. In the opposite case, the investor would prefer to invest into the risk-free asset or into the global minimum variance portfolio which lies in the vertex of the set of feasible portfolios. Assuming that the portfolio asset returns are independent and multivariate normally distributed, we suggest a test on the location of the tangency portfolio on the set of feasible portfolios. The distribution of the test statistic is derived under both hypotheses, which we use to assess the power of the test and construct a confidence interval. Moreover, out-of-sample performance of the test is evaluated based on real data. The robustness to the assumption of normality is investigated via an extensive simulation study where we show that the new test is robust to the violation of the normality assumption and can also be used for heavy-tailed stochastic models. Moreover, in an empirical study we apply the developed theory to real data. We find that when the sample size is relatively large and a stable period is present on the market, then the mean-variance efficiency of the tangency portfolio can be statistically justified.

Keywords: tangency portfolio, feasible portfolios, test theory, power function, out-of-sample performance

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1 Introduction

The question of wealth allocation is relevant for both individuals, e.g. retirement savings, as well as for banks and other institutional investors. How this should be done in practice, however depend on a multitude of factors, not the least the investors view on risk in relation to return. The most influential approach to deal with this problem is the mean-variance analysis proposed by Markowitz (1952). Following Markowitz (1952), the optimal portfolio weights are found by minimizing the risk, i.e. the variance, of the portfolio for a given level of the expected return.

In the case without a risk-free asset, Merton (1972) showed that all optimal solutions of Markowitz's optimization problem lie on the upper limb of the parabola in the mean-variance space. This parabola is known as the efficient frontier and given by

$$V = \frac{a - 2bR + cR^2}{ac - b^2}, \tag{1}$$

where $a = \boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}$, $b = \mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}$, $c = \mathbf{1}\boldsymbol{\Sigma}^{-1}\mathbf{1}$; $R = \mathbf{w}'\boldsymbol{\mu}$ is the expected return of the portfolio with the weights $\mathbf{w}$; $V = \mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}$ is its variance; $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are the expected return vector and the covariance matrix of the asset returns, respectively. The symbol $\mathbf{1}$ denotes the vector of ones of an appropriate order. Unfortunately, the set of parameters $\{a, b, c\}$, known as the efficient set of constants, does not possess an appropriate financial meaning.

Rewriting (1) we obtain an alternative expression of the efficient frontier

$$(R - R_{GMV})^2 = s(V - V_{GMV}) \tag{2}$$

where

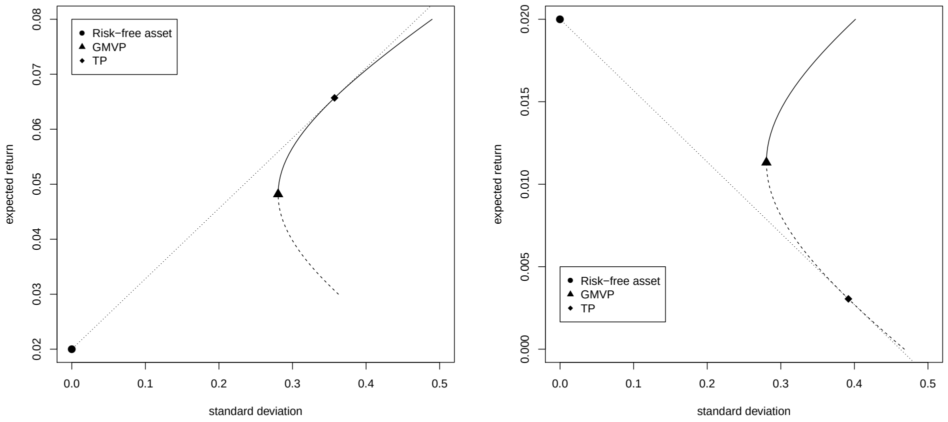
$$R_{GMV} = \frac{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}} \quad \text{and} \quad V_{GMV} = \frac{1}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}} \tag{3}$$

are the expected return and the variance of the global minimum variance portfolio (GMVP), that is, the portfolio with the smallest variance among the efficient portfolios (see, e.g., Frahm (2010); Glombek (2014); Bodnar et al. (2017a,b)). The parameter

$$s = \boldsymbol{\mu}'\mathbf{R}\boldsymbol{\mu} \quad \text{with} \quad \mathbf{R} = \boldsymbol{\Sigma}^{-1} - \frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}\mathbf{1}'\boldsymbol{\Sigma}^{-1}}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}} \tag{4}$$

stands for the slope coefficient of the parabola in the mean-variance space. The properties of the efficient frontier together with the statement about the distribution of the sample efficient frontier were discussed in detail by Bodnar and Schmid (2008); Kan and Zhou (2008); Bodnar and Schmid (2009).

If there is a possibility to invest into a risk-free asset, then the efficient frontier becomes a tangent line in the mean-variance space which is drawn from the return of the risk-free asset to the parabola (2). The tangent point is known as the tangency portfolio



(a) $R_{GMV} \geq r_f$

(b) $R_{GMV} < r_f$

Figure 1: Location of the tangency portfolio on the set of feasible portfolios in the two cases: Figure 1(a) $R_{GMV} \geq r_f$ and Figure1(b) $R_{GMV} < r_f$.

(TP), see e.g., Ingersoll (1987). This portfolio maximizes the Sharpe ratio (SR), $SR = (\mathbf{w}'\boldsymbol{\mu} - r_f)/(\sqrt{\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}})$, and it has recently received a lot of attention in the literature. For instance, its statistical properties under different assumptions imposed on the distribution of the asset returns were discussed in Lo (2002), whereas Britten-Jones (1999) derived an exact test on the TP weights and showed that it is not possible to reject the null hypothesis that the weight of the US market is equal to one in an international portfolio. Further, while Okhrin and Schmid (2006) showed that the estimated weights of this portfolio do not possess the first moment, Schmid and Zabolotskyy (2008) proved that the unbiased estimator of the TP weights does not exist at all. Recently, Bodnar and Zabolotskyy (2017) investigated the risk properties of the TP and showed that this portfolio is a very risky investment opportunity which should be carefully considered in practice.

The location of the TP portfolio on the set of feasible portfolio depends crucially on the relation between the expected return of the GMVP and the return of the risk-free asset (see Figure 1). The TP is mean-variance efficient, i.e. it belongs to the upper part of the efficient frontier as in Figure 1(a) only if the expected return of the GMVP is greater than the return on the risk-free asset return (see, e.g., Ingersoll (1987, chapter 4)). On the other hand, this consideration may not be appropriate in many practical situations where the expected return of the GMVP is inferior to the return of the risk-free asset. In this case the tangent line drawn to the set of feasible portfolios from the return of the risk-free rate has no joint point with the efficient frontier and, consequently, the TP belongs to the set of the feasible portfolios which are located on the lower part of the parabola as shown in Figure 1(b). The investor would then prefer to invest into the risk-free asset or in the GMVP which lies in the vertex of the efficient frontier. We contribute to the existing literature on the TP by deriving an exact test on its location on the set of feasible portfolios. The distribution of the suggested test statistic is obtained under both hypotheses. Moreover, out-of-sample performance of the portfolio determined by implementing the derived test is assessed.

The remainder of the paper is organised as follows. Section 2 contains a detailed description of statistical test theory for the location of the tangency portfolio on the set of feasible portfolios. We concentrate on the derivation of the test statistic, its distribution under both hypotheses, the analysis of the power function, and the construction of a confidence interval. In Section 3, out-of-sample performance is presented. In Section 4, the numerical procedure for investigating the robustness of normality assumption are provided, while empirical results are discussed in Section 5. Final remarks are presented in Section 6. All proofs are found in the appendix.

2 Finite-sample test on the location of the tangency portfolio

The location of the tangency portfolio on the set of feasible portfolios depends on the relation between the risk-free rate r_f and the expected return on the GMVP(R_{GMV}) as shown in Figure 1. If the investor wants to be sure in the investment into the TP, (s)he has to check if $R_{GMV} > r_f$. This problem can be formulated as a statistical test with the hypotheses given by

$$H_0: \ R_{GMV} \le r_f \quad \text{against} \quad H_1: \ R_{GMV} > r_f. \tag{5}$$

The rejection of the null hypothesis means that the TP lies on the upper part of the efficient frontier as shown in Figure 1(a). In contrast, if the null hypothesis in (5) cannot be rejected, then the investor cannot be certain of the optimality of the TP and allocation into the risk-free asset could be considered as a suitable alternative.

Let $\mathbf{X}_1, \ldots, \mathbf{X}_n$ denote an independent k-dimensional sample of the asset returns, where $\mathbf{E}[\mathbf{X}_t] = \boldsymbol{\mu}$ and $\mathbf{cov}[\mathbf{X}_t] = \boldsymbol{\Sigma}$, for $t = 1, \ldots, n$. The test statistic for testing (5) is obtained following the derivation in Bodnar and Schmid (2009) and is given by

$$T = \frac{\sqrt{n-k}}{\sqrt{n-1}} \frac{\hat{R}_{GMV} - r_f}{\sqrt{1 + \frac{n}{n-1}\hat{s}}\sqrt{\frac{\hat{V}_{GMV}}{n}}}, \tag{6}$$

where $\hat{R}_{GMV}$, $\hat{V}_{GMV}$, and $\hat{s}$ are the sample estimators for R_{GMV}, V_{GMV}, and s given by

$$\hat{R}_{GMV} = \frac{\mathbf{1}'\hat{\boldsymbol{\Sigma}}^{-1}\hat{\boldsymbol{\mu}}}{\mathbf{1}'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}} \quad \text{and} \quad \hat{V}_{GMV} = \frac{1}{\mathbf{1}'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}} \tag{7}$$

and

$$\hat{s} = \hat{\boldsymbol{\mu}}'\hat{\mathbf{R}}\hat{\boldsymbol{\mu}}, \quad \hat{\mathbf{R}} = \hat{\boldsymbol{\Sigma}}^{-1} - \frac{\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}\mathbf{1}'\hat{\boldsymbol{\Sigma}}^{-1}}{\mathbf{1}'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}} \tag{8}$$

where

$$\hat{\boldsymbol{\mu}} = \frac{1}{n}\sum_{t=1}^{n}\mathbf{X}_t \quad \text{and} \quad \hat{\boldsymbol{\Sigma}} = \frac{1}{n-1}\sum_{t=1}^{n}(\mathbf{X}_t - \hat{\boldsymbol{\mu}})(\mathbf{X}_t - \hat{\boldsymbol{\mu}})'$$

are the sample mean vector and the sample covariance matrix, respectively. Further, the distribution of T is given by

Proposition 1. *Let $\mathbf{X}_1, \ldots, \mathbf{X}_n$ be independent random vectors of asset returns with $\mathbf{X}_t \sim N_k(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ for $t = 1, \ldots, n$. Assume that $\boldsymbol{\Sigma}$ is positive definite and $n > k$. Then the density of T is given by*

$$f_T(x) = \frac{n(n-k+1)}{(k-1)(n-1)}\int_0^{\infty} f_{t_{n-k,\delta(y)}}(x) f_{F_{k-1,n-k+1,ns}}\left(\frac{n(n-k+1)}{(k-1)(n-1)}y\right)dy \tag{9}$$

where

$$\delta(y) = \sqrt{\frac{n}{1+y(n/(n-1))}}S_{GMV} \quad \text{and where} \quad S_{GMV} = \frac{R_{GMV} - r_f}{\sqrt{V_{GMV}}}$$

is the Sharpe ratio of the GMVP. The slope parameter s is defined in (4).

The proof of Proposition 1 follows from Proposition 1 in Bodnar and Schmid (2009). Hence, from Proposition 1 it is seen that the test statistic T may be represented as a mixture of a non-central t distribution with $n-k$ degrees of freedom and a non-centrality parameter $\delta(y)$. Further, by using Proposition 1 it is possible to derive the critical value for the test (5) at significance level α. The result of this is stated in Proposition 2, whose proof is given in the appendix.

Proposition 2. *Under the conditions of Proposition 1, it holds that*

$$\sup_{V_{GMV}>0, s\geq 0, R_{GMV}\leq r_f} G_{T,\alpha,t_{n-k,1-\alpha}}(S_{GMV}, s) \leq \mathbb{P}_{H_0:R_{GMV}=r_f}(T > t_{n-k,1-\alpha}) = \alpha,$$

where

$$G_{T,\alpha,c}(S_{GMV}, s) = \mathbb{P}(T > c) = \int_c^{\infty} f_T(x)dx\,.$$

Thus, from Proposition 2 it is seen that the test of (5) rejects H_0 in favour of H_1 as soon as $T \geq t_{n-k,1-\alpha}$. Another important characteristic of a statistical test is its power function. It turns out that the power function of the test (5) only depends on $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$

in terms of S_{GMV} and s and is given by

$$G_{T,\alpha,t_{n-k,1-\alpha}}(S_{GMV},s) = \mathbb{P}(T > t_{n-k,1-\alpha})$$
$$= \frac{n(n-k+1)}{(k-1)(n-1)} \int_0^\infty \left(1 - F_{t_{n-k,\delta(y)}}(t_{n-k,1-\alpha})\right) f_{F_{k-1,n-k+1,ns}}\left(\frac{n(n-k+1)}{(k-1)(n-1)}y\right) dy.$$

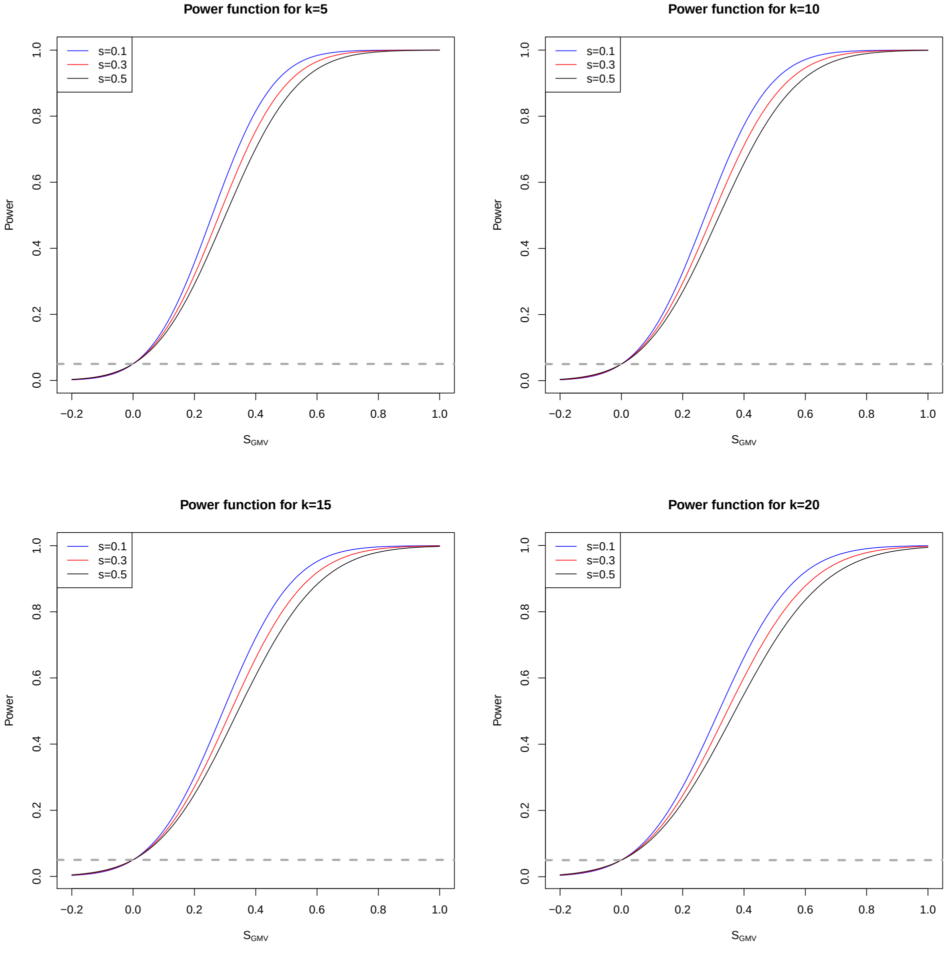


Figure 2: Power function of test (5) for portfolio dimension $k \in \{5, 10, 15, 20\}$ and sample size $n = 50$

This is a nice property of the suggested test which allows us to visualize its power function for fixed values of k and n as a function of s and S_{GMV} only. In Figures 2 and 3, we present the power of the test (5) for $k \in \{5, 10, 15, 20\}$, $n \in \{50, 250\}$, and $s = \{0.1, 0.3, 0.5\}$. The values of S_{GMV} smaller than or equal to 0 corresponds to the null

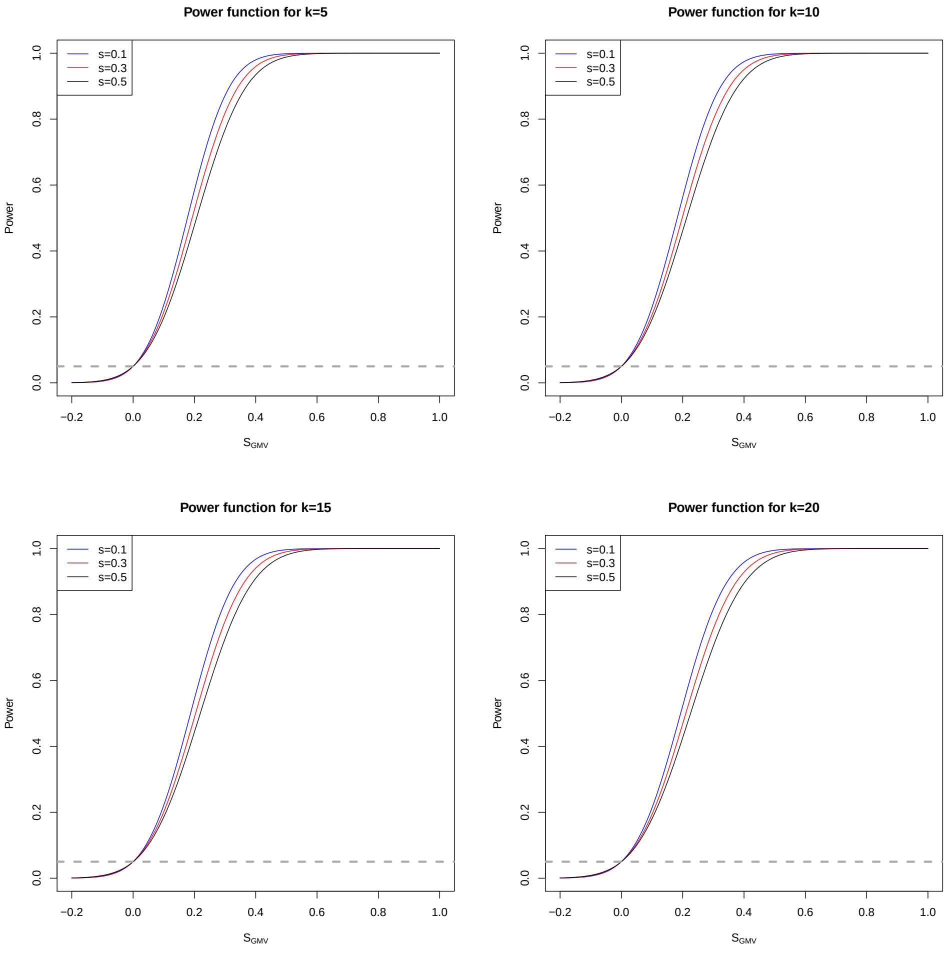


Figure 3: Power function of test (5) for portfolio dimension $k \in \{5, 10, 15, 20\}$ and sample size $n = 100$

hypothesis. We observe that the power increases rapidly as S_{GMV} becomes larger than zero. It reaches one already for moderate values of S_{GMV}. For example, it is close one for S_{GMV} around 0.2 when $n = 250$ corresponding to approximately one year of daily market observations or five years of weekly data. Furthermore, we note that the power increases if s decreases. This result is in line with the behaviour of the non-central F-distribution whose distribution function is decreasing in the non-centrality parameter. This result also has an interesting financial interpretation. If the slope parameter s is smaller, then the optimal portfolio with the same Sharpe ratio and the excess expected return as one in the case of larger s has a higher variance. Consequently, it deviates from the GMVP stronger than in the case of larger s and thus can be easier detected by the test (5).

We conclude this section with the two important remarks:

Remark 1. *Performing a statistical test on the hypotheses* (5)*, one can only draw conclusions about investing into the TP. However, if the null hypothesis cannot be rejected, then we still have no statistical justification about avoiding the wealth allocation into the TP. In order to be sure that the TP belongs to the lower part of the parabola as in Figure 1(b), one has to perform the lower one-sided test with the hypotheses given by*

$$\tilde{H}_0: \; R_{GMV} \geq r_f \quad \text{against} \quad \tilde{H}_1: \; R_{GMV} < r_f. \tag{10}$$

This test reject the null hypothesis, i.e. it confirms that the TP is not efficient, as soon as $T < t_{n-k,\alpha}$ where the statistic T is given in (6)*.*

The power function of the test (10) *is obtained similarly to the power function of the test* (5) *and is given by*

$$\begin{aligned}
&\tilde{G}_{T,\alpha,t_{n-k,\alpha}}(S_{GMV}, s) = \mathbb{P}(T < t_{n-k,\alpha}) \\
&= \frac{n(n-k+1)}{(k-1)(n-1)} \int_0^{\infty} \left(F_{t_{n-k,\delta(y)}}(t_{n-k,\alpha}) \right) f_{F_{k-1,n-k+1,ns}} \left(\frac{n(n-k+1)}{(k-1)(n-1)} y \right) dy.
\end{aligned}$$

which also only depends on $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ through S_{GMV} and s.

Remark 2. *Using a one-to-one correspondence between a statistical test and an interval estimation, we can draw a further important conclusion by using the suggested two tests. Namely, it is possible to specify a $(1-\alpha)$ one-sided confidence interval for the risk-free rate such that if r_f belongs to this interval, a conclusion about the investment into the TP can be drawn.*

In the case of the upper one-sided test this interval is given by

$$I_{1-\alpha} = \left[\hat{R}_{GMV} - t_{n-k,1-\alpha} \frac{\sqrt{n-1}}{\sqrt{n-k}} \sqrt{1 + \frac{n}{n-1} \hat{s}} \sqrt{\frac{\hat{V}_{GMV}}{n}}, +\infty \right),$$

while for the lower one-sided we get

$$\tilde{I}_{1-\alpha} = \left(-\infty, \hat{R}_{GMV} - t_{n-k,\alpha}\frac{\sqrt{n-1}}{\sqrt{n-k}}\sqrt{1+\frac{n}{n-1}\hat{s}}\sqrt{\frac{\hat{V}_{GMV}}{n}}\right],$$

Hence, for all $r_f \notin I_{1-\alpha}$ *we conclude that the TP belongs to the efficient frontier and for all* $r_f \notin \tilde{I}_{1-\alpha}$ *the TP lies on the lower part of the set of feasible portfolios.*

3 Out-of-sample performance

In this section we investigate the behaviour of the realized expected return of the GMVP in the period $n+1$ given by $\hat{R}_{GMV,n+1} = \hat{\mathbf{w}}'_{GMV}\mathbf{X}_{n+1}$ where $\mathbf{X}_{n+1}$ is the vector of asset returns at time point $n+1$ and $\hat{\mathbf{w}}_{GMV} = \hat{\mathbf{\Sigma}}^{-1}\mathbf{1}/(\mathbf{1}'\hat{\mathbf{\Sigma}}^{-1}\mathbf{1})$ are the estimated weights of the GMVP by using asset returns $\mathbf{X}_1, ..., \mathbf{X}_n$. The aim is to provide statements about the two conditional probabilities:

$$P_1 = \mathbb{P}\left(\hat{R}_{GMV,n+1} > r_f | \hat{R}_{GMV} > r_f\right) \tag{11}$$

and

$$P_2 = \mathbb{P}\left(\hat{R}_{GMV,n+1} > r_f | T > t_{n-k,1-\alpha}\right) \tag{12}$$

While the probability in (11) can be considered as a naive approach about forecasting the efficiency of the TP at time point $t+1$ given that the estimated expected return of the GMVP is larger than the return of the risk-free asset, the second probability provides a similar statement which is based on the result of the statistical test developed in Section 2.

In order to determine the conditional probabilities in (11) and (12), we first derive the joint distributions $(\hat{R}_{GMV,n+1}, \hat{R}_{GMV})$ and $(\hat{R}_{GMV,n+1}, T)$ in Theorem 1 presented in terms of their stochastic representations which is a very popular tool in computational statistics (Givens and Hoeting (2012)), frequentist statistics (Gupta et al. (2013)) and Bayesian statistics (Bodnar et al. (2017a)). Let the symbol $\overset{d}{=}$ denote equality in distribution. Then we get the following results.

Theorem 1. *Let* $\mathbf{X}_1, \ldots, \mathbf{X}_n$ *be independent random vectors of asset returns with* $\mathbf{X}_t \sim N_k(\boldsymbol{\mu}, \mathbf{\Sigma})$ *for* $t = 1, \ldots, n$. *Assume that* $\mathbf{\Sigma}$ *is positive definite and* $n > k$. *Then:*

(a) the stochastic representation for $(\hat{R}_{GMV}, \hat{R}_{GMV,n+1})$ *is given by*

$$\hat{R}_{GMV} \overset{d}{=} R_{GMV} + \frac{\sqrt{V_{GMV}}}{\sqrt{n}}z_4 + \sqrt{\frac{1}{n}\xi_3 + \frac{1}{n}\left(\sqrt{ns} + z_5\right)^2}\sqrt{V_{GMV}}\frac{z_1}{\sqrt{\xi_1}} \tag{13}$$

and

$$\hat{R}_{GMV,n+1} \stackrel{d}{=} R_{GMV} + \sqrt{V_{GMV}} z_6 + \sqrt{V_{GMV}} \left(\frac{\sqrt{s}(\sqrt{ns} + z_5)}{\sqrt{\xi_3 + (\sqrt{ns} + z_5)^2}} + z_7 \right) \frac{z_1}{\sqrt{\xi_1}}$$
$$+ \sqrt{V_{GMV}} \sqrt{\xi_4} \left(\frac{z_3}{\sqrt{\xi_2}} \frac{z_1}{\sqrt{\xi_1}} + \frac{z_2}{\sqrt{\xi_2}} \right) \tag{14}$$

where $z_1, z_2, z_3, z_4, z_5, z_6, z_7 \sim \mathcal{N}(0,1)$, $\xi_1 \sim \chi^2_{n-k+1}$, $\xi_2 \sim \chi^2_{n-k+2}$, $\xi_3 \sim \chi^2_{k-2}$, $\xi_4 | z_5, \xi_3 \sim \chi^2_{k-2;\delta^2(s,\xi_3,z_5)}$ *with* $\delta^2(s, \xi_3, z_5) = \frac{s\xi_3}{\xi_3 + \left(\sqrt{ns}+z_5\right)^2}$*;* $z_1, z_2, z_3, z_4, z_6, z_7, \xi_1, \xi_2, (z_5, \xi_3, \xi_4)$ *are mutually independent.*

(b) the stochastic representation for $(T, \hat{R}_{GMV,n+1})$ *is given by* (14) *and*

$$T \stackrel{d}{=} \frac{\sqrt{n-k}}{\sqrt{\xi_5}} \frac{1}{\sqrt{1 + \frac{\xi_3 + \left(\sqrt{ns}+z_5\right)^2}{\xi_1}}} \left(\sqrt{n} \frac{R_{GMV} - r_f}{\sqrt{V_{GMV}}} + z_4 + \sqrt{\frac{\xi_3 + \left(\sqrt{ns}+z_5\right)^2}{\xi_1}} z_1 \right) \tag{15}$$

where $\xi_5 \sim \chi^2_{n-k}$ *independent of* $z_1, z_2, z_3, z_4, z_6, z_7, \xi_1, \xi_2, (z_5, \xi_3, \xi_4)$.

The proof of Theorem 1 is given in the appendix. The stochastic representations of Theorem 1 appear to be a very useful tool to investigate the distributional properties of $(\hat{R}_{GMV}, \hat{R}_{GMV,n+1})$ as well as of $(T, \hat{R}_{GMV,n+1})$. Moreover, they show that the distributions of $(\hat{R}_{GMV}, \hat{R}_{GMV,n+1})$ and of $(T, \hat{R}_{GMV,n+1})$ depend on $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ only through the three parameters of the efficient frontier (R_{GMV}, V_{GMV}, s).

An important application of the stochastic representation for $(\hat{R}_{GMV}, \hat{R}_{GMV,n+1})$ and of the stochastic representation for $(T, \hat{R}_{GMV,n+1})$ is that they allow for computation of the conditional probabilities P_1 and P_2 from (11) and (12) in a simple and efficient way. It is remarkable that a high numerical precision of the approximations of the conditional probabilities can be obtained by increasing the size of the drawn samples.

In the case of $(\hat{R}_{GMV}, \hat{R}_{GMV,n+1})$, the following algorithm can be used to evaluate P_1:

Algorithm 1 : Computing P_1 from (11)

(i) fix the values of r_f and (R_{GMV}, V_{GMV}, s);

(ii) generate independently $z_1^b, z_2^b, z_3^b, z_4^b, z_5^b, z_6^b, z_7^b \sim \mathcal{N}(0,1)$, $\xi_1^b \sim \chi^2_{n-k+1}$, $\xi_2^b \sim \chi^2_{n-k+2}$, $\xi_3^b \sim \chi^2_{k-2}$;

(iii) generate $\xi_4^b \sim \chi^2_{k-2;\delta^2(s,\xi_3^b,z_5^b)}$ with $\delta^2(s,\xi_3^b,z_5^b) = \frac{s\xi_3^b}{\xi_3^b+\left(\sqrt{ns}+z_5^b\right)^2}$;

(iv) compute $(\hat{R}^b_{GMV}, \hat{R}^b_{GMV,n+1})$ as in (13) and (14) by using $z_1^b, z_2^b, z_3^b, z_4^b, z_5^b, z_6^b, z_7^b$, $\xi_1^b, \xi_2^b, \xi_3^b, \xi_4^b$;

(v) determine

$$c_1^b = \mathbb{1}_{\{\hat{R}^b_{GMV} > r_f, \hat{R}^b_{GMV,n+1} > r_f\}} \quad \text{and} \quad c_2^b = \mathbb{1}_{\{\hat{R}^b_{GMV} > r_f\}},$$

where $\mathbb{1}_{\{\mathcal{A}\}}$ is the indicator function of set $\mathcal{A}$;

(vi) repeat steps (i)-(v) for $b = 1, ..., B$ and approximate P_1 by

$$\hat{P}_1 = \frac{\sum_{b=1}^B c_1^b}{\sum_{b=1}^B c_2^b}$$

For $(T, \hat{R}_{GMV,n+1})$, the above algorithm is slightly modified and it is given by

Algorithm 2 : Computing P_2 from (12)

(i) fix the values of r_f and (R_{GMV}, V_{GMV}, s);

(ii) generate independently $z_1^b, z_2^b, z_3^b, z_4^b, z_5^b, z_6^b, z_7^b \sim \mathcal{N}(0,1)$, $\xi_1^b \sim \chi^2_{n-k+1}$, $\xi_2^b \sim \chi^2_{n-k+2}$, $\xi_3^b \sim \chi^2_{k-2}$, $\xi_5^b \sim \chi^2_{n-k}$;

(iii) generate $\xi_4^b \sim \chi^2_{k-2;\delta^2(s,\xi_3^b,z_5^b)}$ with $\delta^2(s,\xi_3^b,z_5^b) = \frac{s\xi_3^b}{\xi_3^b + \left(\sqrt{ns}+z_5^b\right)^2}$;

(iv) compute $(T^b, \hat{R}^b_{GMV,n+1})$ as in (13) and (14) by using $z_1^b, z_2^b, z_3^b, z_4^b, z_5^b, z_6^b, z_7^b$, $\xi_1^b, \xi_2^b, \xi_3^b, \xi_4^b$;

(v) determine

$$c_1^b = \mathbb{1}_{\{T^b > t_{n-k,1-\alpha}, \hat{R}^b_{GMV,n+1} > r_f\}} \text{ and } c_2^b = \mathbb{1}_{\{T^b > t_{n-k,1-\alpha}\}};$$

(vi) repeat steps (i)-(v) for $b = 1, ..., B$ and approximate P_2 by

$$\hat{P}_2 = \frac{\sum_{b=1}^B c_1^b}{\sum_{b=1}^B c_2^b}$$

In Figure 4 we present the approximated conditional probabilities $\hat{P}_1$ and $\hat{P}_2$ as a function of $R_{GMV} - r_f$ for $r_f = 0.001$, $V_{GMV} = 0.001$, and $s = 0.22$. The values of r_f, V_{GMV}, and s corresponds to the considered data sets of the empirical illustration of Section 5.1 in Bodnar and Schmid (2009). We also put $n = 50$ (Figure 4) and consider $k \in \{5, 10, 15, 20\}$. We observe that the probability $\hat{P}_2$ is always larger than $\hat{P}_1$ and, consequently, the realized expected return of the GMVP at time $(n+1)$ is larger than the risk-free rate with a higher probability when the decision about this investment opportunity is based on the test (5). Furthermore, we note that the distance between the two curves in the figures is larger for smaller values of $R_{GMV} - r_f$ and for larger values of k.

4 Robustness to the assumption of normality

In this section we investigate the robustness of the test procedure presented in Section 2 when the assumption of normality is violated. The empirical power of the test is computed via simulations by generating samples from the multivariate normal distribution and the standardized multivariate t-distribution with 5 and 10 degrees of freedom, where the standardization of the t-distribution is done in order to have samples with the same mean vector and covariance matrices. Recall that as a result of Proposition 2 it is seen that the power function of the test depends on the mean vector and the covariance matrix only through the slope parameter s of the efficient frontier and the Sharpe ratio S_{GMV} of the

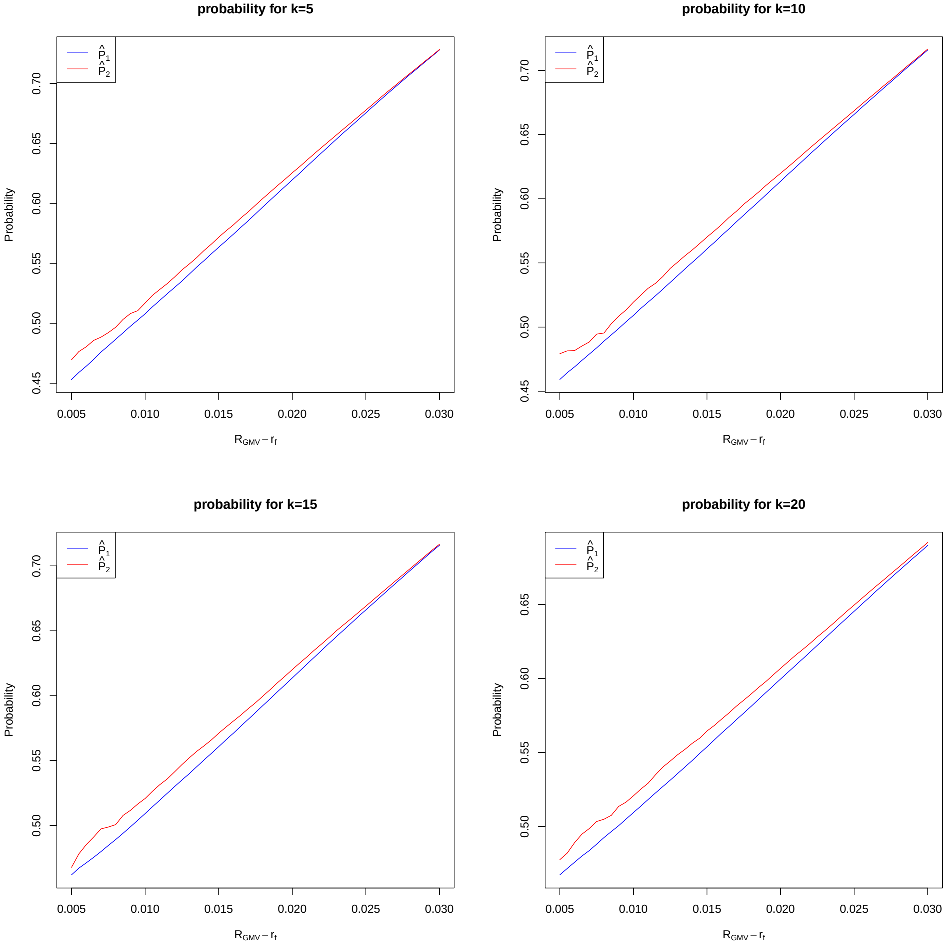


Figure 4: Probabilities $\hat{P}_1$ and $\hat{P}_2$ for portfolio dimension $k \in \{5, 10, 15, 20\}$ and sample size $n = 50$

GMVP. Due to this, we set $\boldsymbol{\Sigma} = \boldsymbol{I}_k$, an identity matrix of appropriate dimension k in the simulation study, and consider several values of $\boldsymbol{\mu}$ given by[2]

- $\boldsymbol{\mu}_1 = (0.1, 0, \ldots, 0)'$;
- $\boldsymbol{\mu}_2 = (0.1, 0.1, 0, \ldots, 0)'$;
- $\boldsymbol{\mu}_3 = (0.1, 0.1, 0.1, 0 \ldots, 0)'$;
- $\boldsymbol{\mu}_4 = (0.1, 0.1, 0.1, 0.1, 0, \ldots, 0)'$;
- $\boldsymbol{\mu}_5 = (0.1, 0.1, 0.1, 0.1, 0.1, 0, \ldots, 0)'$.

The resulting values of s and S_{GMV} are summarized in Table 1. The values with $S_{GMV} \leq 0$ corresponds to the null hypothesis in (5), while $S_{GMV} > 0$ favours the alternative hypothesis. For the cases $S_{GMV} = 0$ we expected the empirical significance level of the test obtained via simulations to be at the nominal significance level $\alpha = 0.05$. Further, the risk-free rate is set to be equal to 0.01 and the portfolio size is $k \in \{5, 10, 15, 20\}$. Moreover, we observe that the slope parameter s becomes larger as k increases, while the Sharpe ratio S_{GMV} increases when the number of non-zero elements in the mean vector becomes larger.

k	s&S_{GMV}	$\boldsymbol{\mu}_1$	$\boldsymbol{\mu}_2$	$\boldsymbol{\mu}_3$	$\boldsymbol{\mu}_4$	$\boldsymbol{\mu}_5$
	s	0.0080	0.0120	0.0120	0.0080	0.0000
5	S_{GMV}	0.0224	0.0671	0.1118	0.1565	0.2012
	s	0.0090	0.0160	0.0210	0.0240	0.0250
10	S_{GMV}	0.0000	0.0316	0.0632	0.0949	0.1265
	s	0.0093	0.0173	0.0240	0.0293	0.0333
15	S_{GMV}	-0.0129	0.0129	0.0387	0.0645	0.0904
	s	0.0095	0.018	0.0255	0.0320	0.0375
20	S_{GMV}	-0.0224	0.0000	0.0224	0.0447	0.0671

Table 1: Slope parameter s and Sharpe ratio S_{GMV} for the portfolio dimension $k \in \{5, 10, 15, 20\}$ and several values of $\boldsymbol{\mu}$

In Tables 2, 3 and 4 the results of the simulation study are presented for $k \in \{5, 10, 15, 20\}$ and $n \in \{50, 100, 250\}$. Each value of the power function presented in the tables is obtained by drawing $B = 10^6$ independent samples from the corresponding model. The simulation study suggests that even though data are generated using a heavy tailed t-distribution the tests are performing well. This observation remains true independently of the considered sample size n and portfolio dimension k. Furthermore, we observe that the power grows as the number of non-zero elements in the mean vector

[2]From Proposition 2 it is seen that the power of test (5) only depends on $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ through S_{GMV} and s, hence any choice of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ with the same values of S_{GMV} and s will not affect the power of the test if the asset returns are multivariate normally distributed.

becomes larger and decreases for larger values of k. The power is not larger than the nominal significance level of the test, namely 5% in all cases where S_{GMV} is non-positive and it is always larger than 5% for $S_{GMV} > 0$. This statements remains valid independently if data are generated from the normal distribution or from the t-distribution. Finally, we note that the empirical power obtained under the t-distribution is always smaller than the one obtained for the normal distribution and, thus, the test becomes slightly conservative when data are drawn from a heavy-tailed distribution, but it always keeps the nominal significance level.

k	Distribution	$\boldsymbol{\mu}_1$	$\boldsymbol{\mu}_2$	$\boldsymbol{\mu}_3$	$\boldsymbol{\mu}_4$	$\boldsymbol{\mu}_5$
	Normal	0.0669	0.1151	0.1833	0.2731	0.3827
5	t_5	0.0638	0.1061	0.1670	0.2473	0.3438
	t_{10}	0.0660	0.1110	0.1767	0.2622	0.3655
	Normal	0.0497	0.0738	0.1055	0.1456	0.1952
10	t_5	0.0464	0.0683	0.0952	0.1316	0.1752
	t_{10}	0.0485	0.0710	0.1012	0.1402	0.1866
	Normal	0.0426	0.0582	0.0781	0.1020	0.1310
15	t_5	0.0392	0.0526	0.0699	0.0912	0.1163
	t_{10}	0.0414	0.0561	0.0745	0.0969	0.1244
	Normal	0.0389	0.0497	0.0634	0.0801	0.0991
20	t_5	0.0348	0.0447	0.0567	0.0707	0.0873
	t_{10}	0.0371	0.0480	0.0607	0.0762	0.0941

Table 2: Power function for the portfolio dimension $k \in \{5, 10, 15, 20\}$ and the sample size $n = 50$. The nominal significance level of the test is $\alpha = 0.05$.

k	Distribution	$\boldsymbol{\mu}_1$	$\boldsymbol{\mu}_2$	$\boldsymbol{\mu}_3$	$\boldsymbol{\mu}_4$	$\boldsymbol{\mu}_5$
	Normal	0.0771	0.1592	0.2869	0.4492	0.6234
5	t_5	0.0715	0.1443	0.2546	0.4004	0.5631
	t_{10}	0.0744	0.1533	0.2733	0.4285	0.5985
	Normal	0.0500	0.0889	0.1464	0.2239	0.3220
10	t_3	0.0463	0.0801	0.1301	0.1976	0.2833
	t_5	0.0484	0.0856	0.1395	0.2129	0.3058
	Normal	0.0388	0.0636	0.0981	0.1447	0.2041
15	t_5	0.0352	0.0563	0.0864	0.1261	0.1772
	t_{10}	0.0374	0.0609	0.0934	0.1372	0.1928
	Normal	0.0327	0.0499	0.0735	0.1053	0.1453
20	t_5	0.0293	0.0442	0.0646	0.0909	0.1241
	t_{10}	0.0312	0.0479	0.0700	0.0990	0.1360

Table 3: Power function for the portfolio dimension $k \in \{5, 10, 15, 20\}$ and the sample size $n = 100$. The nominal significance level of the test is $\alpha = 0.05$.

k	Distribution	$\boldsymbol{\mu}_1$	$\boldsymbol{\mu}_2$	$\boldsymbol{\mu}_3$	$\boldsymbol{\mu}_4$	$\boldsymbol{\mu}_5$
	Normal	0.0966	0.2735	0.5372	0.7859	0.9339
5	t_5	0.0877	0.2394	0.4732	0.7204	0.8921
	t_{10}	0.0939	0.2596	0.5114	0.7608	0.9185
	Normal	0.0502	0.1231	0.2496	0.4240	0.6136
10	t_5	0.0454	0.1075	0.2145	0.3650	0.5412
	t_{10}	0.0482	0.1167	0.2346	0.3988	0.5837
	Normal	0.0326	0.0736	0.1445	0.2517	0.3892
15	t_5	0.0295	0.0639	0.1234	0.2119	0.3295
	t_{10}	0.0316	0.0700	0.1356	0.2345	0.3637
	Normal	0.0236	0.0498	0.0945	0.1643	0.2587
20	t_5	0.0211	0.0428	0.0797	0.1360	0.2145
	t_{10}	0.0226	0.0471	0.0881	0.1530	0.2401

Table 4: Power function for the portfolio dimension $k \in \{5, 10, 15, 20\}$ and the sample size $n = 250$. The nominal significance level of the test is $\alpha = 0.05$.

5 Empirical Study

In order to get a better understanding of the findings obtained in the previous sections, we apply the derived theoretical results to real data. Weekly returns on 29 stocks listed on Dow Jones Industrial (DJI) index are considered for the period from 0.1.01.2006 to 31.12.2015.[3] The 13 weeks US treasury bill covering the aforementioned period is considered as a risk-free asset. The results are obtained for different portfolio dimension $k \in \{5, 10, 15, 20\}$ and sample size $n \in \{50, 100, 250\}$. The chosen values of n roughly correspond to one year, two years, and five years of weekly data.

5.1 Empirical distribution of p-values

In order to provide some general statements about the location of the TP on the efficient frontier independently of the chosen stocks, we perform the test (5) for 1000 randomly selected sets of stocks listed in the DJI index for each $k \in \{5, 10, 15, 20\}$ and $n \in \{50, 100, 250\}$. Namely, for all pairs of k and n we choose randomly k stocks listed in DJI and their n most recent returns. Then, using these data we perform the test on the hypothesis (5) and calculate the corresponding p-value. The procedure is repeated 1000 times resulting in a sample of p-values calculated from different sets of stocks with fixed k and n. From these samples the histograms are constructed which are shown in Figure 5 for $n = 50$, in Figure 6 for $n = 100$, and in Figure 7 for $n = 250$.

We observe that the number of rejection of the null hypothesis depends crucially on the

[3]In comparison to daily returns Fama (1976) showed that the distribution of monthly returns is approximately normal. On the other hand, the application of monthly data may result to the bias due to time-varying dynamics in model parameters. For this reason, weekly returns are used as a trade-off between daily and monthly returns.

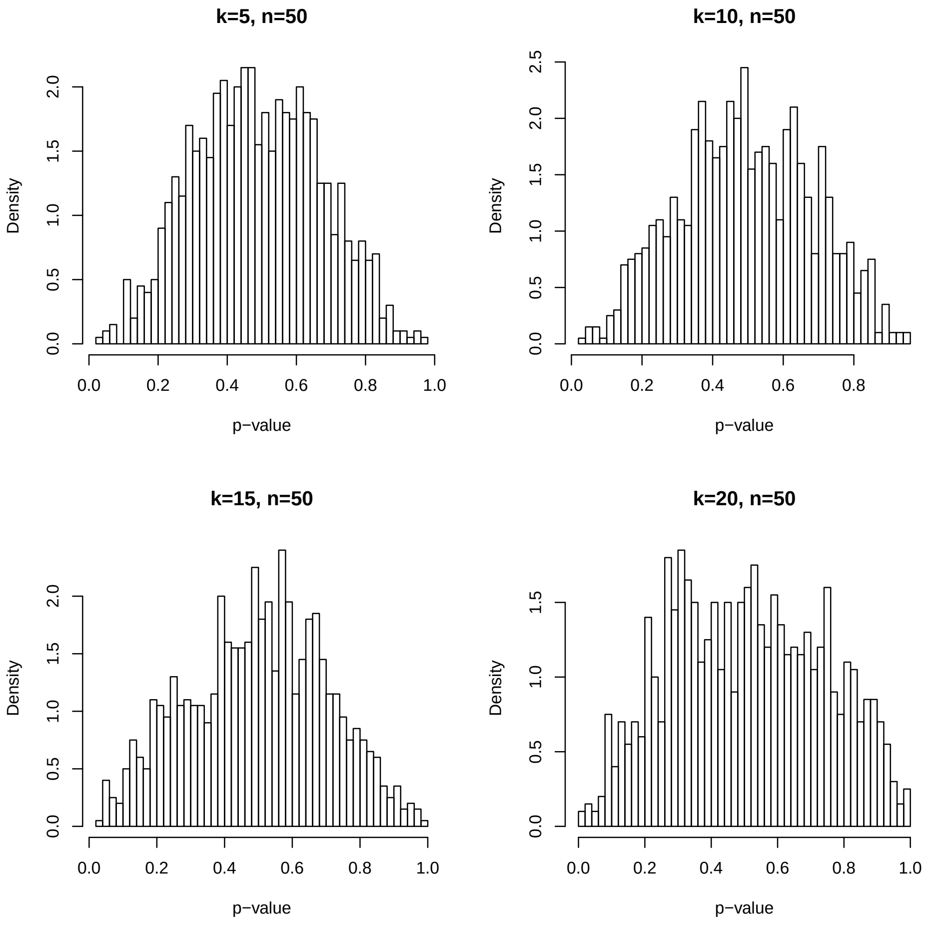


Figure 5: Histograms of p-values for 1000 randomly sampled sets of stocks listed in the DJI index in the case of $k = 5$ (top left), $k = 10$ (top right), $k = 15$ (bottom left), and $k = 20$ (bottom right). For each chosen set of stocks $n = 50$ most recent returns are used.

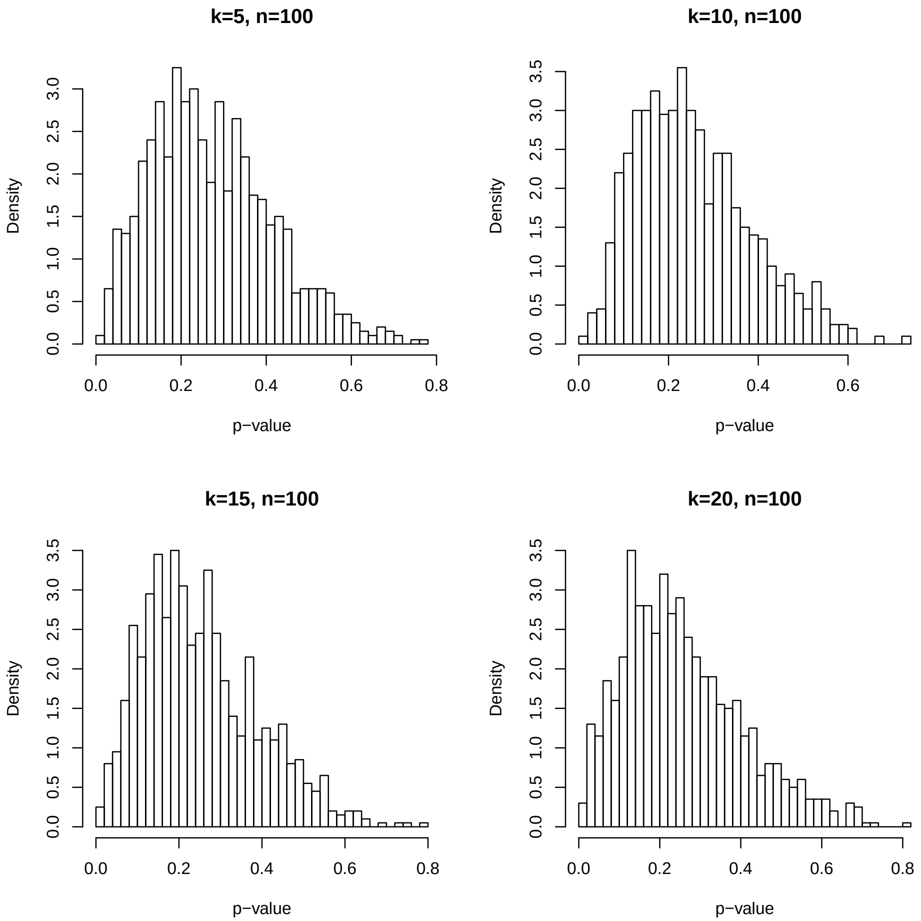


Figure 6: Histograms of p-values for 1000 randomly sampled sets of stocks listed in the DJI index in the case of $k = 5$ (top left), $k = 10$ (top right), $k = 15$ (bottom left), and $k = 20$ (bottom right). For each chosen set of stocks $n = 100$ most recent returns are used.

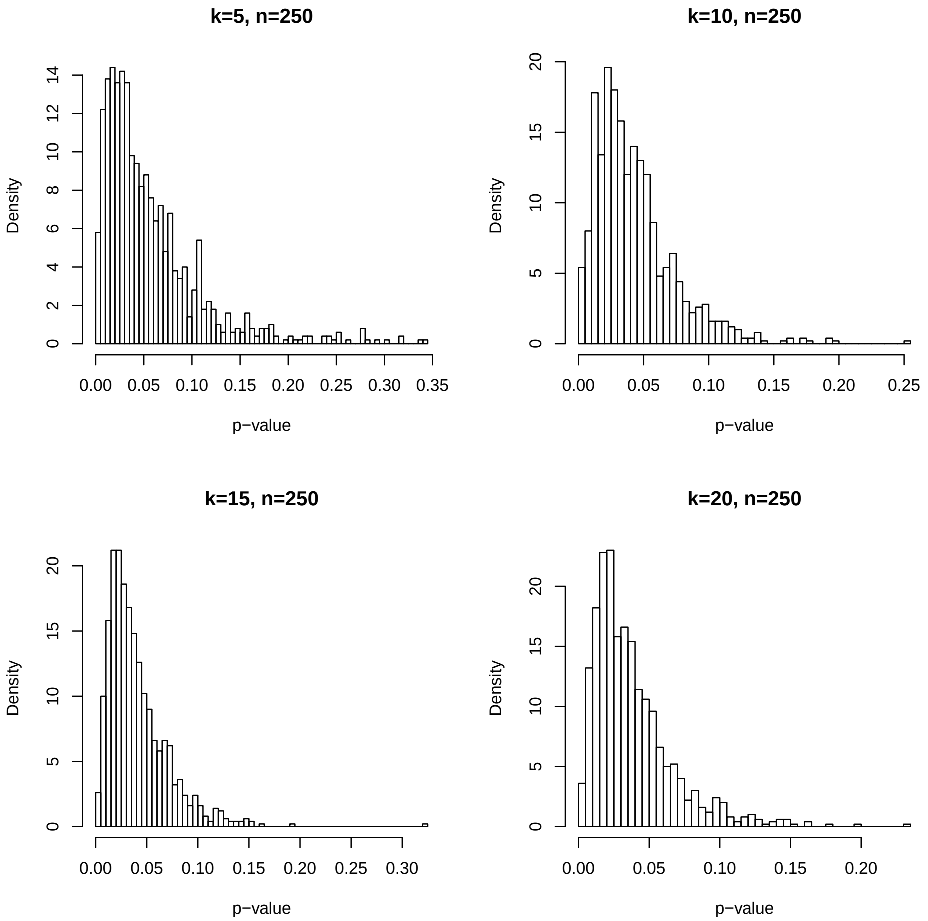


Figure 7: Histograms of p-values for 1000 randomly sampled sets of stocks listed in the DJI index in the case of $k = 5$ (top left), $k = 10$ (top right), $k = 15$ (bottom left), and $k = 20$ (bottom right). For each chosen set of stocks $n = 250$ most recent returns are used.

sample size. For $n = 50$, we are not able to reject the null hypothesis at 5% significance level in most of the considered cases. That is, it is not possible to conclude that the TP is a suitable alternative to both the GMVP and the investment into the risk-free asset as it might be located on the lower part of the feasible set of optimal portfolio. However, when n increases, the p-values become smaller and, in particular, they are almost all below 10% for $n = 250$. Table 5 provides further insight into the behavior of the p-values. Here, the number of rejections of the null hypothesis (5) for the significance levels of 1%, 5%, and 10% are present. The number of rejections dramatically increases when n becomes larger. Also, we observe an increase when k is larger. To this end, we conclude that the decision about the location of the tangency portfolio on the feasible set of portfolios depends crucially on the amount of information used to make a decision. If the sample size is small, then the test (5) is not powerful enough to reject the null hypothesis and to be able to draw a conclusion about investing in the TP. This finding is in line with the results of the simulation study presented in Section 4.

k/n	5			10			15			20		
	0.01	0.05	0.1	0.01	0.05	0.1	0.01	0.05	0.1	0.01	0.05	0.1
50	0	2	6	0	2	8	0	6	18	1	7	26
100	0	25	98	0	13	89	2	29	123	3	44	124
250	90	575	846	67	685	946	63	719	956	84	753	957

Table 5: Number of rejections of the null hypothesis in (5) for 1000 randomly sampled sets of stocks listed in the DJI index in the case of $k \in \{5, 10, 15, 20\}$ and $n \in \{50, 100, 250\}$. The significance level of the test is set to $\alpha \in \{0.01, 0.05, 0.1\}$

5.2 Time series behavior of the p-values

In order to investigate the performance of the suggested test on the location of the tangency portfolio at several time points, we apply the rolling window estimation (testing) technique with sample size (window length) of $n \in \{50, 100, 250\}$. In all cases we choose $k = \{5, 10, 15, 20\}$ stocks listed in the DJI index following their alphabetical order.

In Figure 8 we present the values of the Sharpe ratio calculated for the estimated GMVP. A very volatile behavior is present, especially when the window length is small. If k increases, then the values of the calculated Sharpe ratio become larger showing a positive effect of diversification, a well-known result in portfolio theory. Finally, larger values of the Sharpe ratio are present at the end of the considered time period leading to the conclusion that the capital market recovers after the financial crisis in 2008, while negative values of the Sharpe ratio are present around the period of the financial crisis. Finally, we point out, that larger values red of the Sharpe ratio can be obtained for smaller sample sizes when most recent data are used in the construction of the GMVP. However, in this case we also see more volatile behaviour of the estimated characteristics of the GMVP which leads to higher risk.

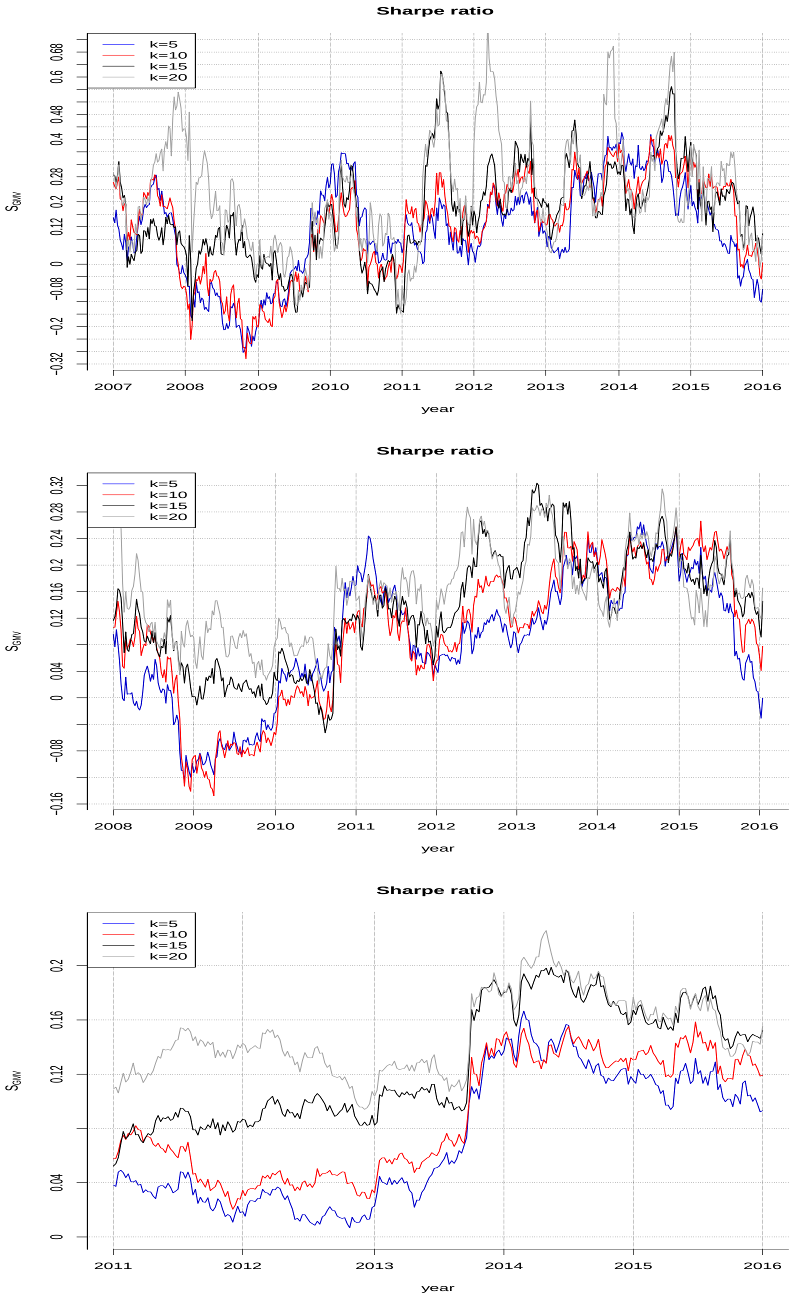


Figure 8: Empirical Sharpe ratio calculated for the GMVP constructed by using first (in alphabet order) $k \in \{5, 10, 15, 20\}$ stocks listed in the DJI index. The size of the rolling window is $n = 50$ (upper), $n = 100$ (middle), and $n = 250$ (button).

In Figures 9, 10, and 11, the p-values (blue lines) are shown for the test (5) in the case of $k \in \{5, 10, 15, 20\}$ and $n \in \{50, 100, 250\}$. In addition, we also present the p-values (red line) of the test (10) (see Remark 1 in Section 2) where we test if the TP lies on the lower part of the efficient frontier under H_1, i.e. we check if the TP is not mean-variance efficient. Similarly to the Sharpe ratio, the p-values show high fluctuation over time when using smaller sample size, while they are quite stable for larger sample sizes. For $n = 50$ the p-values of both tests are larger than the nominal significance level of 5% and, hence, no decision about the investment into the TP could be done since both null hypotheses cannot be rejected. This point is fully related to the power properties of the tests, i.e. the window length is too small for drawing a conclusive decision. By increasing the value of n, the situation improves and we may draw conclusions concerning the mean-variance efficiency of the TP in almost the whole period starting at the end of 2013 for $n = 250$. In contrast, the decision about the inefficiency of the TP can be drawn at the end of 2008 when $n = 50$ and $k \in \{5, 10\}$. We also note that in all cases where the empirical Sharpe ratio is negative, we are not able to reject the null hypothesis of the test (5).

Finally, we present the values of the conditional probabilities $\hat{P}_1$ and $\hat{P}_2$ in Table 6 which are defined in Section 3 as the probabilities that the realized expected return of the GMVP is larger than the risk-free rate in the consequent period provided that the estimated expected return of this portfolio is larger than the risk-free rate (for $\hat{P}_1$) or the test (5) at significance level 5% rejects the null hypothesis (for $\hat{P}_2$). Note that the number of cases used in the computation of $\hat{P}_1$ and $\hat{P}_2$ depends on the occurrence of the events $\{\hat{R}_{GMV} > r_f\}$ and $\{T > t_{n-k,1-\alpha}\}$, respectively. The number of rejections of the null hypothesis by test (5) are summarized in Table 6. In these cases, $\hat{P}_2$ were computed, while slightly larger samples were used for the calculation of $\hat{P}_1$. Table 6 documents that $\hat{P}_2$ outperforms $\hat{P}_1$ for $n = 50$ and $n = 100$, while they have the same performance for $n = 250$. Hence, the best strategy to forecast the efficiency of the TP is to use the statistical approach developed in Section 2. Furthermore, the results of Table 6 are in line with the findings of the simulation study of Section 3 where similar performance is documented.

k/n	5			10			15			20		
	Rej	$\hat{P}_1$	$\hat{P}_2$	Rej	$\hat{P}_1$	$\hat{P}_2$	Rej	$\hat{P}_1$	$\hat{P}_2$	Rej	$\hat{P}_1$	$\hat{P}_2$
50	87	0.9861	1	61	0.9744	1	70	0.9702	1	83	0.9862	1
100	96	0.9853	1	99	0.9823	1	124	0.9875	1	103	1	1
250	107	1	1	118	1	1	130	1	1	248	1	1

Table 6: Empirical probabilities $\hat{P}_1$ and $\hat{P}_2$ of the realized return of the GMVP to be positive calculated for the first $k \in \{5, 10, 15, 20\}$ stocks listed in the DJI index in the alphabetical order. Rolling window estimation is used with the window length equal to $n \in \{50, 100, 250\}$. The nominal significance level of the test (5) used in the calculations of $\hat{P}_2$ is $\alpha = 0.05$.

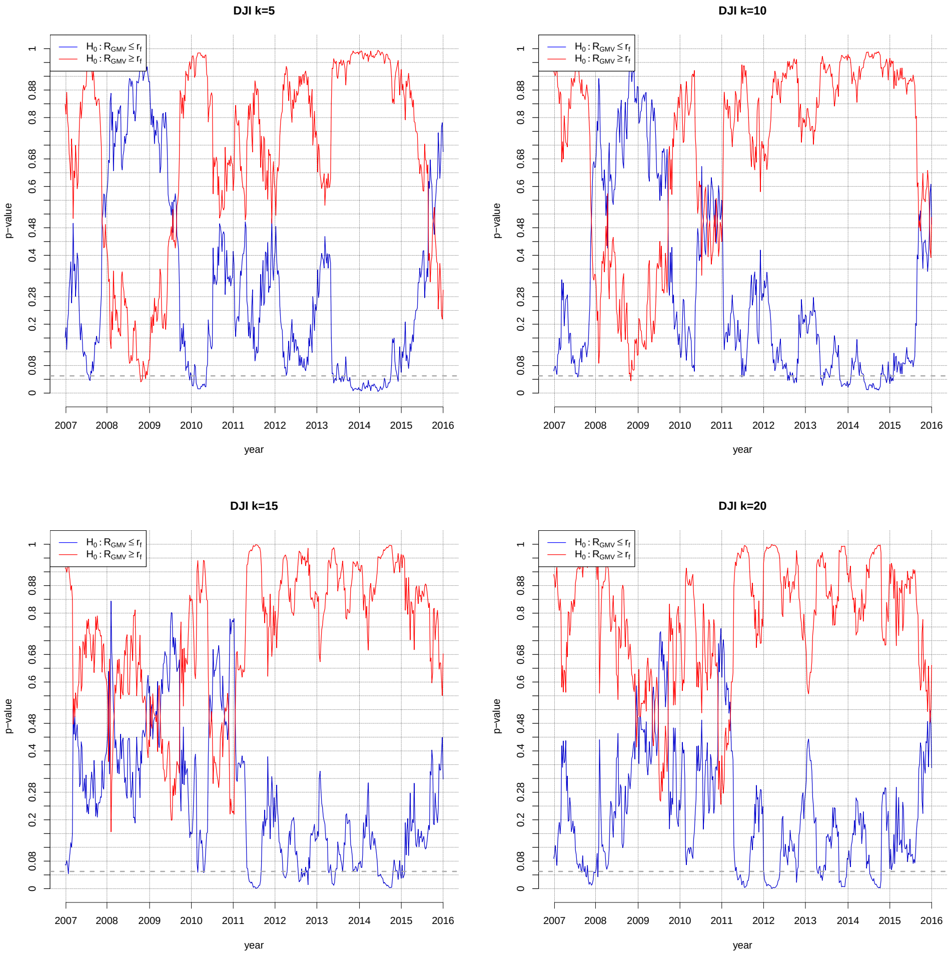


Figure 9: p-values calculated for the test (5) (blue line) and for the test (10) (red line) for the first $k \in \{5, 10, 15, 20\}$ stocks listed in the DJI index in the alphabetical order. Rolling window estimation is used with the window length equal to $n = 50$.

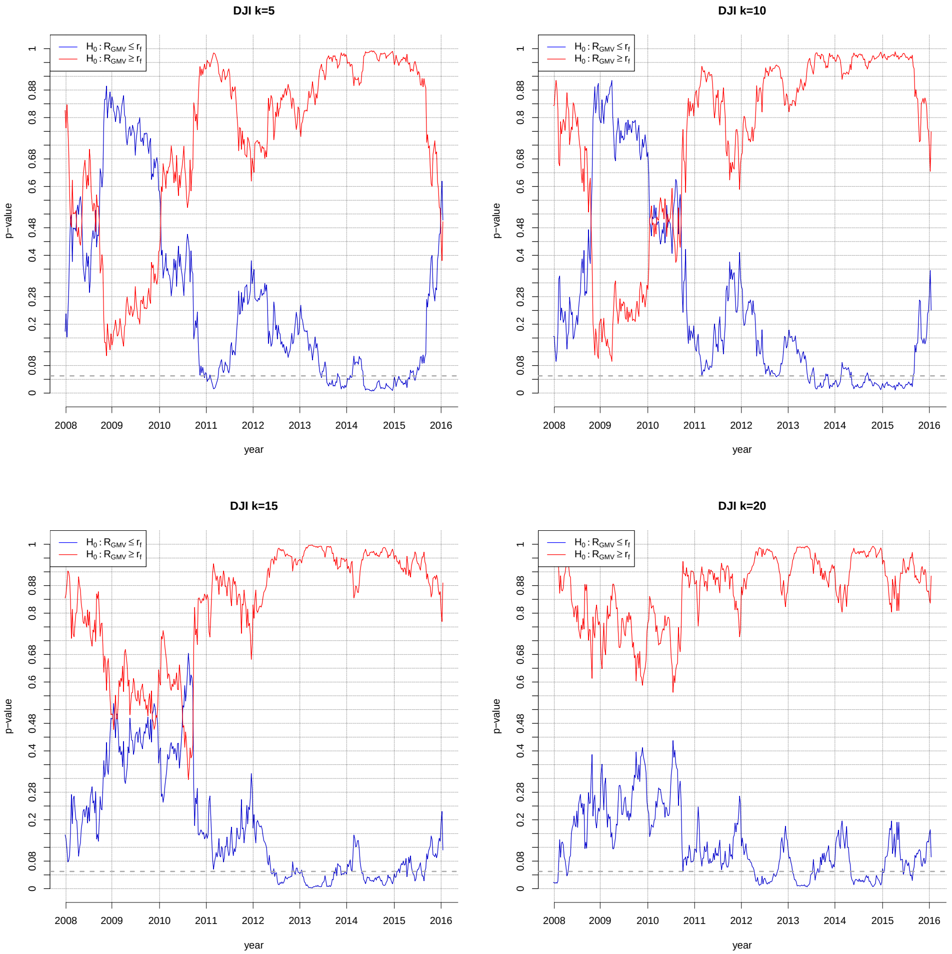


Figure 10: p-values calculated for the test (5) (blue line) and for the test (10) (red line) for the first $k \in \{5, 10, 15, 20\}$ stocks listed in the DJI index in the alphabetical order. Rolling window estimation is used with the window length equal to $n = 100$.

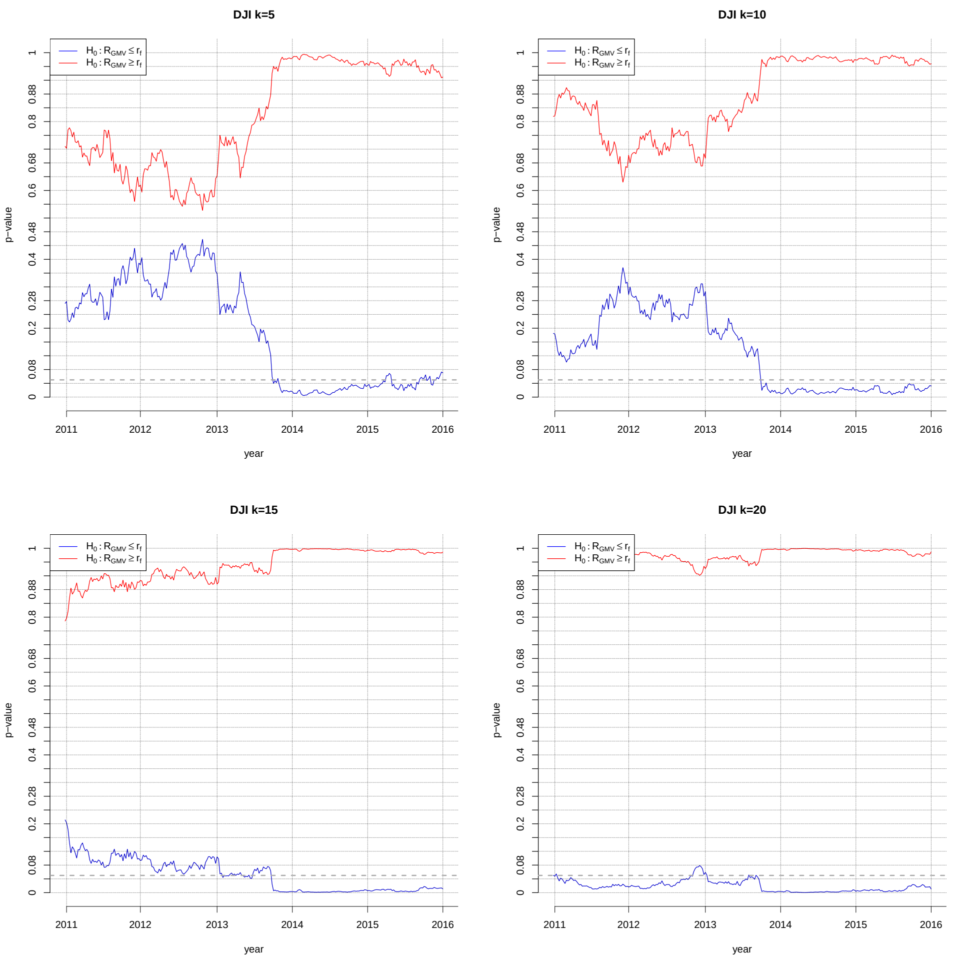


Figure 11: p-values calculated for the test (5) (blue line) and for the test (10) (red line) for the first $k \in \{5, 10, 15, 20\}$ stocks listed in the DJI index in the alphabetical order. Rolling window estimation is used with the window length equal to $n = 250$.

6 Summary

The tangency portfolio plays an important role in the financial literature and is usually used as a market portfolio in the capital asset pricing model. However, due to the way how the TP is constructed together with the large amount of uncertainty that is present in financial markets, the TP might not be mean-variance efficient at all. Although a number of studies is devoted to the estimation of the TP weights and investigating the distributional properties of the tangency portfolio (see, Ingersoll (1987); Britten-Jones (1999); Okhrin and Schmid (2006); Schmid and Zabolotskyy (2008); Bodnar and Zabolotskyy (2017)), the problem of the location of the TP on the set of feasible portfolios has not been treated in the literature to the best of our knowledge.

In this paper we introduce a finite-sample test on the mean-variance efficiency of the tangency portfolio. The distribution of the test statistic is also derived under both hypotheses. Further, it is shown that the suggested test is easily performed in practice by comparing the value of the test statistic with the quantile of a t-distribution. Moreover, the result under the alternative hypothesis is used to investigate the test power. Within an extensive simulation study, we show that the new test is robust to the violation of the normality assumption and can also be used for heavy-tailed stochastic models. Finally, the theoretical results are applied to recent data based on the returns on the stocks included into the DJI index. We conclude, empirically, that the TP is not mean-variance efficient during some parts of the financial crisis. On the other hand, we are not able to accept the efficiency of the TP when the sample size is small because of a large amount of uncertainty present in the financial markets. However, if the sample size is relatively large and a stable period is present on market, then the mean-variance efficiency of the TP can be statistically justified.

7 Appendix

Proof of Proposition 2. For a given constant c, we get that

$$
\begin{aligned}
&G_{T,\alpha}(S_{GMV}, s) = \mathbb{P}(T > c) = \int_c^\infty f_T(x)dx \\
&= \frac{n(n-k+1)}{(k-1)(n-1)} \int_c^\infty \int_0^\infty f_{t_{n-k,\delta(y)}}(x) f_{F_{k-1,n-k+1,ns}}\left(\frac{n(n-k+1)}{(k-1)(n-1)}y\right) dydx \\
&= \frac{n(n-k+1)}{(k-1)(n-1)} \int_0^\infty \left(\int_c^\infty f_{t_{n-k,\delta(y)}}(x)dx\right) f_{F_{k-1,n-k+1,ns}}\left(\frac{n(n-k+1)}{(k-1)(n-1)}y\right) dy \\
&= \frac{n(n-k+1)}{(k-1)(n-1)} \int_0^\infty \left(1 - F_{t_{n-k,\delta(y)}}(c)\right) f_{F_{k-1,n-k+1,ns}}\left(\frac{n(n-k+1)}{(k-1)(n-1)}y\right) dy.
\end{aligned}
$$

In using that $1 - F_{t_{n-k,\delta(y)}}(c) > 1 - F_{t_{n-k,0}}(c)$ for all $y \geq 0$ and $R_{GMV} < r_f$, we get

$$\begin{aligned}
G_{T,\alpha}(S_{GMV}, s) &\leq \frac{n(n-k+1)}{(k-1)(n-1)} \int_0^\infty \left(1 - F_{t_{n-k,0}}(c)\right) f_{F_{k-1,n-k+1,ns}}\left(\frac{n(n-k+1)}{(k-1)(n-1)} y\right) dy \\
&= \left(1 - F_{t_{n-k,0}}(c)\right) \underbrace{\frac{n(n-k+1)}{(k-1)(n-1)} \int_0^\infty f_{F_{k-1,n-k+1,ns}}\left(\frac{n(n-k+1)}{(k-1)(n-1)} y\right) dy}_{1} \\
&= 1 - F_{t_{n-k}}(c) = \alpha.
\end{aligned}$$

with $c = t_{n-k,1-\alpha}$ where $t_{n-k,1-\alpha}$ denotes the $(1-\alpha)$ quantile of the t-distribution with $n-k$ degrees of freedom. $\square$

Proof of Theorem 1. From Theorem 3.1.2 and Corollary 3.2.2 in Muirhead (1982), we get $\hat{\boldsymbol{\mu}} \sim \mathcal{N}_k(\boldsymbol{\mu}, \boldsymbol{\Sigma}/n)$, $(n-1)\hat{\boldsymbol{\Sigma}} \sim \mathcal{W}_k(n-1, \boldsymbol{\Sigma})$ (k-dimensional Wishart distribution with $n-1$ degrees of freedom and the parameter matrix $\boldsymbol{\Sigma}$); $\hat{\boldsymbol{\mu}}$ and $\hat{\boldsymbol{\Sigma}}$ are independently distributed. Moreover, we get $\mathbf{X}_{n+1}$ is independent of both $\hat{\boldsymbol{\mu}}$ and $\hat{\boldsymbol{\Sigma}}$ by the assumptions of the theorem.

Let

$$\hat{\boldsymbol{\Omega}} = \begin{bmatrix} \hat{\boldsymbol{\mu}}' \\ \mathbf{X}'_{n+1} \\ \mathbf{1}'_k \end{bmatrix} \hat{\boldsymbol{\Sigma}}^{-1} \begin{bmatrix} \hat{\boldsymbol{\mu}} & \mathbf{X}_{n+1} & \mathbf{1}_k \end{bmatrix}$$

Since $\hat{\boldsymbol{\Sigma}}$ is independent of $\hat{\boldsymbol{\mu}}$ and $\mathbf{X}_{n+1}$, the conditional distribution of $\hat{\boldsymbol{\Omega}}$ given $\hat{\boldsymbol{\mu}} = \boldsymbol{\mu}_0$ and $\mathbf{X}_{n+1} = \mathbf{X}_0$ is equal to $\tilde{\boldsymbol{\Omega}}$ expressed as

$$\tilde{\boldsymbol{\Omega}} = \begin{bmatrix} \boldsymbol{\mu}'_0 \\ \mathbf{X}'_0 \\ \mathbf{1}'_k \end{bmatrix} \hat{\boldsymbol{\Sigma}}^{-1} \begin{bmatrix} \boldsymbol{\mu}_0 & \mathbf{X}_0 & \mathbf{1}_k \end{bmatrix} = \begin{bmatrix} \boldsymbol{\mu}'_0\hat{\boldsymbol{\Sigma}}^{-1}\boldsymbol{\mu}_0 & \boldsymbol{\mu}'_0\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{X}_0 & \boldsymbol{\mu}'_0\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}_k \\ \mathbf{X}'_0\hat{\boldsymbol{\Sigma}}^{-1}\boldsymbol{\mu}_0 & \mathbf{X}'_0\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{X}_0 & \mathbf{X}'_0\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}_k \\ \mathbf{1}'_k\hat{\boldsymbol{\Sigma}}^{-1}\boldsymbol{\mu}_0 & \mathbf{1}'_k\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{X}_0 & \mathbf{1}'_k\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}_k \end{bmatrix}$$

Defining

$$\boldsymbol{\Omega} = \begin{bmatrix} \boldsymbol{\mu}'_0\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_0 & \boldsymbol{\mu}'_0\boldsymbol{\Sigma}^{-1}\mathbf{X}_0 & \boldsymbol{\mu}'_0\boldsymbol{\Sigma}^{-1}\mathbf{1}_k \\ \mathbf{X}'_0\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_0 & \mathbf{X}'_0\boldsymbol{\Sigma}^{-1}\mathbf{X}_0 & \mathbf{X}'_0\boldsymbol{\Sigma}^{-1}\mathbf{1}_k \\ \mathbf{1}'_k\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_0 & \mathbf{1}'_k\boldsymbol{\Sigma}^{-1}\mathbf{X}_0 & \mathbf{1}'_k\boldsymbol{\Sigma}^{-1}\mathbf{1}_k \end{bmatrix}$$

and using Theorem 3.2.11 by Muirhead (1982), we get that $(n-1)^{-1}\tilde{\boldsymbol{\Omega}}^{-1} \sim \mathcal{W}_3(n-k+2, \boldsymbol{\Omega}^{-1})$. Hence, $(n-1)\tilde{\boldsymbol{\Omega}} \sim \mathcal{W}_3^{-1}(n-k+6, \boldsymbol{\Omega})$.

Let

$$s_0 = \boldsymbol{\mu}'_0\hat{\mathbf{R}}\boldsymbol{\mu}_0, \quad h_0 = \mathbf{X}'_0\hat{\mathbf{R}}\boldsymbol{\mu}_0, \quad v_0 = \mathbf{X}'_0\hat{\mathbf{R}}\mathbf{X}_0.$$

From Theorem 3.(b) in Bodnar and Okhrin (2008) we get

$$\left(\begin{array}{c}\frac{\boldsymbol{\mu}_0'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}_k}{\mathbf{1}_k'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}_k}\\ \frac{\mathbf{X}_0'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}_k}{\mathbf{1}_k'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}_k}\end{array}\right)\Bigg| s_0, h_0, v_0 \sim \mathcal{N}_2\left(\left(\begin{array}{c}\frac{\boldsymbol{\mu}_0'\boldsymbol{\Sigma}^{-1}\mathbf{1}_k}{\mathbf{1}_k'\boldsymbol{\Sigma}^{-1}\mathbf{1}_k}\\ \frac{\mathbf{X}_0'\boldsymbol{\Sigma}^{-1}\mathbf{1}_k}{\mathbf{1}_k'\boldsymbol{\Sigma}^{-1}\mathbf{1}_k}\end{array}\right), \frac{(n-1)^{-1}}{\mathbf{1}_k'\boldsymbol{\Sigma}^{-1}\mathbf{1}_k}\left(\begin{array}{cc} s_0 & h_0\\ h_0 & v_0\end{array}\right)\right), \tag{16}$$

where $\mathbf{1}_k'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}_k = \hat{V}_{GMV}^{-1}$ is independent of $\left(\frac{\boldsymbol{\mu}_0'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}_k}{\mathbf{1}_k'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}_k}, \frac{\mathbf{X}_0'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}_k}{\mathbf{1}_k'\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}_k}, s_0, h_0, v_0\right)$ and (see, e.g., Lemma A1 in Bodnar and Schmid (2009))

$$(n-1)\frac{\hat{V}_{GMV}}{V_{GMV}} \sim \chi^2_{n-k}. \tag{17}$$

Moreover, we get that (Theorem 3.(b) in Bodnar and Okhrin (2008))

$$(n-1)^{-1}\left(\begin{array}{cc} s_0 & h_0\\ h_0 & v_0\end{array}\right) \sim \mathcal{W}_2^{-1}\left(n-k+5, \left(\begin{array}{cc}\boldsymbol{\mu}_0'\mathbf{R}\boldsymbol{\mu}_0 & \boldsymbol{\mu}_0'\mathbf{R}\mathbf{X}_0\\ \mathbf{X}_0'\mathbf{R}\boldsymbol{\mu}_0 & \mathbf{X}_0'\mathbf{R}\mathbf{X}_0\end{array}\right)\right)$$

and, consequently,

$$\begin{aligned}
(n-1)\frac{\boldsymbol{\mu}_0'\mathbf{R}\boldsymbol{\mu}_0}{s_0} &\sim \chi^2_{n-k+1}, & (18)\\
\frac{h_0}{s_0}|v_0 - h_0^2/s_0 &\sim \mathcal{N}\left(\frac{\mathbf{X}_0'\mathbf{R}\boldsymbol{\mu}_0}{\boldsymbol{\mu}_0'\mathbf{R}\boldsymbol{\mu}_0}, \frac{(n-1)^{-1}}{\boldsymbol{\mu}_0'\mathbf{R}\boldsymbol{\mu}_0}\left(v_0 - h_0^2/s_0\right)\right), & (19)\\
(n-1)\frac{\mathbf{X}_0'\mathbf{R}\mathbf{X}_0 - (\mathbf{X}_0'\mathbf{R}\boldsymbol{\mu}_0)^2/\boldsymbol{\mu}_0'\mathbf{R}\boldsymbol{\mu}_0}{v_0 - h_0^2/s_0} &\sim \chi^2_{n-k+2} & (20)
\end{aligned}$$

as well as s_0 is independent of h_0/s_0 and $v_0 - h_0^2/s_0$. Let

$$\hat{s} = \hat{\boldsymbol{\mu}}'\hat{\mathbf{R}}\hat{\boldsymbol{\mu}}, \quad \hat{h} = \mathbf{X}_{n+1}'\hat{\mathbf{R}}\hat{\boldsymbol{\mu}}, \quad \hat{v} = \mathbf{X}_{n+1}'\hat{\mathbf{R}}\mathbf{X}_{n+1}.$$

Then the unconditional distributions of

$$\xi_1 = (n-1)\frac{\hat{\boldsymbol{\mu}}'\mathbf{R}\hat{\boldsymbol{\mu}}}{\hat{s}} \quad \text{and} \quad \xi_2 = (n-1)\frac{\mathbf{X}_{n+1}'\mathbf{R}\mathbf{X}_{n+1} - (\mathbf{X}_{n+1}'\mathbf{R}\hat{\boldsymbol{\mu}})^2/\hat{\boldsymbol{\mu}}'\mathbf{R}\hat{\boldsymbol{\mu}}}{\hat{v} - \hat{h}^2/\hat{s}}$$

coincide with the corresponding conditional ones as given in (18) and (20) as well as ξ_1 is independent of $\hat{h}/\hat{s}$ and ξ_2.

(a) The application of (16)-(20) leads to the stochastic representation for $\left(\hat{R}_{GMV}, \hat{R}_{GMV,n+1}\right)$ given by

$$\begin{aligned}
\hat{R}_{GMV} &\stackrel{d}{=} \frac{\hat{\boldsymbol{\mu}}'\boldsymbol{\Sigma}^{-1}\mathbf{1}_k}{\mathbf{1}_k'\boldsymbol{\Sigma}^{-1}\mathbf{1}_k} + \sqrt{\frac{(n-1)^{-1}}{\mathbf{1}_k'\boldsymbol{\Sigma}^{-1}\mathbf{1}_k}}\sqrt{s}z_1\\
&\stackrel{d}{=} \frac{\hat{\boldsymbol{\mu}}'\boldsymbol{\Sigma}^{-1}\mathbf{1}_k}{\mathbf{1}_k'\boldsymbol{\Sigma}^{-1}\mathbf{1}_k} + \sqrt{\frac{\hat{\boldsymbol{\mu}}'\mathbf{R}\hat{\boldsymbol{\mu}}}{\mathbf{1}_k'\boldsymbol{\Sigma}^{-1}\mathbf{1}_k}}\frac{z_1}{\sqrt{\xi_1}}
\end{aligned}$$

and

$$
\begin{aligned}
\hat{R}_{GMV,n+1} &\overset{d}{=} \frac{\mathbf{X}_{n+1}'\mathbf{\Sigma}^{-1}\mathbf{1}_k}{\mathbf{1}_k'\mathbf{\Sigma}^{-1}\mathbf{1}_k} + \sqrt{\frac{(n-1)^{-1}}{\mathbf{1}_k'\mathbf{\Sigma}^{-1}\mathbf{1}_k}}\left(\frac{h}{\sqrt{s}}z_1 + \sqrt{v-h^2/s}\,z_2\right)\\
&\overset{d}{=} \frac{\mathbf{X}_{n+1}'\mathbf{\Sigma}^{-1}\mathbf{1}_k}{\mathbf{1}_k'\mathbf{\Sigma}^{-1}\mathbf{1}_k} + \sqrt{\frac{\hat{\boldsymbol{\mu}}'\mathbf{R}\hat{\boldsymbol{\mu}}}{\mathbf{1}_k'\mathbf{\Sigma}^{-1}\mathbf{1}_k}}\frac{\mathbf{X}_{n+1}'\mathbf{R}\hat{\boldsymbol{\mu}}}{\hat{\boldsymbol{\mu}}'\mathbf{R}\hat{\boldsymbol{\mu}}}\frac{z_1}{\sqrt{\xi_1}}\\
&+ \sqrt{\frac{\mathbf{X}_{n+1}'\mathbf{R}\mathbf{X}_{n+1} - \frac{(\mathbf{X}_{n+1}'\mathbf{R}\hat{\boldsymbol{\mu}})^2}{\hat{\boldsymbol{\mu}}'\mathbf{R}\hat{\boldsymbol{\mu}}}}{\mathbf{1}_k'\mathbf{\Sigma}^{-1}\mathbf{1}_k}}\left(\frac{z_3}{\sqrt{\xi_2}}\frac{z_1}{\sqrt{\xi_1}} + \frac{z_2}{\sqrt{\xi_2}}\right)
\end{aligned}
$$

where $z_1, z_2, z_3 \sim \mathcal{N}(0,1)$, $\xi_1 \sim \chi^2_{n-k+1}$, $\xi_2 \sim \chi^2_{n-k+2}$; $z_1, z_2, z_3, \xi_1, \xi_2$ are mutually independent.

Since

$$
\mathbf{R}\mathbf{\Sigma}\frac{\mathbf{\Sigma}^{-1}\mathbf{1}_k}{\mathbf{1}_k'\mathbf{\Sigma}^{-1}\mathbf{1}_k} = \frac{\mathbf{R}\mathbf{1}_k}{\mathbf{1}_k'\mathbf{\Sigma}^{-1}\mathbf{1}_k} = \mathbf{0},
$$

we get that (see Corollary 7.8.6.1 in Gupta and Nagar (2000))

$$
\begin{pmatrix} \frac{\hat{\boldsymbol{\mu}}'\mathbf{\Sigma}^{-1}\mathbf{1}_k}{\mathbf{1}_k'\mathbf{\Sigma}^{-1}\mathbf{1}_k} \\ \frac{\mathbf{X}_{n+1}'\mathbf{\Sigma}^{-1}\mathbf{1}_k}{\mathbf{1}_k'\mathbf{\Sigma}^{-1}\mathbf{1}_k} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \hat{\boldsymbol{\mu}}'\mathbf{R}\hat{\boldsymbol{\mu}} & \hat{\boldsymbol{\mu}}'\mathbf{R}\mathbf{X}_{n+1} \\ \mathbf{X}_{n+1}'\mathbf{R}\hat{\boldsymbol{\mu}} & \mathbf{X}_{n+1}'\mathbf{R}\mathbf{X}_{n+1} \end{pmatrix}
$$

are independently distributed with

$$
\begin{pmatrix} \frac{\hat{\boldsymbol{\mu}}'\mathbf{\Sigma}^{-1}\mathbf{1}_k}{\mathbf{1}_k'\mathbf{\Sigma}^{-1}\mathbf{1}_k} \\ \frac{\mathbf{X}_{n+1}'\mathbf{\Sigma}^{-1}\mathbf{1}_k}{\mathbf{1}_k'\mathbf{\Sigma}^{-1}\mathbf{1}_k} \end{pmatrix} \sim \mathcal{N}_2\left(R_{GMV}\mathbf{1}_2, \begin{pmatrix} V_{GMV}/n & 0 \\ 0 & V_{GMV} \end{pmatrix}\right).
$$

Moreover, using that $\hat{\boldsymbol{\mu}}$ and $\mathbf{X}_{n+1}$ are independent, we get that

$$
\begin{aligned}
&\frac{\mathbf{X}_{n+1}'\mathbf{R}\hat{\boldsymbol{\mu}}}{\hat{\boldsymbol{\mu}}'\mathbf{R}\hat{\boldsymbol{\mu}}}|\hat{\boldsymbol{\mu}} \sim \mathcal{N}\left(\frac{\boldsymbol{\mu}'\mathbf{R}\hat{\boldsymbol{\mu}}}{\hat{\boldsymbol{\mu}}'\mathbf{R}\hat{\boldsymbol{\mu}}}, \frac{1}{\hat{\boldsymbol{\mu}}'\mathbf{R}\hat{\boldsymbol{\mu}}}\right)\\
&\mathbf{X}_{n+1}'\mathbf{R}\mathbf{X}_{n+1} - \frac{(\mathbf{X}_{n+1}'\mathbf{R}\hat{\boldsymbol{\mu}})^2}{\hat{\boldsymbol{\mu}}'\mathbf{R}\hat{\boldsymbol{\mu}}}|\hat{\boldsymbol{\mu}} \sim \chi^2_{k-2;\delta^2(\hat{\boldsymbol{\mu}})} \ \text{ with}\\
&\delta^2(\hat{\boldsymbol{\mu}}) = \boldsymbol{\mu}\mathbf{R}\boldsymbol{\mu} - \frac{(\boldsymbol{\mu}'\mathbf{R}\hat{\boldsymbol{\mu}})^2}{\hat{\boldsymbol{\mu}}'\mathbf{R}\hat{\boldsymbol{\mu}}} = \frac{\boldsymbol{\mu}'\mathbf{R}\boldsymbol{\mu}}{\hat{\boldsymbol{\mu}}'\mathbf{R}\hat{\boldsymbol{\mu}}}\hat{\boldsymbol{\mu}}'\left(\mathbf{R} - \frac{\mathbf{R}\boldsymbol{\mu}\boldsymbol{\mu}'\mathbf{R}}{\boldsymbol{\mu}'\mathbf{R}\boldsymbol{\mu}}\right)\hat{\boldsymbol{\mu}},
\end{aligned}
$$

and the two quantities given $\hat{\boldsymbol{\mu}}$ are independently distributed. These results follow from Corollary 5.1.3a and Theorem 5.5.1 of Mathai and Provost (1992) since

$$
\left(\mathbf{R} - \frac{\mathbf{R}\hat{\boldsymbol{\mu}}\hat{\boldsymbol{\mu}}'\mathbf{R}}{\hat{\boldsymbol{\mu}}'\mathbf{R}\hat{\boldsymbol{\mu}}}\right)\mathbf{\Sigma}\frac{\mathbf{R}\hat{\boldsymbol{\mu}}}{\hat{\boldsymbol{\mu}}'\mathbf{R}\hat{\boldsymbol{\mu}}} = \mathbf{0}
$$

and

$$\left(\mathbf{R}-\frac{\mathbf{R}\hat{\boldsymbol{\mu}}\hat{\boldsymbol{\mu}}'\mathbf{R}}{\hat{\boldsymbol{\mu}}'\mathbf{R}\hat{\boldsymbol{\mu}}}\right)\boldsymbol{\Sigma}\left(\mathbf{R}-\frac{\mathbf{R}\hat{\boldsymbol{\mu}}\hat{\boldsymbol{\mu}}\mathbf{R}}{\hat{\boldsymbol{\mu}}'\mathbf{R}\hat{\boldsymbol{\mu}}}\right)=\mathbf{R}-\frac{\mathbf{R}\hat{\boldsymbol{\mu}}\hat{\boldsymbol{\mu}}'\mathbf{R}}{\hat{\boldsymbol{\mu}}'\mathbf{R}\hat{\boldsymbol{\mu}}}$$

with $rank\left(\left(\mathbf{R}-\frac{\mathbf{R}\hat{\boldsymbol{\mu}}\hat{\boldsymbol{\mu}}'\mathbf{R}}{\hat{\boldsymbol{\mu}}'\mathbf{R}\hat{\boldsymbol{\mu}}}\right)\boldsymbol{\Sigma}\right)=k-2$.

In using that

$$\hat{\boldsymbol{\mu}}'\mathbf{R}\hat{\boldsymbol{\mu}}=\hat{\boldsymbol{\mu}}'\left(\mathbf{R}-\frac{\mathbf{R}\boldsymbol{\mu}\boldsymbol{\mu}'\mathbf{R}}{\boldsymbol{\mu}'\mathbf{R}\boldsymbol{\mu}}\right)\hat{\boldsymbol{\mu}}+\boldsymbol{\mu}'\mathbf{R}\boldsymbol{\mu}\left(\frac{\hat{\boldsymbol{\mu}}'\mathbf{R}\boldsymbol{\mu}}{\boldsymbol{\mu}'\mathbf{R}\boldsymbol{\mu}}\right)^2$$

and applying Corollary 5.1.3a and Theorem 5.5.1 of Mathai and Provost (1992), we get that

$$\hat{\boldsymbol{\mu}}'\left(\mathbf{R}-\frac{\mathbf{R}\boldsymbol{\mu}\boldsymbol{\mu}'\mathbf{R}}{\boldsymbol{\mu}'\mathbf{R}\boldsymbol{\mu}}\right)\hat{\boldsymbol{\mu}}\ \text{ and }\ \frac{\hat{\boldsymbol{\mu}}'\mathbf{R}\boldsymbol{\mu}}{\boldsymbol{\mu}'\mathbf{R}\boldsymbol{\mu}}$$

are independent with

$$\frac{\hat{\boldsymbol{\mu}}'\mathbf{R}\boldsymbol{\mu}}{\boldsymbol{\mu}'\mathbf{R}\boldsymbol{\mu}}\sim\mathcal{N}\left(1,\frac{n^{-1}}{\boldsymbol{\mu}'\mathbf{R}\boldsymbol{\mu}}\right)$$

and

$$n\hat{\boldsymbol{\mu}}'\left(\mathbf{R}-\frac{\mathbf{R}\boldsymbol{\mu}\boldsymbol{\mu}'\mathbf{R}}{\boldsymbol{\mu}'\mathbf{R}\boldsymbol{\mu}}\right)\hat{\boldsymbol{\mu}}\sim\chi^2_{k-2}.$$

Hence, the stochastic representation for $\left(\hat{R}_{GMV},\hat{R}_{GMV,n+1}\right)$ expressed as

$$\hat{R}_{GMV}\overset{d}{=}R_{GMV}+\frac{\sqrt{V_{GMV}}}{\sqrt{n}}z_4+\sqrt{\frac{1}{n}\xi_3+\frac{1}{n}\left(\sqrt{ns}+z_5\right)^2}\sqrt{V_{GMV}}\frac{z_1}{\sqrt{\xi_1}}$$

and

$$\begin{aligned}\hat{R}_{GMV,n+1}&\overset{d}{=}R_{GMV}+\sqrt{V_{GMV}}z_6+\sqrt{V_{GMV}}\left(\frac{\sqrt{s}(\sqrt{ns}+z_5)}{\sqrt{\xi_3+(\sqrt{ns}+z_5)^2}}+z_7\right)\frac{z_1}{\sqrt{\xi_1}}\\&+\sqrt{V_{GMV}}\sqrt{\xi_4}\left(\frac{z_3}{\sqrt{\xi_2}}\frac{z_1}{\sqrt{\xi_1}}+\frac{z_2}{\sqrt{\xi_2}}\right)\end{aligned}$$

where $z_1,z_2,z_3,z_4,z_5,z_6,z_7\sim\mathcal{N}(0,1)$, $\xi_1\sim\chi^2_{n-k+1}$, $\xi_2\sim\chi^2_{n-k+2}$, $\xi_3\sim\chi^2_{k-2}$, $\xi_4|z_5,\xi_3\sim\chi^2_{k-2;\delta^2(s,\xi_3,z_5)}$ with $\delta^2(s,\xi_3,z_5)=\frac{s\xi_3}{\xi_3+\left(\sqrt{ns}+z_5\right)^2}$; $z_1,z_2,z_3,z_4,z_6,z_7,\xi_1,\xi_2,(z_5,\xi_3,\xi_4)$ are mutually independent.

(b) Let

$$a=\frac{\sqrt{n-k}}{\sqrt{n-1}}\frac{1}{\sqrt{1+\frac{n}{n-1}\hat{s}_0}\sqrt{\frac{\hat{V}_{GMV}}{n}}}.$$

Given $\hat{\boldsymbol{\mu}} = \boldsymbol{\mu}_0$ and $\mathbf{X}_{n+1} = \mathbf{X}_0$, we get

$$\left(\begin{array}{c} a\left(\frac{\mathbf{X}_0'\hat{\mathbf{\Sigma}}^{-1}\mathbf{1}_k}{\mathbf{1}_k'\hat{\mathbf{\Sigma}}^{-1}\mathbf{1}_k} - r_f\right) \\ \frac{\mathbf{X}_0'\hat{\mathbf{\Sigma}}^{-1}\mathbf{1}_k}{\mathbf{1}_k'\hat{\mathbf{\Sigma}}^{-1}\mathbf{1}_k} \end{array} \right) \Big| s_0, h_0, v_0$$
$$\sim \mathcal{N}_2\left(\left(\begin{array}{c} a\left(\frac{\boldsymbol{\mu}_0'\mathbf{\Sigma}^{-1}\mathbf{1}_k}{\mathbf{1}_k'\mathbf{\Sigma}^{-1}\mathbf{1}_k} - r_f\right) \\ \frac{\mathbf{X}_0'\mathbf{\Sigma}^{-1}\mathbf{1}_k}{\mathbf{1}_k'\mathbf{\Sigma}^{-1}\mathbf{1}_k} \end{array} \right), \frac{(n-1)^{-1}}{\mathbf{1}_k'\mathbf{\Sigma}^{-1}\mathbf{1}_k} \left(\begin{array}{cc} a^2 s_0 & a h_0 \\ a h_0 & v_0 \end{array} \right)\right),$$

Then, using the derivation of part (a) and (17) we get a stochastic representation for $\hat{R}_{GMV,n+1}$ as in part (a) and a stochastic representation of T given by

$$\begin{aligned} T &\stackrel{d}{=} \frac{\sqrt{n-k}}{\sqrt{n-1}} \frac{1}{\sqrt{1 + \frac{\xi_3 + \left(\sqrt{ns} + z_5\right)^2}{\xi_1}} \sqrt{\xi_5 \frac{V_{GMV}}{(n-1)n}}} \\ &\times \left(R_{GMV} - r_f + \frac{\sqrt{V_{GMV}}}{\sqrt{n}} z_4 + \sqrt{\frac{1}{n}\xi_3 + \frac{1}{n}\left(\sqrt{ns} + z_5\right)^2} \sqrt{V_{GMV}} \frac{z_1}{\sqrt{\xi_1}} \right) \\ &= \frac{\sqrt{n-k}}{\sqrt{\xi_5}} \frac{1}{\sqrt{1 + \frac{\xi_3 + \left(\sqrt{ns} + z_5\right)^2}{\xi_1}}} \left(\sqrt{n} \frac{R_{GMV} - r_f}{\sqrt{V_{GMV}}} + z_4 + \sqrt{\frac{\xi_3 + \left(\sqrt{ns} + z_5\right)^2}{\xi_1}} z_1 \right), \end{aligned}$$

where $\xi_5 \sim \chi^2_{n-k}$ independent of $z_1, z_2, z_3, z_4, z_6, z_7, \xi_1, \xi_2, (z_5, \xi_3, \xi_4)$.

□

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II

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A TEST ON MEAN-VARIANCE EFFICIENCY OF THE TANGENCY PORTFOLIO IN HIGH-DIMENSIONAL SETTING

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Abstract. In this paper we derive the asymptotic distribution of the test of the efficiency of the tangency portfolio in high-dimensional settings, namely when both the portfolio dimension and the sample size grow to infinity. Moreover, we propose a new test based on the estimator for the slope parameter of the efficient frontier in the mean-variance space when there is a possibility in investing into the riskless asset, and derive the asymptotic distribution of that test statistic under both the null and alternative hypotheses. Additionally, we study the finite sample performance of the derived theoretical results via simulations.

Key words and phrases. Tangency Portfolio, Mean-variance portfolio, High-dimensional settings .

2010 *Mathematics Subject Classification.* Primary: 62H10, 60E05; Secondary: 60E10.

1. Introduction

Since the introduction of mean-variance theory by Markowitz (1952), a large number of papers devoted to the optimal portfolio selection have been published and brought remarkable contributions in different avenues of finance, be it in research or in practice. Since then, the problem of testing the efficiency of a given portfolio has gained a lot of attention (Bodnar and Schmid (2008); Bodnar et al. (2019a); Gibbons et al. (1989); Glombek (2014); Britten-Jones (1999); Muhinyuza et al. (2017)) to just name a few. In this regard, the aim of the investor is to find an optimal portfolio that minimizes the risk, i.e, the variance of the portfolio for a given level of the expected return. In the absence of a risk-free asset, the risk aversion strategy leads to minimal variance portfolio. Merton (1972) showed that all Markowitz' optimal portfolios lie on the upper part of the parabola in the mean-variance space (known as Efficient Frontier(EF)) and its equation is given by

$$(R - R_{GMV})^2 = s(V - V_{GMV}) \tag{1}$$

where

$$R_{GMV} = \frac{\mathbf{1}\mathbf{\Sigma}^{-1}\mu}{\mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{1}}, V_{GMV} = \frac{1}{\mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{1}}, \text{ and } s = \mu'\mathbf{R}\mu \text{ with } \mathbf{R} = \mathbf{\Sigma}^{-1} - \frac{\mathbf{\Sigma}^{-1}\mathbf{1}\mathbf{1}'\mathbf{\Sigma}^{-1}}{\mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{1}}. \tag{2}$$

Whereas, in the presence of a risk-free asset, the tangency portfolio (TP), i.e, a linear combination of risky assets and a risk-free asset needs to be considered and the equation of the efficiency frontier in case of the presence of risk-free asset is given by

$$(R - r_f)^2 = \lambda V \tag{3}$$

where

$$\lambda = (\mu - r_f\mathbf{1})'\mathbf{\Sigma}^{-1}(\mu - r_f\mathbf{1}). \tag{4}$$

In practice, this investment theory appears to be challenging because of the presence of the sampling error while estimating the unknown theoretical quantities. A big number of the literature in this area of research treats the case of classical asymptotics (the sample size n increases while the size of the portfolio k remains constant). In this situation, the plug-in estimator (sample estimator) of the optimal portfolio turns to be a good estimator due to its attractive properties, namely the consistency and asymptotic normality.

Nowadays, in several applications the number of assets k in a portfolio are comparable to the sample size n, i.e., the portfolio size k and the sample size n grow to infinity at the same order, that means $\frac{k}{n} \to c \in (0, \infty)$. In this situation the traditional asymptotic theory cannot be applied because of the failure in delivering consistent estimators of the unknown parameters of the assets returns, namely, the mean vector and the covariance matrix. A number of papers treat the high-dimensional asymptotics in portfolio theory by the help of the results from random matrix theory (see, e.g.,Frahm and Jaekel (2008); Glombek (2014); Bodnar et al. (2019b, 2016b, 2018)). Recently, Bodnar et al. (2019a) studied the distributional properties of the estimated TP weights and suggested inference procedures in small and high-dimensions. Furthermore, they delivered the high-dimensional asymptotic distribution of the estimated TP weights and they proposed a test statistic when both the population and sample covariance matrices are singular. This paper complements the existing literature in different ways. It provides the asymptotics of the test statistic for testing the existence of the EF and for testing the efficiency of the TP under high-dimensional regime.

The rest of the paper is structured as follow: Section 2 discusses the main results of the two provided tests including the stochastic representations of their test statistics. These stochastic representations are later used to obtain the high-dimensional asymptotic distributions of the test statistics under both the null and the alternative hypotheses. It also provides the power functions of the suggested tests for different values of c. Section 3 presents the results of the simulation study, in which we compare the performance of the two proposed tests, while the concluding remarks are given in Section 4.

2. Test theory on the location of the TP in high-dimension

Through-out this section, we assume $\mathbf{x}_1, \ldots, \mathbf{x}_n$ to be an independent k-dimensional sample of asset returns, with $\mathbf{E}(\mathbf{x}_t) = \mu$ and $\mathbf{cov}(\mathbf{x}_t) = \mathbf{\Sigma}$, for $t = 1, \ldots, n$, where $\mathbf{\Sigma}$ is assumed to be positive definite, we also assume that $\mathbf{x}_t \sim \mathcal{N}_k(\mu, \mathbf{\Sigma})$.

2.1. A test on the existence of the EF based on the slope parameter.

The slope parameter plays an important role in the construction of the efficient frontier. It shows how the market is profitable, i.e. how large is the increase in the portfolio profit in relation to the unit increase of the portfolio variance. If the slope parameters is zero, the population efficient frontier reduces to a straight line. In this case, the GMV portfolio is the only available investment. If there is a possibility to invest in the risk-free asset with return r_f, a part of the investor wealth may be invested into the riskless asset and it may reduce the variance, whereas the rest of the wealth can be invested into the risky assets. In this case, a test for the existence of the efficient frontier would be of importance and its hypotheses are given by

$$H_0 : \lambda = 0 \text{ against } H_1 : \lambda > 0, \tag{5}$$

where λ is defined in (4). The rejection of the null hypothesis ensures the existence of the efficient frontier, i.e. it confirms the positiveness of the slope parameter($\lambda > 0$), and the investor has a number of investment options to choose from including the TP. On the other hand,

the non rejection of the null hypothesis means that the slope coefficient of the efficient frontier is equal to zero. In this case the GMV portfolio is the only available portfolio for investment, and also the allocation of the whole wealth into the riskless asset could be considered as a suitable alternative.

The test statistic for testing (5) is based on the derivation in Bodnar and Schmid (2009) and is given by

$$T_\lambda = \frac{n(n-k)}{(n-1)k} (\hat{\mu} - r_f \mathbf{1})' \hat{\mathbf{\Sigma}}^{-1} (\hat{\mu} - r_f \mathbf{1}), \tag{6}$$

with

$$\hat{\mu} = \frac{1}{n}\sum_{t=1}^{n}\mathbf{x}_t \text{ and } \hat{\mathbf{\Sigma}} = \frac{1}{n-1}\sum_{t=1}^{n}(\mathbf{x}_t - \hat{\mu})(\mathbf{x}_t - \hat{\mu})' \tag{7}$$

are the sample mean vector and the sample covariance matrix, respectively. The distribution of the test statistic T_λ in the equation (6) is given in the following proposition. In the following the symbols $F_{a,b,d}$ denotes a non-central F-distribution with a, b degrees of freedom and non-centrality parameter d, while $F_{a,b}$ stands for a central F-distribution with a, b degrees of freedom.

Proposition 1. *Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ be i.i.d random vectors with $\mathbf{x}_1 \sim \mathcal{N}_k(\mu, \mathbf{\Sigma})$, $k < n$. $\mathbf{\Sigma}$ is assumed to be positive definite. Then it holds that*

a) $T_\lambda \sim \mathcal{F}_{k,n-k,\delta_T}$ under H_1, where $\delta_T = n\lambda$ with λ as defined in (4);
b) $T_\lambda \sim \mathcal{F}_{k,n-k}$ under H_0.

Proof of proposition 1. From Theorem 3.1.2 of Muirhead (1982), it follows that

$$\hat{\mu} - r_f\mathbf{1} \sim \mathcal{N}\left(\mu - r_f\mathbf{1}, \frac{1}{n}\mathbf{\Sigma}\right) \text{ and } (n-1)\mathbf{S} \sim \mathcal{W}_k(n-1, \mathbf{\Sigma}),$$

where $W_k(n-1, \mathbf{\Sigma})$ denotes a k-dimensional Wishart distribution with $n-1$ degree of freedom and the parameter matrix $\mathbf{\Sigma}$. On top of that, $(\hat{\mu} - r_f\mathbf{1})$ and $(n-1)\mathbf{S}$ are independent. Applying the results of Theorem 6.7a.1 in Mathai and Provost (1992), we get the statement of part (a) of the proposition. The statement of Proposition 1(b) follows by setting $\lambda = 0$ under the null hypothesis. □

Alternatively, one can represent the distribution of T_λ using the following stochastic representation

$$T_\lambda \overset{d}{=} \frac{n-k}{k}\frac{(\sqrt{n\lambda} + z_1)^2 + \zeta_1}{\zeta_2} \tag{8}$$

where $z_1 \sim \mathcal{N}(0,1), \zeta_1 \sim \chi^2_{k-1}$ and $\zeta_2 \sim \chi^2_{n-k}$. Moreover, z_1, ζ_1 and ζ_2 are independent. The symbol $\overset{d}{=}$ stands for equality in distribution.

From Proposition 1, it is remarkable that the density function of the statistic T_λ depends on the parameters μ and $\mathbf{\Sigma}$ only over the non-centrality parameter δ_T. Thus, the exact power function of the test can be easily computed using any mathematical software package. However, some numerical difficulties may be encountered when the power function of the test is computed for large values of k and n.

To address this problem, we derive the asymptotic distribution of T_λ for high-dimensional setting. This result is given in Theorem 1. We note that for finite case, this result has been used to test the efficiency of any portfolio from the efficient frontier with respect to the GMV portfolio (Bodnar and Schmid (2009); Bodnar and Bodnar (2010)).
Because of positive definiteness of the covariance matrix, the null hypothesis $H_0 : \lambda = 0$ occurs only if $\mu = r_f\mathbf{1}$. Note that a similar test statistic is used when testing the hypothesis $H_0 : \mu_1 = \cdots = \mu_k$ (see, e.g., Rencher and Christensen (2012)). This is not a surprise since both hypotheses are equivalent.

Theorem 1. *Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ be i.i.d random vectors with $\mathbf{x}_1 \sim \mathcal{N}_k(\mu, \mathbf{\Sigma})$, $k < n$. $\mathbf{\Sigma}$ is assumed to be positive definite. Then it holds that*

$$\sqrt{k}\left(\frac{T_\lambda - 1 - \frac{n}{k}\lambda}{\sigma_{T_\lambda}}\right) \xrightarrow{d} \mathcal{N}(0,1)$$

where

$$\sigma^2_{T_\lambda} = 2 + 4\frac{\lambda}{c} + 2\frac{c}{1-c}\left(1 + \frac{\lambda}{c}\right)^2$$

for $k/n \to c \in (0,1)$ as $n \to \infty$. Under the null hypothesis, $\sqrt{k}\,(T_\lambda - 1) \xrightarrow{d} \mathcal{N}\left(0, 2/(1-c)\right)$ for $k/n \to c \in (0,1)$ as $n \to \infty$.

Proof of Theorem 1. Using the stochastic representation given in equation (8), it holds that

$$
\begin{aligned}
T_\lambda - 1 - \frac{n}{k}\lambda &= \frac{n-k}{k}\frac{(\sqrt{n\lambda}+z_1)^2+\zeta_1}{\zeta_2} - 1 - \frac{n}{k}\lambda \\
&= \frac{n-k}{\zeta_2}\left(\frac{\zeta_1 + n\lambda + 2\sqrt{n\lambda}z_1 + z_1^2}{k} - \left(1+\frac{n}{k}\lambda\right)\frac{\zeta_2}{n-k}\right) \\
&= \frac{n-k}{\zeta_2}\left(\frac{\zeta_1 + n\lambda + 2\sqrt{n\lambda}z_1 + z_1^2}{k} - \left(1+\frac{n}{k}\lambda\right) + \left(1+\frac{n}{k}\lambda\right) - \left(1+\frac{n}{k}\lambda\right)\frac{\zeta_2}{n-k}\right) \\
&= \frac{n-k}{\zeta_2}\left(\frac{\zeta_1}{k} + \frac{n}{k}\lambda + 2\frac{\sqrt{n\lambda}}{k}z_1 + \frac{z_1^2}{k} - 1 - \frac{n}{k}\lambda - \left(1+\frac{n}{k}\lambda\right)\left(\frac{\zeta_2}{n-k} - 1\right)\right) \\
&= \frac{n-k}{\zeta_2}\left(\left(\frac{\zeta_1}{k} - 1\right) - \left(1+\frac{n}{k}\lambda\right)\left(\frac{\zeta_2}{n-k} - 1\right) + 2\frac{\sqrt{n\lambda}}{k}z_1 + \frac{z_1^2}{k}\right)
\end{aligned}
$$

We then have

$$
\sqrt{k}\left(T_\lambda - 1 - \frac{n}{k}\lambda\right) = \frac{n-k}{\zeta_2}\left(\sqrt{k}\left(\frac{\zeta_1}{k} - 1\right) - \left(1+\frac{n}{k}\lambda\right)\sqrt{k}\left(\frac{\zeta_2}{n-k} - 1\right) + 2\frac{\sqrt{n\lambda}}{\sqrt{k}}z_1 + \frac{z_1^2}{\sqrt{k}}\right)
$$

Using Lemma 3 in Bodnar and Reiß (2016) and the proof of Lemma 4 in Bodnar et al. (2016b), we obtain the following results:

$$
\frac{\zeta_2}{n-k} \xrightarrow{a.s} 1, 1+\frac{n}{k}\lambda \xrightarrow{a.s} 1+\frac{\lambda}{c}, \sqrt{k}\left(\frac{\zeta_1}{k} - 1\right) \xrightarrow{d} \mathcal{N}\left(0, 2\right), \sqrt{k}\left(\frac{\zeta_2}{n-k} - 1\right) \xrightarrow{d} \mathcal{N}\left(0, 2\frac{c}{1-c}\right),
$$

$$
2\frac{\sqrt{n\lambda}}{\sqrt{k}}z_1 \xrightarrow{d} \mathcal{N}\left(0, 4\frac{\lambda}{c}\right) \text{ and } \frac{z_1^2}{\sqrt{k}} \xrightarrow{a.s} 0.
$$

The fact that z_1, ζ_1 and ζ_2 are independent and the application of Slutsky's lemma (see, e.g., Theorem 2.8 in Van der Vaart (2000)) gives us

$$
\sqrt{k}\left(\frac{T_\lambda - 1 - \frac{n}{k}\lambda}{\sigma_{T_\lambda}}\right) \xrightarrow{d} \mathcal{N}\left(0, 1\right)
$$

where

$$
\sigma^2_{T_\lambda} = 2 + 4\frac{\lambda}{c} + 2\frac{c}{1-c}\left(1+\frac{\lambda}{c}\right)^2
$$

□

The application of Theorem 1 leads to an asymptotic expression of the power function given by

$$
\begin{aligned}
G_{T_\lambda}(s, S_{GMV}) = P\left(\frac{\sqrt{k}(T_\lambda - 1)}{\sqrt{2/(1-c)}} > z_{1-\alpha}\right) &= 1 - P\left(\sqrt{k}\frac{T_\lambda - 1 - \frac{n}{k}\lambda}{\sigma_{T_\lambda}} \le \frac{\sqrt{2/(1-c)}z_{1-\alpha} - \sqrt{k}\frac{n}{k}\lambda}{\sigma_{T_\lambda}}\right) &(9) \\
&\approx 1 - \Phi\left(\frac{\sqrt{2/(1-c)}z_{1-\alpha} - \sqrt{k}\frac{\lambda}{c}}{\sigma_{T_\lambda}}\right) &(10)
\end{aligned}
$$

where $z_{1-\alpha}$ is the $(1-\alpha)$-quantile of the standard normal distribution.

In Figures 1 and 2 we plot the power function (10) as a function of λ for different values of c and n as a solid line. We additionally plot the empirical power of the test for the same values of c and n as dashed line, and it is interpreted as the number of rejections of the null hypothesis obtained via a simulation study. As it can be seen from the proof of Theorem 1, instead of generating the huge random matrix of order $k \times n$ of asset returns in each simulation run, we instead simulate three independent random variables from standard univariate distributions and compute the statistic T_λ for given values of λ in the stochastic representation given in equation (8). The following algorithm can be used to compute the asymptotic power:

(i) generate independently $z_1^{(b)} \sim \mathcal{N}(0,1)$, $\zeta_1^{(b)} \sim \chi^2_{k-1}$ and $\zeta_2^{(b)} \sim \chi^2_{n-k}$;

(ii) for fixed λ, compute

$$T_s^{(b)} \stackrel{d}{=} \frac{n-k}{k} \frac{(\sqrt{n\lambda} + z_1^{(b)})^2 + \zeta_1^{(b)}}{\zeta_2^{(b)}}$$

(iii) repeat steps (i)-(ii) for $b = 1, \ldots, B$ and approximate P by

$$\hat{P} = \frac{1}{B} \sum_{b=1}^{B} \mathbb{1}_{\{z_{1-\alpha}, +\infty\}} \left(\sqrt{k} \frac{(T_\lambda^{(b)} - 1)}{\sqrt{2/(1-c)}} \right)$$

where $\mathbb{1}_{\{\mathcal{A}\}}$ is the indicator function of set $\mathcal{A}$

In Figures 1 and 2 we observe that the high-dimensional expression of the power function provides a reliable approximation of the true power function. We also note that for small values of the concentration constant the two power functions are indistinguishable while a moderate discrepancy is present for large values of the concentration coefficient c. The observed discrepancy comes from the fact that, as the concentration ratio c is neighbouring to one, the estimator produces a high bias which consequently leads to their inconsistency.

2.2. **Test on the location of the TP on EF in high-dimensions.** The location of the TP on the EF depends crucially on the relation between the risk-free rate r_f and the expected return of the global minimum variance portfolio (GMVP). At each time point, the investor wants to check whether holding TP is mean-variance efficient or it has to be reconstructed. As it can be seen from Figure 3, a TP is at the tangency point of the parabola and the line passing through the risk-free rate. It is seen that the TP lies on the upper part of the parabola since the expected return of the GMVP is greater than the return of risk-free asset. However, a lower TP may also occur when the riskless return is less than the expected return of the GMVP (see, e.g., Ingersoll (1987)).

For that reason, this problem can be formalised as a statistical test problem, with the following hypotheses

$$H_0 : R_{GMV} \leq r_f \text{ versus } H_1 : R_{GMV} > r_f \tag{11}$$

Rejecting H_0 in (11), means that the TP is mean-variance efficient while the non rejection of H_0 in (5) does not guarantee the efficiency of the TP. In the case of non rejection of H_0, the investor cannot be sure of the optimality of the TP and the investment in the risk-free rate could be considered as a suitable alternative.

Muhinyuza et al. (2017) proposed the following test statistics for (11)

$$T = \frac{\sqrt{n-k}}{\sqrt{n-1}} \frac{\hat{R}_{GMV} - r_f}{\sqrt{1 + \frac{n}{n-1}\hat{s}}\sqrt{\frac{\hat{V}_{GMV}}{n}}} \tag{12}$$

where $\hat{R}_{GMV}, \hat{V}_{GMV}$ and $\hat{s}$ are the sample estimators for R_{GMV}, V_{GMV} and s given by

$$\hat{R}_{GMV} = \frac{\mathbf{1}'\hat{\mathbf{\Sigma}}^{-1}\hat{\mu}}{\mathbf{1}'\hat{\mathbf{\Sigma}}^{-1}\mathbf{1}} \text{ and } \hat{V}_{GMV} = \frac{1}{\mathbf{1}'\hat{\mathbf{\Sigma}}^{-1}\mathbf{1}} \tag{13}$$

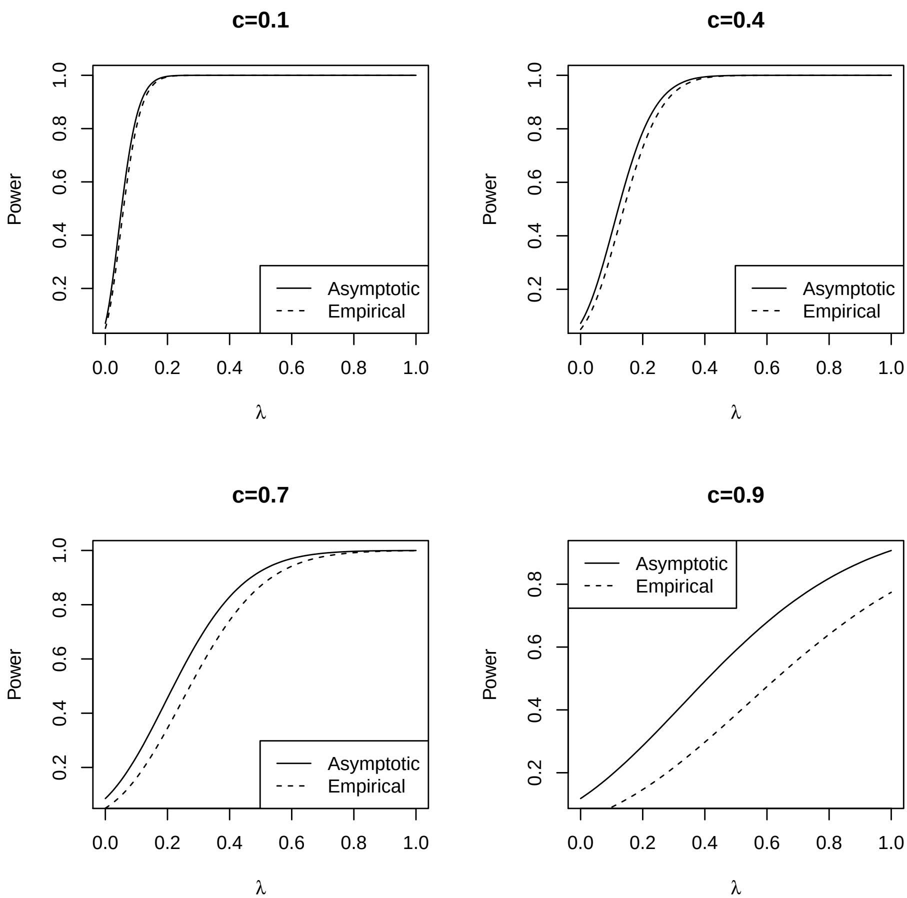

FIGURE 1. Asymptotic power versus empircal power for different values of c as a function of λ, significance level 5% and $n = 250$.

and

$$\hat{s} = \hat{\mu}'\hat{\mathbf{R}}\hat{\mu} \text{ with } \hat{\mathbf{R}} = \hat{\mathbf{\Sigma}}^{-1} - \frac{\hat{\mathbf{\Sigma}}^{-1}\mathbf{1}\mathbf{1}'\hat{\mathbf{\Sigma}}^{-1}}{\mathbf{1}'\hat{\mathbf{\Sigma}}^{-1}\mathbf{1}}. \tag{14}$$

Moreover, they provided the distribution of T in form of a density function and a stochastic representation under both the null and alternative hypotheses (see, Muhinyuza et al. (2017, Proposition 1,Theorem 1)). The following proposition summarizes the distribution of T.

Proposition 2. *Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ be i.i.d random vectors with $\mathbf{x}_1 \sim \mathcal{N}_k(\mu, \mathbf{\Sigma})$, $k < n$. $\mathbf{\Sigma}$ is assumed to be positive definite. Then*

(a) the density of T is given by

$$f_T(x) = \frac{n(n-k+1)}{(k-1)(n-1)} \int_0^\infty f_{t_{n-k,\delta(y)}}(x) f_{F_{k-1,n-k+1,ns}}\left(\frac{n(n-k+1)}{(k-1)(n-1)} y\right) dy \tag{15}$$

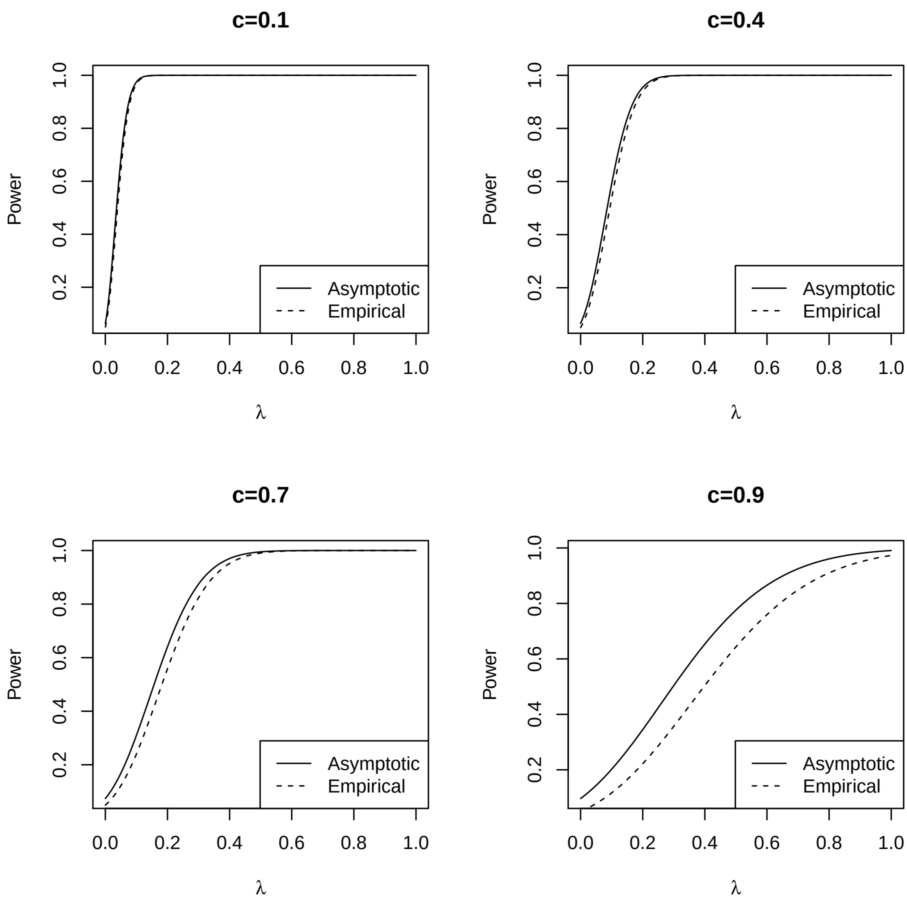

FIGURE 2. Asymptotic power versus empircal power for different values of c as a function of λ, significance level 5% and $n = 500$.

where

$$\delta(y) = \frac{\sqrt{n}}{\sqrt{1+\frac{n}{n-1}y}} S_{GMV} \quad \text{with} \quad S_{GMV} = \frac{R_{GMV} - r_f}{\sqrt{V_{GMV}}}$$

being the Sharpe ratio of the GMVP. The slope parameter $s = \mu' \mathbf{R} \mu$ *with* $\mathbf{R} = \mathbf{\Sigma}^{-1} - \mathbf{\Sigma}^{-1}\mathbf{1}\mathbf{1}'\mathbf{\Sigma}^{-1}/\mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{1}$.

(b) the stochastic representation of T *is given by*

$$T \stackrel{d}{=} \frac{\sqrt{n-k}}{\sqrt{\xi}} \frac{1}{\sqrt{1+\frac{\xi_3+\left(\sqrt{ns}+z_1\right)^2}{\xi_2}}} \left(\sqrt{n} S_{GMV} + z_2 + \sqrt{\frac{\xi_3 + \left(\sqrt{ns}+z_1\right)^2}{\xi_2}} z_3 \right) \tag{16}$$

where $z_1, z_2, z_3 \sim \mathcal{N}(0,1)$, $\xi \sim \chi^2_{n-k}$, $\xi_2 \sim \chi^2_{n-k+1}$, $\xi_3 \sim \chi^2_{k-2}$ *are mutually independent.*

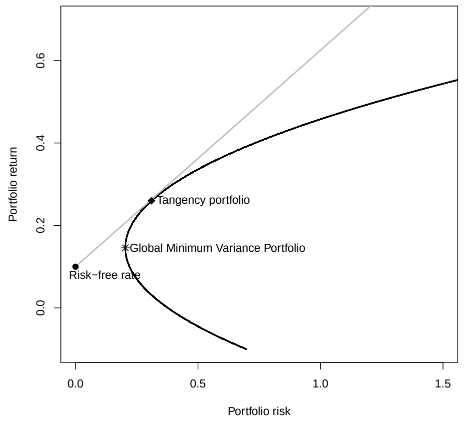

FIGURE 3. A graphical illustration of the location of a tangency portfolio on the efficient frontier .

It is seen that the distribution of test statistic in equation (12) can be represented as a mixture of a non-central t distribution with $n-k$ degrees of freedom and a non-centrality parameter $\delta(y)$, or it can be given in the form of a stochastic representation as in equation (16).

Furthermore, using the results from Proposition 2 the power function of the test can be easily obtained and it is easy to see that it depends on the parameters μ and $\mathbf{\Sigma}$ through s and S_{GMV}. However, this approach may encounter some difficulties for large values of k and n, since $\mathbf{\Sigma}$ becomes unstable for large values of k. To deal with this problem, we derive the asymptotic distribution of T in a high-dimensional environment. We assume that $k/n \to c \in (0,1)$ as $n \to \infty$. No further relation is imposed between k and n. Note that, under high-dimensional setting, the usual estimators for the precision matrix (Inverse of the covariance matrix) performs poorly and are not consistent anymore (Bodnar et al. (2016a)). Therefore, it is worth to study the behaviour of the test statistic developed by Muhinyuza et al. (2017) under high-dimensional environment and propose an alternative test which takes into account the correction of the estimated precision matrix.

The next theorem presents the asymptotic distribution of T under double asymptotic regime.

Theorem 2. *Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ be i.i.d random vectors with $\mathbf{x}_1 \sim \mathcal{N}_k(\mu, \mathbf{\Sigma})$, $k < n$. $\mathbf{\Sigma}$ is assumed to be positive definite. Let $\frac{k}{n} \to c \in (0,1)$ as $n \to \infty$. Then, it holds that the asymptotic distribution of T is given by*

(a)

$$\sigma_T^{-1}\left(T - \sqrt{n}\frac{S_{GMV}}{\sqrt{1 + \frac{k-1}{n-k+1}\left(1 + \frac{n}{k-1}s\right)}}\right) \xrightarrow{d} \mathcal{N}(0,1) \tag{17}$$

where

$$\sigma_T^2 = 1 + \frac{S_{GMV}^2}{1+s}\left(\frac{1}{2} + \frac{1}{2}\frac{s^2+2s+c}{(1+s)^2}\right) \tag{18}$$

for $\frac{k}{n} \to c \in (0,1)$ as $n \to \infty$.

(b) Under the null hypothesis it holds that $T \sim N(0,1)$

Proof of Theorem 2. From the stochastic representation given in (16) and using the properties of a normally distributed random variable, we have that

$$T \stackrel{d}{=} \sqrt{\frac{n-k}{\xi}} \frac{\sqrt{n} S_{GMV}}{\sqrt{1+\frac{\zeta}{\xi_2}}} + z_0 \sqrt{\frac{n-k}{\xi}} \tag{19}$$

where $z_0 \sim \mathcal{N}(0,1)$, $\xi \sim \chi^2_{n-k}$, $\xi_2 \sim \chi^2_{n-k+1}$, $\zeta \sim \chi^2_{k-1,ns}$, and on top of that z_0, ζ, ξ and ξ_2 are mutually independent.

We then have

$$T - \sqrt{n}\frac{S_{GMV}}{\sqrt{1+\frac{k-1}{n-k+1}\left(1+\frac{n}{k-1}s\right)}} = \sqrt{\frac{n-k}{\xi}} z_0 + \sqrt{n}\sqrt{\frac{n-k}{\xi}} \frac{S_{GMV}}{\sqrt{1+\frac{\zeta}{\xi_2}}} - \sqrt{n}\frac{S_{GMV}}{\sqrt{1+\frac{k-1}{n-k+1}\left(1+\frac{n}{k-1}s\right)}}$$

Adding and subtracting $\sqrt{n}\frac{S_{GMV}}{\sqrt{1+\frac{\zeta}{\xi_2}}}$ and factoring out $\frac{S_{GMV}}{\sqrt{1+\frac{\zeta}{\xi_2}}}$ and rearranging we get

$$\begin{aligned} & T - \sqrt{n}\frac{S_{GMV}}{\sqrt{1+\frac{k-1}{n-k+1}\left(1+\frac{n}{k-1}s\right)}} \\ = \quad & \sqrt{\frac{n-k}{\xi}} z_0 + \frac{S_{GMV}}{\sqrt{1+\frac{\zeta}{\xi_2}}} \left(\sqrt{n}\left(\sqrt{\frac{n-k}{\xi}} - 1\right) + \sqrt{n}\left(1 - \frac{\sqrt{1+\frac{\zeta}{\xi_2}}}{\sqrt{1+\frac{k-1}{n-k+1}\left(1+\frac{n}{k-1}s\right)}}\right)\right) \end{aligned}$$

Putting the last expression on common denominator and multiplying the numerator by its conjugate, we obtain

$$\sqrt{1+\frac{k-1}{n-k+1}\left(1+\frac{n}{k-1}s\right)} - \sqrt{1+\frac{\zeta}{\xi_2}} = \frac{\frac{k-1}{n-k+1}\left(1+\frac{n}{k-1}s\right) - \frac{\zeta}{\xi_2}}{\sqrt{1+\frac{k-1}{n-k+1}\left(1+\frac{n}{k-1}s\right)} + \sqrt{1+\frac{\zeta}{\xi_2}}}$$

We then have

$$\frac{k-1}{n-k+1}\left(1+\frac{n}{k-1}s\right) - \frac{\zeta}{\xi_2} = \frac{k-1}{n-k+1}\left(1+\frac{n}{k-1}s - \frac{\zeta/(k-1)}{\xi_2/(n-k+1)}\right) \tag{20}$$

Putting (20) on common denominator and rearranging it, we get

$$1+\frac{n}{k-1}s - \frac{\zeta/(k-1)}{\xi_2/(n-k+1)} = \frac{n-k+1}{\xi_2}\left[\left(1+\frac{n}{k-1}s\right)\left(\frac{\xi_2}{n-k+1}-1\right) - \left(\frac{\zeta}{k-1} - 1 - \frac{n}{k-1}s\right)\right] \tag{21}$$

We also have that

$$\sqrt{\frac{n-k}{\xi}} - 1 = \left(\sqrt{\frac{n-k}{\xi}} - 1\right)\frac{\sqrt{\frac{n-k}{\xi}}+1}{\sqrt{\frac{n-k}{\xi}}+1} = \frac{\frac{n-k}{\xi}-1}{\sqrt{\frac{n-k}{\xi}}+1} = \frac{\left(1-\frac{\xi}{n-k}\right)\frac{n-k}{\xi}}{\sqrt{\frac{n-k}{\xi}}+1} \tag{22}$$

Considering (21) and (22), we then get

$$\begin{aligned} T \quad - \quad & \sqrt{n}\frac{S_{GMV}}{\sqrt{1+\frac{k-1}{n-k+1}\left(1+\frac{n}{k-1}s\right)}} = \sqrt{\frac{n-k}{\xi}} z_0 + \frac{S_{GMV}}{\sqrt{1+\frac{\zeta}{\xi_2}}}\left(\frac{\sqrt{n}\left(1-\frac{\xi}{n-k}\right)\frac{n-k}{\xi}}{\sqrt{\frac{n-k}{\xi}}+1}\right) \\ + \quad & \frac{S_{GMV}}{\sqrt{1+\frac{\zeta}{\xi_2}}}\left(\frac{\frac{k-1}{n-k+1}\left[\frac{n-k+1}{\xi_2}\left(\left(1+\frac{n}{k-1}s\right)\sqrt{n}\left(\frac{\xi_2}{n-k+1}-1\right) - \sqrt{n}\left(\frac{\zeta}{k-1}-1-\frac{n}{k-1}s\right)\right)\right]}{\sqrt{1+\frac{k-1}{n-k+1}\left(1+\frac{n}{k-1}s\right)}\left(\sqrt{1+\frac{k-1}{n-k+1}\left(1+\frac{n}{k-1}s\right)}+\sqrt{1+\frac{\zeta}{\xi_2}}\right)}\right) \end{aligned}$$

Using Lemma 3(a) in Bodnar and Reiß (2016), we obtain

$$\frac{\xi}{n-k} \xrightarrow{a.s} 1, \frac{\xi_2}{n-k+1} \xrightarrow{a.s} 1 \text{ and } \frac{\zeta}{k-1} - 1 - \frac{n}{k-1}s \xrightarrow{a.s} 0 \Rightarrow \frac{\zeta}{k-1} \xrightarrow{a.s} 1 + \frac{s}{c} \tag{23}$$

Consequently,

$$\sqrt{1+\frac{\zeta}{\xi_2}} = \sqrt{1+\frac{\zeta}{\xi_2}\frac{k-1}{k-1}\frac{n-k+1}{n-k+1}} = \sqrt{1+\frac{\zeta}{k-1}\frac{n-k+1}{\xi_2}\frac{k-1}{n-k+1}} \xrightarrow{a.s} \sqrt{\frac{1+s}{1-c}} \tag{24}$$

and

$$\sqrt{1+\frac{k-1}{n-k+1}\left(1+\frac{n}{k-1}s\right)} \xrightarrow{a.s} \sqrt{\frac{1+s}{1-c}} \tag{25}$$

By using Lemma 3(b) in Bodnar and Reiß (2016) and the proof of Lemma 4 in Bodnar et al. (2016b), we get

$$\sqrt{n}\left(\frac{\xi}{n-k}-1\right) \xrightarrow{d} \mathcal{N}(0, 2/(1-c)), \tag{26}$$

$$\left(1+\frac{n}{k-1}s\right)\sqrt{n}\left(\frac{\xi_2}{n-k+1}-1\right) \xrightarrow{d} \mathcal{N}\left(0, \frac{2}{1-c}\left(1+\frac{s}{c}\right)^2\right), \tag{27}$$

and

$$\sqrt{n}\left(\frac{\zeta}{k-1}-1-\frac{n}{k-1}s\right) \xrightarrow{d} \mathcal{N}\left(0, \frac{2}{c}\left(1+2\frac{s}{c}\right)\right) \tag{28}$$

for $k/n \to c \in (0,1)$ as $n \to \infty$.
We also know that

$$\sqrt{n}\left(z_0/\sqrt{n}\right) \xrightarrow{d} \mathcal{N}(0,1) \tag{29}$$

Since z_0, ξ, ξ_2 and ζ are independent and taking into account to equations (23)-(29), we obtain the following asymptotic result

$$T - \sqrt{n}\frac{S_{GMV}}{\sqrt{1+\frac{k-1}{n-k+1}\left(1+\frac{n}{k-1}s\right)}} \xrightarrow{d} \overline{z}_0 + \frac{S_{GMV}}{\sqrt{\frac{1+s}{1-c}}}\frac{\overline{z}_1}{2} + \frac{S_{GMV}}{\sqrt{\frac{1+s}{1-c}}}\left(\frac{c/(1-c)\left(\overline{z}_2+\overline{z}_3\right)}{2\frac{1+s}{1-c}}\right)$$

where $\overline{z}_0 \sim \mathcal{N}(0,1)$, $\overline{z}_1 \sim \mathcal{N}\left(0, \frac{2}{1-c}\right)$, $\overline{z}_2 \sim \mathcal{N}\left(0, \frac{2}{1-c}\left(1+\frac{s}{c}\right)^2\right)$ and $\overline{z}_3 \sim \mathcal{N}\left(0, \frac{2}{c}\left(1+2\frac{s}{c}\right)\right)$. Moreover, $\overline{z}_0, \overline{z}_1, \overline{z}_2$ and $\overline{z}_4$ are independently distributed.

Finally, the application of Slutsky's lemma (see, e.g., Theorem 2.8 in Van der Vaart (2000)) leads to

$$\sigma_T^{-1}\left(T - \sqrt{n}\frac{S_{GMV}}{\sqrt{1+\frac{k-1}{n-k+1}\left(1+\frac{n}{k-1}s\right)}}\right) \xrightarrow{d} \mathcal{N}(0,1)$$

with

$$\begin{aligned}
\sigma_T^2 &= 1 + \frac{S_{GMV}^2}{1+s}(1-c)\left(\frac{1}{2(1-c)} + \frac{(1-c)^2}{4(1+s)^2}\left(\frac{c^2}{(1-c)^2}\left(\frac{2}{1-c}\left(1+\frac{s}{c}\right)^2 + \frac{2}{c}\left(1+2\frac{s}{c}\right)\right)\right)\right) \\
&= 1 + \frac{S_{GMV}^2}{1+s}(1-c)\left(\frac{1}{2(1-c)} + \frac{s^2+2s+c}{2(1+s)^2(1-c)}\right) \\
&= 1 + \frac{S_{GMV}^2}{1+s}\left(\frac{1}{2} + \frac{s^2+2s+c}{2(1+s)^2}\right)
\end{aligned}$$

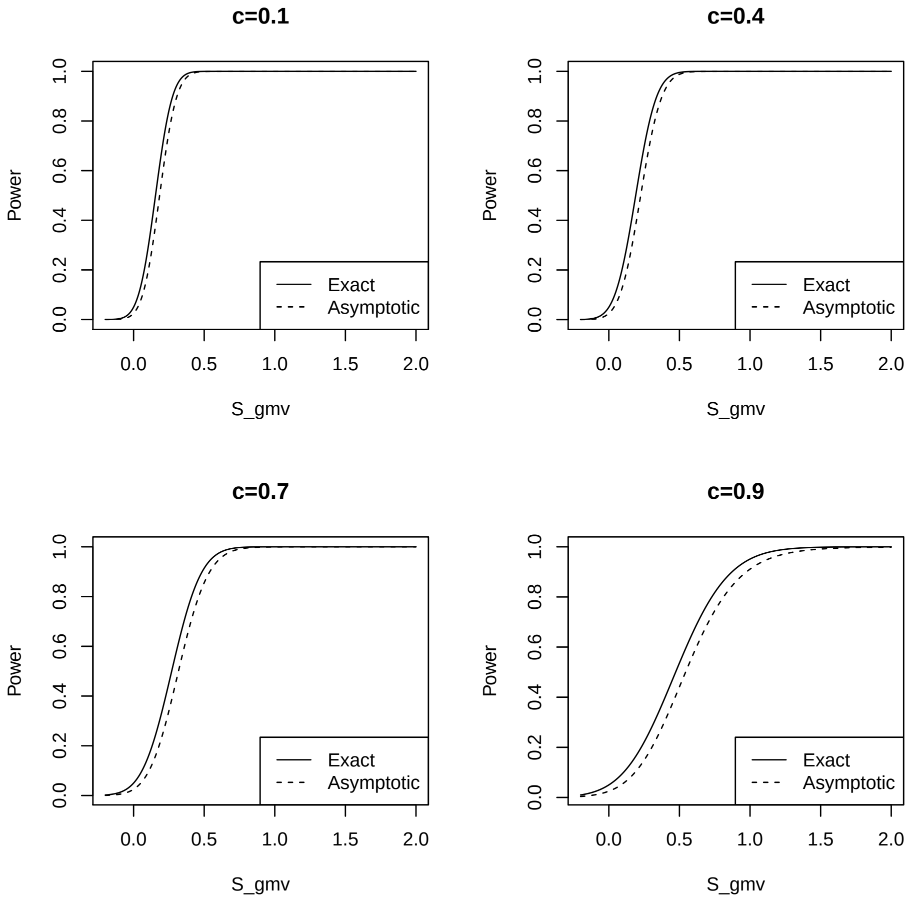

FIGURE 4. Asymptotic power versus Exact power for different values of c as a function of S_{GMV}, significance level 5% and $n = 250$. We set $s = 1$.

The statement of Theorem 2(b) follows by setting $S_{GMV} = 0$ under the null hypothesis. □

From Theorem 2, we obtain the asymptotic expression of the power which is given by

$$G_T(s, S_{GMV}) = P\left(\frac{T - \sqrt{n}\frac{S_{GMV}}{\sqrt{1+\frac{k-1}{n-k+1}(1+\frac{n}{k-1}s)}}}{\sigma_T} > z_{1-\alpha}\right) = 1 - P\left(\frac{T - \sqrt{n}\frac{S_{GMV}}{\sqrt{1+\frac{k-1}{n-k+1}(1+\frac{n}{k-1}s)}}}{\sigma_T} < z_{1-\alpha}\right)$$

$$\approx 1 - \Phi\left(\frac{T - \sqrt{n}\frac{S_{GMV}}{\sqrt{1+\frac{k-1}{n-k+1}(1+\frac{n}{k-1}s)}}}{\sigma_T}\right)$$

In Figures 4 and 5, we deliver the results of the power functions of the exact test and of the high-dimensional asymptotic test that was obtained in Theorem 2 for different values of $c \in \{0.1, 0.4, 0.7, 0.9\}$

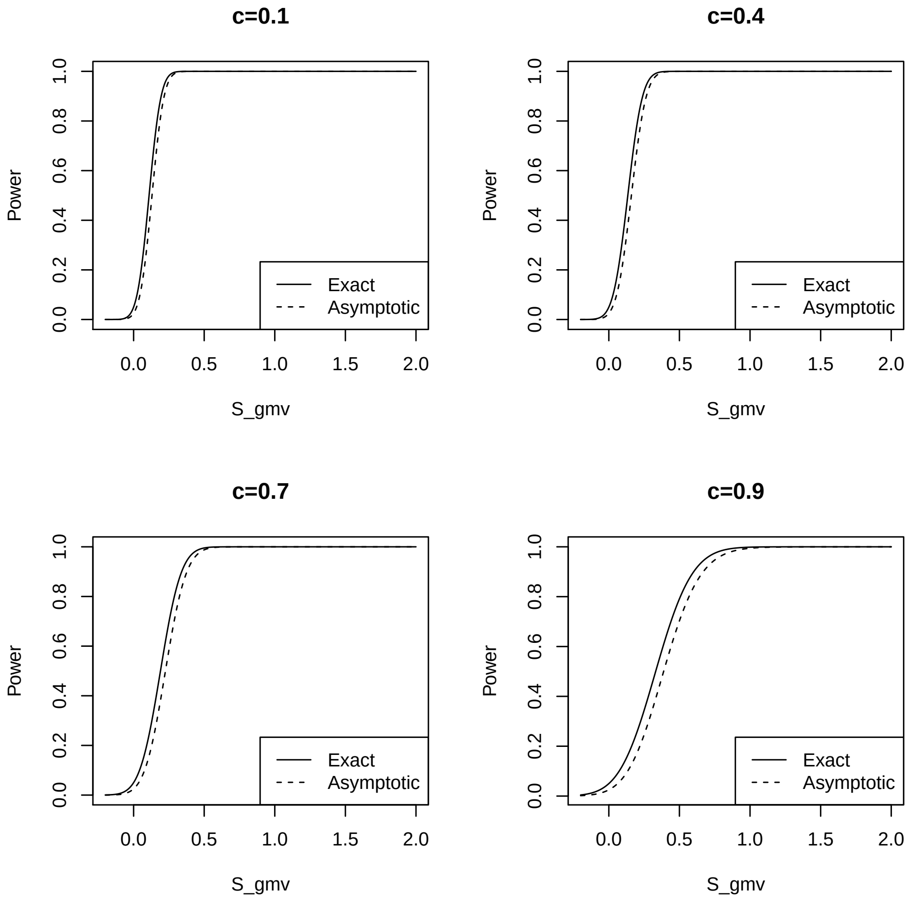

FIGURE 5. Asymptotic power versus Exact power for different values of c as a function of S_{GMV}, significance level 5% and $n = 500$. We set $s = 1$.

and $\alpha = 5\%$. Since the power of the test depends on s and S_{GMV}, for a good visualization of the power we fix $s = 1$. The dashed black line represents the power function of the exact test, while the power function of the high-dimensional test is indicated by a solid black line. A good performance of the asymptotic power is observed for all considered values of c.

3. Comparison of the tests

In this section we examine the performance of developed test by comparing their asymptotic powers and we also study how robust are the two approaches to the violation of normality assumption. we firstly compare the performance of the power function of the test derived in Section 2.1 and the power function of the statistical test developed in Muhinyuza et al. (2017) for $k/n \to c \in (0, 1)$ as $n \to \infty$. Results of both Theorem 2 and Muhinyuza et al. (2017, Proposition 1) show that the power functions of the tests depend on the mean vector and the covariance matrix only through the slope parameter s of the efficient frontier and the Sharpe ratio S_{GMV} of the GMVP. For that reason, we put $\mathbf{\Sigma} = \boldsymbol{I}_k$, an identity matrix of dimension k in the simulation study, and consider several values of μ chosen as follows:

- for μ_1, 15% of its first values are 0.1 and the remaining values are set to zero;
- for μ_2, 20% of its first values are 0.1 and the remaining values are set to zero;
- for μ_3, 25% of its first values are 0.1 and the remaining values are set to zero;
- for μ_4, 30% of its first values are 0.1 and the remaining values are set to zero;
- for μ_5, 35% of its first values are 0.1 and the remaining values are set to zero.

In order to have the equality $\boldsymbol{\mu} = r_f\mathbf{1}$, we choose $\boldsymbol{\mu}_0$ with all its components equal to 0.01. Table 1 contains several values of s and S_{GMV} obtained using the aforementioned values of μ and Σ. We note that the values with $S_{GMV} = 0$ corresponds to the null hypothesis in (11) and expect the empirical significance level of the test obtained through-out the simulation to be the nominal significance level 0.05. Whereas the other values designate the alternative hypothesis. In addition, we set the risk-free rate to be 0.01 and the portfolio size is $k \in \{50, 200, 350, 450\}$. We observe that the slope parameter s becomes larger as k increases, while the Sharpe ratio S_{GMV} increases when the number of non-zero elements in the mean vector becomes larger, the increase is also noted when the portfolio size gets larger. On the other hand, the values with $\lambda = 0$ are equivalent to the null hypothesis in (5)(which occurs only when $\boldsymbol{\mu} = r_f\mathbf{1}$) while the other values favours the alternative hypothesis. We also note that λ grows as either k or the number of non-zero elements in the mean vector become larger.

k		μ_0	$\boldsymbol{\mu}_1$	$\boldsymbol{\mu}_2$	$\boldsymbol{\mu}_3$	$\boldsymbol{\mu}_4$	$\boldsymbol{\mu}_5$
	s	0.0000	0.0672	0.0800	0.0962	0.1050	0.1152
50	λ	0.0000	0.0690	0.0850	0.1090	0.1250	0.1490
	S_{GMV}	0.0000	0.0424	0.0707	0.1131	0.1414	0.1838
	s	0.0000	0.2550	0.3200	0.3750	0.4200	0.4550
200	λ	0.0000	0.2600	0.3400	0.4200	0.500	0.5800
	S_{GMV}	0.0000	0.0707	0.1414	0.2121	0.2828	0.3536
	s	0.0000	0.4497	0.5600	0.6587	0.7350	0.7977
350	λ	0.0000	0.4590	0.5950	0.7390	0.8750	1.0190
	S_{GMV}	0.0000	0.0962	0.1871	0.2833	0.3742	0.4704
	s	0.0000	0.5772	0.7200	0.8462	0.9450	1.0252
450	λ	0.0000	0.5890	0.7650	0.9490	1.1250	1.3090
	S_{GMV}	0.0000	0.1084	0.2121	0.3206	0.4243	0.5327

TABLE 1. Slope parameters s, λ and Sharpe ratio S_{GMV} for the portfolio dimension $k \in \{50, 200, 350, 450\}$ and several values of μ.

	Power	T	T_λ	T	T_λ	T	T_λ	T	T_λ	T	T_λ	T	T_λ
k	Distribution	$\boldsymbol{\mu}_0$		$\boldsymbol{\mu}_1$		$\boldsymbol{\mu}_2$		$\boldsymbol{\mu}_3$		$\boldsymbol{\mu}_4$		$\boldsymbol{\mu}_5$	
	Normal	0.0420	0.0400	0.2260	0.8230	0.4240	0.9190	0.7680	0.9810	0.8760	0.9940	0.9830	1.0000
50	t_5	0.0270	0.0390	0.1480	0.7460	0.3390	0.8630	0.6060	0.9570	0.7860	0.9830	0.9510	0.9940
	t_{10}	0.0420	0.0500	0.1770	0.8040	0.3900	0.9060	0.7020	0.9690	0.8630	0.9880	0.9710	0.9940
	Normal	0.0480	0.0460	0.3020	0.9920	0.6760	1.0000	0.9310	1.0000	0.9950	1.0000	1.0000	1.0000
200	t_5	0.0140	0.0330	0.1270	0.9730	0.4730	0.9970	0.7770	1.0000	0.9520	1.0000	0.9920	1.0000
	t_{10}	0.0290	0.0540	0.2210	0.9870	0.5700	0.9970	0.8810	1.0000	0.9800	1.0000	0.9980	1.0000
	Normal	0.0480	0.0500	0.2710	0.9810	0.5850	1.0000	0.8500	1.0000	0.9560	1.0000	0.9930	1.0000
350	t_5	0.0130	0.0420	0.1090	0.9540	0.2820	0.9910	0.5870	0.9980	0.8050	1.0000	0.9450	1.0000
	t_{10}	0.0240	0.0450	0.1690	0.9640	0.4370	0.9960	0.7460	1.0000	0.9140	1.0000	0.9840	1.0000
	Normal	0.0420	0.049 0	0.1280	0.7350	0.2900	0.9020	0.4930	0.9610	0.6700	0.9840	0.8230	0.9980
450	t_5	0.0160	0.0480	0.0740	0.6500	0.1460	0.7670	0.2850	0.8750	0.4400	0.9370	0.6310	0.9770
	t_{10}	0.0440	0.0500	0.1140	0.6790	0.2310	0.8390	0.3650	0.9120	0.5620	0.9690	0.7010	0.9910

TABLE 2. Power function for the portfolio dimension $k \in \{50, 200, 350, 450\}$ and the sample size $n = 500$. The nominal significance level of the test is $\alpha = 0.05$.

In Table 2, we present the results from the simulation study of the powers of the tests for the hypotheses (5) and (11) for different values of $k \in \{50, 200, 350, 450\}$ and $n = 500$. Each value of the power function given in the table was obtained by generating 10^6 independent sample from the corresponding model. All obtained results show a good performance even for data generated from heavy tailed t-distribution. Moreover, we observe an increase of the power functions a the number of non-zero elements of the mean vector becomes larger. To this end, we note that the asymptotic power function of T_λ grows faster compared to the asymptotic power function of T. We also note that for both hypotheses, our tests are moderately conservative when data are generated from a heavy-tailed distribution, since the powers obtained under the $t-$distribution are in all cases smaller than the one obtained for the normal distribution. We also observe that the powers for T and T_λ are not larger that the nominal significance level of the tests, in all cases where $S_{GMV} = 0$ and $\lambda = 0$, respectively. This behaviour is maintained if data are generated from the normal distribution or from the t-distribution.

Moreover, in order to compare the two tests, we first establish the relation between λ, s and S_{GMV} as it can be seen in the next lemma.

Lemma 1. *Under the assumption of of Proposition 1, we have that* $\lambda = s + S_{GMV}^2$.

Proof of Lemma 1 .

$$
\begin{aligned}
\lambda &= (\mu - r_f\mathbf{1})'\mathbf{\Sigma}^{-1}(\mu - r_f\mathbf{1}) \\
&= \mu'\mathbf{\Sigma}^{-1}\mu - 2r_f\mathbf{1}'\mathbf{\Sigma}^{-1}\mu + r_f^2\mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{1} \\
&= \mu'\mathbf{\Sigma}^{-1}\mu - \mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{1}\left(2r_f\frac{\mathbf{1}'\mathbf{\Sigma}^{-1}\mu}{\mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{1}} - r_f^2\right) \\
&= \mu'\mathbf{\Sigma}^{-1}\mu - \mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{1}\left(\left(\frac{\mathbf{1}'\mathbf{\Sigma}^{-1}\mu}{\mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{1}}\right)^2 - \left(\frac{\mathbf{1}'\mathbf{\Sigma}^{-1}\mu}{\mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{1}}\right)^2 + 2r_f\frac{\mathbf{1}'\mathbf{\Sigma}^{-1}\mu}{\mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{1}} - r_f^2\right) \\
&= \mu'\mathbf{\Sigma}^{-1}\mu - \mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{1}\left(\frac{\mathbf{1}'\mathbf{\Sigma}^{-1}\mu}{\mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{1}}\right)^2 + \mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{1}\left(\left(\frac{\mathbf{1}'\mathbf{\Sigma}^{-1}\mu}{\mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{1}}\right)^2 - 2r_f\frac{\mathbf{1}'\mathbf{\Sigma}^{-1}\mu}{\mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{1}} + r_f^2\right) \\
&= \mu'\mathbf{\Sigma}^{-1}\mu - \frac{(\mathbf{1}\mathbf{\Sigma}^{-1}\mu)^2}{\mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{1}} + \mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{1}\left(\frac{\mathbf{1}\mathbf{\Sigma}^{-1}\mu}{\mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{1}} - r_f\right)^2
\end{aligned}
$$

Lemma 1 is established. □

Lemma 1 helps us to compute the difference of two power functions given below

$$\Delta(s, S_{GMV}) = G_{T_\lambda}(s, S_{GMV}) - G_T(s, S_{GMV}). \tag{30}$$

To clearly visualize the difference of these two powers a contour-plot was used. In Figure 6, it is seen that the difference becomes smaller as the concentration ratio gets larger. It also clearly shows that: when s is small and S_{GMV} is large, then the test based on T_λ performs better. On the other hand, if s is large and S_{GMV} is small, then the test T is preferable.

4. Conclusion

In this paper we focus on the property of the TP in high dimension. Especially, we provide the high dimensional asymptotic distribution of the test statistic for testing the existence of the EF and for testing the efficiency of the TP under high-dimensional regime. In either test the asymptotic distribution is obtained under the null and alternative hypotheses. With an extensive simulation study, we have shown that both tests are robust to the violation of the normality assumption and perform well for heavy-tailed t-distribution.

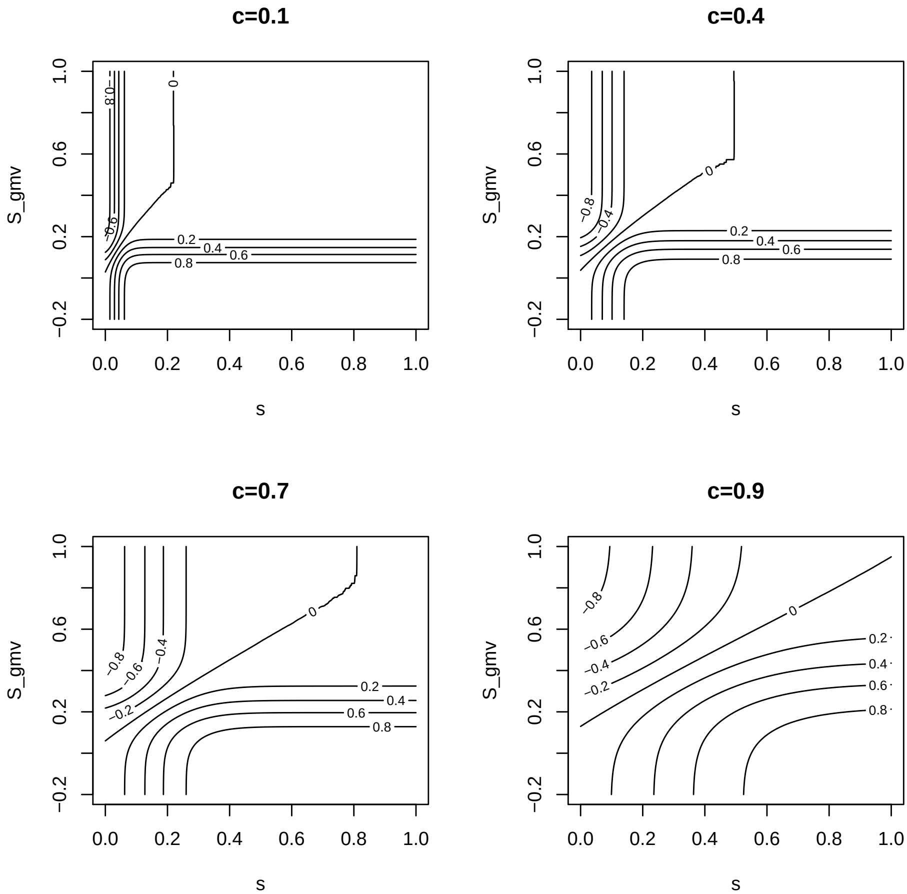

FIGURE 6. Contour plot of $\Delta(s, S_{GMV})$ for $c \in \{0.1, 0.4, 0.7, 0.9\}$ and $n = 500$.

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III

Statistical Inference for the Tangency Portfolio in High Dimension

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Abstract

In this paper, we study the distributional properties of the tangency portfolio (TP) weights assuming a normal distribution of the logarithmic returns. We derive a stochastic representation of the TP weights that fully describes their distribution. Under a high-dimensional asymptotic regime, i.e. the dimension of the portfolio, k, and the sample size, n, approach infinity such that $k/n \to c \in (0, 1)$, we deliver the asymptotic distribution of the TP weights. Moreover, we consider test about the elements of the TP and derive the asymptotic distribution of the test statistic under the null and alternative hypotheses. In a simulation study, we compare the asymptotic distribution of the TP weights with the exact finite sample density. We also compare the high-dimensional asymptotic test with exact one. We document a good performance of the asymptotic approximations except for small sample sizes combined with c close to one. In an empirical study, we analyze the TP weights in portfolios containing stocks from the S&P 500 index.

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Keywords: Tangency portfolio, high-dimensional asymptotics, hypothesis testing.

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1 Introduction

The fundamental goal of the portfolio theory introduced by Markowitz (1952) is to efficiently allocate investments among various assets. The mean-variance optimization technique serves as a quantitative tool that considers the trade-off between the risk of the portfolio and its return. In the formulation of the mean-variance analysis the investor selects a portfolio with the highest expected return for a given level of risk or the smallest risk for a given level of the expected return. The risk aversion strategy in the absence of risk-free assets (bonds) leads to the minimum variance portfolio, whereas in the presence of risk-free assets, the tangency portfolio (TP) is constructed and it consists of both risky and risk-free assets. Moreover, it is the only portfolio that maximizes the Sharpe ratio. Because of its significant role in finance for both researchers and practitioners, having a full understanding of the properties of the TP becomes vital for any financial actor.

The statistical properties of the estimated TP weights are investigated in a number of papers. Britten-Jones (1999) developed an exact finite sample F-test for TP weights. Okhrin and Schmid (2006), under the assumption of independently and multivariate normally distributed returns, derived the univariate density of the TP weights as well as its asymptotic distribution. Bodnar (2009) proposed a sequential monitoring procedures for the TP weights, while Bodnar and Okhrin (2011) suggested several exact test of general linear hypotheses about the elements of the portfolio weights. In Kotsiuba and Mazur (2015), the asymptotic distribution and the approximate density function, based on a third order Taylor series approximation, of the TP weights are derived. A test of the existence of TP on the set of feasible portfolio is developed by Muhinyuza et al. (2017). Bodnar and Zabolotskyy (2017) considered the risk properties of the TP and concluded that this portfolio is a very risky investment opportunity which should be carefully considered in practice. Bauder et al. (2018) studied different distributional properties of TP weights from Bayesian statistics point of view. Bodnar et al. (2019b) analyzed the distributional properties of the estimated TP weights and proposed inference procedures in small and high dimensions when both the population and the sample covariance matrices are singular. More recently, higher-order moments of the estimated TP weights are obtained by Javed et al. (2020).

The present paper complements the existing literature by delivering the stochastic representation and asymptotic distribution of the estimated TP weights as well as the asymptotic distribution of the statistical test about the elements of the TP. Asymptotic results are delivered under a high-dimensional asymptotic regime, i.e. $k/n \to c \in (0, 1)$ as $k \to \infty$ and $n \to \infty$, and assuming positive definiteness of the population covariance matrix.

The remaining parts of this paper are organized as follow. In Section 2, we present a very useful stochastic representation of the estimated TP weights that depicts their distribution. The obtained stochastic representation is then used in the derivation of the high-dimensional asymptotic distribution of the estimated TP weights and high-dimensional asymptotic test on the TP weights. In Section 3, we present the results of the simulation and empirical studies, while the summary and concluding remarks are given in Section 4.

2 Main Results

We consider a portfolio that consists of k risky assets. Let $\mathbf{x}_t = (x_{1t}, \ldots, x_{kt})^T$ be the k-dimensional vector of log-returns of these assets at time point $t = 1, \ldots, n$, and $\mathbf{w}_t = (w_1, \ldots, w_k)^T$ be a vector of weights, where w_i denotes the portion of the wealth allocated to the ith asset. Let also the mean vector of the asset returns be denoted by $\boldsymbol{\mu}$ and the covariance matrix by $\mathbf{\Sigma}$ which assumed to be positive definite. Following the mean-variance theory introduced by Markowitz (1952), an investor allocates her/his wealth among k risky assets by maximizing the portfolio expected return for a given level of the portfolio risk or, equivalently, by minimizing the risk given some predetermined level of the portfolio expected return. In this context, the risk is measured by the variance of the portfolio return. Levy and Markowitz (1979) and Kroll et al. (1984) showed that the mean-variance portfolio problem is equivalent to maximizing the expected quadratic utility. In the absence of a risk-free asset, the optimal portfolio is obtained by maximizing the expected quadratic utility under the constraint $\mathbf{w}^T\mathbf{1}_k = 1$, where $\mathbf{1}_k$ denotes the vector of ones. On the other hand, if short selling is allowed and there is a possibility to invest in the risk free-asset with return r_f, a portion of an investor's wealth may be invested in the risk-free asset and it may reduce the variance, while the rest of the wealth can be invested in the risky assets. In this case, the expected return of the risky assets is given by $\mu_p = \mathbf{w}^T(\boldsymbol{\mu} - r_f\mathbf{1}_k) + r_f$ with the variance $\sigma_p^2 = \mathbf{w}^T\mathbf{\Sigma}\mathbf{w}$. The optimal portfolio composition of the tangency portfolio (TP) is obtained by solving the following optimization problem

$$\mu_p - \frac{\alpha}{2}\sigma_p^2 \to \max_{\mathbf{w}} \tag{1}$$

where the coefficient α describes the investor's attitude towards risk or risk aversion. All portfolios from the tangent line are obtained by varying $\alpha \in (0, \infty)$. The higher value of the risk aversion representing lesser tolerance to risk. The risk aversion level can be looked as a characteristic of the investor's indifference curve which represents the investor's preference for risk and return. How to choose or fix the value of α in practice is not obvious and a number of papers have suggested different approaches to estimating the risk aversion coefficient (see, e.g.,Chetty (2003); Campo et al. (2011); Bodnar and Okhrin (2013); Bodnar et al. (2018b)).

When solving the maximization problem defined in (1), we note that short sales are allowed and there are no restrictions on the portfolio weights, therefore, the optimization problem is unconstrained. Consequently, it is easy to see that the global maximum, i.e.

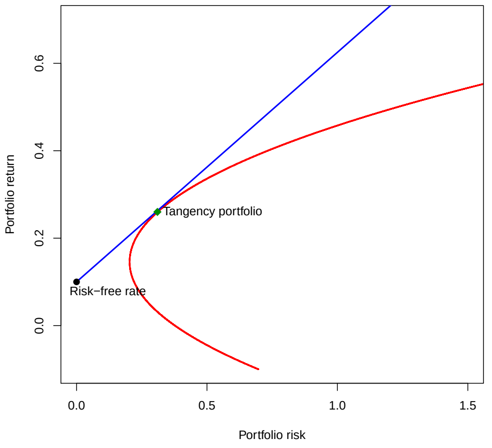


Figure 1: A graphical illustration of the efficient frontier in the presence of risk-free asset.

the TP weights, is given by

$$\mathbf{w}_{TP} = \alpha^{-1}\boldsymbol{\Sigma}^{-1}\left(\boldsymbol{\mu} - r_f\mathbf{1}_k\right). \tag{2}$$

Equation (2) gives the structure of the optimal portfolio composition corresponding to the risky assets only, whereas the portion invested into the risk-free asset is determined by $1-\mathbf{w}_{TP}^T\mathbf{1}_k$. Ingersoll (1987) defined a TP as a tangent point which lies on the intersection of the mean-variance frontier and the tangency line drawn from the return of the risk-free asset (see Figure 1).

The optimal portfolio weights depend on the unknown parameters $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ and in practice they need to be estimated. Using the random sample we estimate the parameters by their sample counterparts as

$$\overline{\mathbf{x}} = \frac{1}{n}\sum_{t=1}^{n}\mathbf{x}_t \qquad \text{and} \qquad \mathbf{S} = \frac{1}{n-1}\sum_{t=1}^{n}\left(\mathbf{x}_t - \overline{\mathbf{x}}\right)\left(\mathbf{x}_t - \overline{\mathbf{x}}\right)^T.$$

Replacing $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ with $\overline{\mathbf{x}}$ and $\mathbf{S}$ in (2), we get the sample estimator of the TP weights given by

$$\hat{\mathbf{w}}_{TP} = \alpha^{-1}\mathbf{S}^{-1}\left(\overline{\mathbf{x}} - r_f\mathbf{1}_k\right). \tag{3}$$

In practice, interest is often focused on just a few weights. In addition, analysis of the whole vector becomes impractical as the dimensions k increases. We will hence focus on the linear combination of $\mathbf{w}_{TP}$ that is given by

$$\theta = \mathbf{l}^T \mathbf{w}_{TP} = \alpha^{-1} \mathbf{l}^T \boldsymbol{\Sigma}^{-1} \left(\boldsymbol{\mu} - r_f \mathbf{1}_k \right),$$

where $\mathbf{l}$ is a k-dimensional vector of constants. Consequently, the sample estimator of θ is expressed as

$$\hat{\theta} = \mathbf{l}^T \hat{\mathbf{w}}_{TP} = \alpha^{-1} \mathbf{l}^T \mathbf{S}^{-1} \left(\overline{\mathbf{x}} - r_f \mathbf{1}_k \right).$$

By choosing different vectors $\mathbf{l}$ we are able to provide information about different linear combinations of the TP weights and more insights into the behaviour of the TP. For example, by choosing $\mathbf{l} = (1, 0, \ldots, 0)^T$, an investor is able to study the behaviour of the first asset in the portfolio. Similarly, if $\mathbf{l} = (1, 1, 0, \ldots, 0)^T$ an investor is interested in the behaviour of the TP weights the two first assets of the portfolio. Taking $\mathbf{l} = \mathbf{1}_k$ an investor can study the share of the portfolio invested in risky assets.

In the following proposition, we derive a stochastic representation of $\hat{\theta}$. The stochastic representation is a powerful tool in the theory of multivariate statistics, it can be used to determine the distribution of random quantity as the distribution of functions of independent random variables with the standard probability distributions. It also plays an important role in both frequentist and Bayesian statistics (see Givens and Hoeting (2012), Bodnar et al. (2017a), Bauder et al. (2018)). Its usefulness is frequently remarkable in Monte Carlo simulations (Givens and Hoeting (2012)) as well as in elliptical contoured distributions (Gupta et al. (2013)).

Proposition 1. *Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ be identically and independently distributed random vectors with $\mathbf{x}_1 \sim \mathcal{N}_k(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, $k < n$. Also, let $\mathbf{l}$ be a k-dimensional vector of constants. Then the stochastic representation of $\hat{\theta} = \mathbf{l}^T \hat{\mathbf{w}}_{TP}$ is given by*

$$\hat{\theta} \stackrel{d}{=} \frac{n-1}{\xi} \left(\theta + \alpha^{-1} z_0 \sqrt{\left(\frac{1}{n} + \frac{k-1}{n(n-k+1)} u \right) \mathbf{l}^T \boldsymbol{\Sigma}^{-1} \mathbf{l}} \right),$$

where $\xi \sim \chi^2_{n-k}$, $z_0 \sim \mathcal{N}(0,1)$ and $u \sim \mathcal{F}(k-1, n-k+1, ns)$ with $s = (\boldsymbol{\mu} - r_f \mathbf{1}_k)^T \mathbf{R}_{\mathbf{l}} (\boldsymbol{\mu} - r_f \mathbf{1}_k)$ and $\mathbf{R}_{\mathbf{l}} = \boldsymbol{\Sigma}^{-1} - \boldsymbol{\Sigma}^{-1} \mathbf{l} \mathbf{l}^T \boldsymbol{\Sigma}^{-1} / \mathbf{l}^T \boldsymbol{\Sigma}^{-1} \mathbf{l}$. Moreover, the random variables ξ, z_0 and u are mutually independently distributed.

Proof of Proposition 1. From Theorem 3.1.2 of Muirhead (1982), it follows that

$$\overline{\mathbf{x}} \sim \mathcal{N}_k \left(\boldsymbol{\mu}, \frac{1}{n} \boldsymbol{\Sigma} \right) \qquad \text{and} \qquad (n-1)\mathbf{S} \sim \mathcal{W}_k(n-1, \boldsymbol{\Sigma}),$$

where $\mathcal{W}_k(n-1, \boldsymbol{\Sigma})$ denotes a k-dimensional Wishart distribution with $n-1$ degrees of freedom and the parameter matrix $\boldsymbol{\Sigma}$. Moreover, $\overline{\mathbf{x}}$ and $(n-1)\mathbf{S}$ are independent. Applying Corollary 1 of Bodnar and Okhrin (2011), we get the statement of the proposition. $\square$

From Proposition 1, we have a stochastic representation of $\hat{\theta}$ given as a function of independently distributed χ^2, standard normal and non-central $\mathcal{F}$ random variables. It is worth noting that the application of Proposition 1 speeds up the simulation of $\hat{\theta}$, since it is sufficient to simulate only three univariate random quantities instead of generating a sample mean vector and sample covariance matrix that can have high dimensions. Proposition 1 also plays an important role in the derivation of the distribution of the linear combination of the estimated TP weights in high dimensions.

Remark 1. According to (3), the sample estimator of the TP weights $\hat{\mathbf{w}}_{TP}$ depends on the inverse of the sample covariance matrix $\mathbf{S}$. In Proposition 1, it is assumed that $k < n$ and this assumption ensures that the distribution of $\mathbf{S}$ is non-singular, therefore, the regular inverse of $\mathbf{S}$ can be taken. If $k > n$, the distribution of $\mathbf{S}$ is singular and the regular inverse cannot be used. This issue is discussed in the portfolio context by utilizing Moore-Penrose inverse (see Bodnar et al. (2016, 2017b), Tsukuma (2016), Bodnar et al. (2019b)). Alternatively, one can use different regularization techniques such as the ridge-type approach (Tikhonov and Arsenin (1977)), the Landweber Fridman iterative algorithm (Kress (1999)), the spectral cut-off (Chernousova and Golubev (2014)), a form of Lasso (Brodie et al. (2009)), and an iterative method based on a second order damped dynamical systems (Gulliksson and Mazur (2019)).

Remark 2. In the Bayesian setting, the posterior distribution of the covariance matrix $\mathbf{\Sigma}$ has the inverse Wishart distribution. Consequently, the posterior distribution of $\mathbf{w}_{TP}$ can be expressed as the product of the (singular) Wishart matrix and a normal vector. The distributional properties of this product are well studied by Bodnar et al. (2013, 2014), Bodnar et al. (2018a), Bodnar et al. (2019a).

Next, we study the asymptotic behaviour of $\hat{\theta} = \mathbf{l}^T \hat{\mathbf{w}}_{TP}$ under a high-dimensional asymptotic regime, that is, the portfolio size k increases together with the sample sizes n and they all tend to infinity. More precisely, we assume that $k_n \equiv k(n)$ and $c_n := k_n/n \to c \in (0, 1)$ as $k \to \infty$ and $n \to \infty$. The following condition is needed for ensuring the validity of the asymptotic results presented in this section:

(A1) there exists m and M such that $0 < m \leq \boldsymbol{\mu}^T \mathbf{\Sigma}^{-1} \boldsymbol{\mu} \leq M < \infty$, $0 < m \leq \mathbf{1}^T \mathbf{\Sigma}^{-1} \mathbf{1} \leq M < \infty$ and $0 < m \leq \mathbf{l}^T \mathbf{\Sigma}^{-1} \mathbf{l} \leq M < \infty$ uniformly in k.

Let us note that we don't have assumptions about the eigenvalues of the population covariance matrix $\mathbf{\Sigma}$. Consequently, one can consider the case when $\mathbf{\Sigma}$ has unbounded spectrum.

In the next theorem we deliver the high-dimensional asymptotic distribution of a linear combination of the estimated TP weights for normally distributed data.

Theorem 1. *Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ be identically and independently distributed random vectors with $\mathbf{x}_1 \sim \mathcal{N}_k(\boldsymbol{\mu}, \mathbf{\Sigma})$, $k < n$. Let $c_n := k_n/n \to c \in (0, 1)$ as $n \to \infty$. Also, let $\mathbf{l}$*

be a k-dimensional vector of constants. Then, under (A1), it holds that the asymptotic distribution of $\hat{\theta} = \mathbf{l}^T \hat{\mathbf{w}}_{TP}$ is given by

$$\sqrt{n-k_n}\left(\hat{\theta} - \frac{n-1}{n-k_n}\theta\right) \xrightarrow{\mathcal{D}} \mathcal{N}(0,\sigma^2)$$

where

$$\sigma^2 = \frac{\alpha^{-2}}{(1-c)^2}\left[\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l} + (\alpha\theta)^2 + \mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l}(\boldsymbol{\mu} - r_f\mathbf{1}_k)^T\mathbf{\Sigma}^{-1}(\boldsymbol{\mu} - r_f\mathbf{1}_k)\right].$$

Proof of Theorem 1. Using the stochastic representation obtained in Proposition 1, we get

$$\begin{aligned}\hat{\theta} - \frac{n-1}{n-k_n}\theta &= \frac{n-1}{\xi}\theta + \alpha^{-1}\frac{n-1}{\xi}\frac{z_0}{\sqrt{n}}\sqrt{\left(1+\frac{k_n-1}{n-k_n+1}u\right)\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l}} - \frac{n-1}{n-k_n}\theta \\ &= \frac{n-1}{n-k_n}\left(\frac{n-k_n}{\xi} - 1\right)\theta + \alpha^{-1}\frac{n-1}{\xi}\frac{z_0}{\sqrt{n}}\sqrt{\left(1+\frac{k_n-1}{n-k_n+1}u\right)\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l}},\end{aligned}$$

where $\xi \sim \chi^2_{n-k}$, $z_0 \sim \mathcal{N}(0,1)$ and $u \sim \mathcal{F}(k-1, n-k+1, ns)$ with $s = (\boldsymbol{\mu} - r_f\mathbf{1}_k)^T\mathbf{R}_\mathbf{l}(\boldsymbol{\mu} - r_f\mathbf{1}_k)$ and $\mathbf{R}_\mathbf{l} = \mathbf{\Sigma}^{-1} - \mathbf{\Sigma}^{-1}\mathbf{l}\mathbf{l}^T\mathbf{\Sigma}^{-1}/\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l}$. Let us also note that ξ, u and z_0 are mutually independently distributed.

Since $\xi \sim \chi^2_{n-k}$, the application of Lemma 3 in Bodnar and Reiß (2016) leads us to

$$\frac{\xi}{n-k_n} - 1 \xrightarrow{a.s.} 0 \qquad \text{and} \qquad \sqrt{n-k_n}\left(\frac{\xi}{n-k_n} - 1\right) \xrightarrow{\mathcal{D}} \mathcal{N}(0,2) \tag{4}$$

for $k_n/n \to c \in (0,1)$ as $n \to \infty$.

We also have that

$$\sqrt{n-k_n}\frac{z_0}{\sqrt{n}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, 1-c). \tag{5}$$

Using the stochastic representation of a non-central $\mathcal{F}$-distributed random variable, i.e. $u = \frac{\zeta_1/(k_n-1)}{\zeta_2/(n-k_n+1)}$ with independent variables $\zeta_1 \sim \chi^2_{k_n-1,ns}$ and $\zeta_2 \sim \chi^2_{n-k_n+1}$, we obtain that

$$\begin{aligned}u - 1 - \frac{ns}{k_n-1} &= \frac{\zeta_1/(k_n-1)}{\zeta_2/(n-k_n+1)} - 1 - \frac{ns}{k_n-1} \\ &= \frac{n-k_n+1}{\zeta_2}\left[\left(\frac{\zeta_1}{k_n-1} - 1 - \frac{ns}{k_n-1}\right) - \left(1+\frac{ns}{k_n-1}\right)\left(\frac{\zeta_2}{n-k_n+1} - 1\right)\right].\end{aligned}$$

From Lemma 3(a) in Bodnar and Reiß (2016) and using the assumption (A1), we have that

$$\frac{\zeta_1}{k_n-1}-1-\frac{ns}{k_n-1}\stackrel{a.s.}{\to}0 \qquad \text{and} \qquad \frac{\zeta_2}{n-k_n+1}-1\stackrel{a.s.}{\to}0.$$

Consequently, it holds that

$$u-1-\frac{ns}{k_n-1}\stackrel{a.s.}{\to}0 \Rightarrow u\stackrel{a.s.}{\to}1+\frac{s}{c}.$$

Hence, we get

$$\sqrt{\left(1+\frac{k_n-1}{n-k_n+1}u\right)\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l}} \stackrel{a.s.}{\to} \sqrt{\frac{1+s}{1-c}\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l}} \tag{6}$$

for $k_n/n\to c\in(0,1)$ as $n\to\infty$.

We also have that

$$\frac{n-1}{\xi}=\frac{n-1}{n-k}\frac{n-k}{\xi} \stackrel{a.s.}{\to} \frac{1}{1-c} \tag{7}$$

for $k_n/n\to c\in(0,1)$ as $n\to\infty$.

Taking into account (4), (5), (6) and (7), we get

$$\begin{aligned}\sqrt{n-k_n}\left(\hat{\theta}-\frac{n-1}{n-k_n}\theta\right) &= \frac{n-1}{\xi}\theta\sqrt{n-k_n}\left(1-\frac{\xi}{n-k_n}\right)\\ &\quad+\alpha^{-1}\frac{n-1}{\xi}\sqrt{\left(1+\frac{k_n-1}{n-k_n+1}u\right)\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l}}\sqrt{n-k_n}\frac{z_0}{\sqrt{n}}\\ &\stackrel{\mathcal{D}}{\to} \frac{1}{1-c}\theta z_1+\alpha^{-1}\frac{1}{1-c}\sqrt{\frac{1+s}{1-c}\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l}}z_2\end{aligned}$$

where $z_1\sim\mathcal{N}(0,2)$ and $z_2\sim\mathcal{N}(0,1-c)$ and they are independently distributed.

Finally, the application of the properties of normal random variables leads to

$$\sqrt{n-k_n}\left(\hat{\theta}-\frac{n-1}{n-k_n}\theta\right)\stackrel{\mathcal{D}}{\to}\mathcal{N}(0,\sigma^2),$$

where

$$\sigma^2 = \frac{2}{(1-c)^2}\theta^2+\frac{1}{(1-c)^2}\alpha^{-2}\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l}+\frac{1}{(1-c)^2}\alpha^{-2}s\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l}.$$

Let us note that

$$\begin{aligned} s &= (\boldsymbol{\mu} - r_f \mathbf{1}_k)^T \mathbf{R}_{\mathbf{l}} (\boldsymbol{\mu} - r_f \mathbf{1}_k) \\ &= (\boldsymbol{\mu} - r_f \mathbf{1}_k)^T \boldsymbol{\Sigma}^{-1} (\boldsymbol{\mu} - r_f \mathbf{1}_k) - (\boldsymbol{\mu} - r_f \mathbf{1}_k)^T \boldsymbol{\Sigma}^{-1} \mathbf{l}\mathbf{l}^T \boldsymbol{\Sigma}^{-1} (\boldsymbol{\mu} - r_f \mathbf{1}_k) / \mathbf{l}^T \boldsymbol{\Sigma}^{-1} \mathbf{l} \end{aligned}$$

and, therefore, we get that

$$\alpha^{-2} s \mathbf{l}^T \boldsymbol{\Sigma}^{-1} \mathbf{l} = \alpha^{-2} \mathbf{l}^T \boldsymbol{\Sigma}^{-1} \mathbf{l} (\boldsymbol{\mu} - r_f \mathbf{1}_k)^T \boldsymbol{\Sigma}^{-1} (\boldsymbol{\mu} - r_f \mathbf{1}_k) - \theta^2.$$

Hence, we obtain that

$$\sigma^2 = \frac{\alpha^{-2}}{(1-c)^2} \left[\mathbf{l}^T \boldsymbol{\Sigma}^{-1} \mathbf{l} + (\alpha\theta)^2 + \mathbf{l}^T \boldsymbol{\Sigma}^{-1} \mathbf{l} (\boldsymbol{\mu} - r_f \mathbf{1}_k)^T \boldsymbol{\Sigma}^{-1} (\boldsymbol{\mu} - r_f \mathbf{1}_k) \right].$$

The statement of the theorem is proved.

$\square$

From Theorem 1, we can observe that the sample estimator of θ is biased and, therefore, bias correction can be applied. In Corollary 1, we construct an unbiased estimator of θ and deliver its central limit theorem in the high-dimensional setting. The statement of the corollary follows immediately from Theorem 1.

Corollary 1. *Let $\widetilde{\theta} = \frac{n-k_n}{n-1}\hat{\theta}$. Under the assumptions of Theorem 1, $\widetilde{\theta}$ is asymptotically unbiased with asymptotic distribution*

$$\sqrt{n-k_n} \left(\widetilde{\theta} - \theta \right) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \breve{\sigma}^2)$$

where

$$\breve{\sigma}^2 = \alpha^{-2} \mathbf{l}^T \boldsymbol{\Sigma}^{-1} \mathbf{l} + \theta^2 + \alpha^{-2} \mathbf{l}^T \boldsymbol{\Sigma}^{-1} \mathbf{l} (\boldsymbol{\mu} - r_f \mathbf{1}_k)^T \boldsymbol{\Sigma}^{-1} (\boldsymbol{\mu} - r_f \mathbf{1}_k).$$

Having established the asymptotic distribution we next consider testing the hypothesis

$$H_0 : \mathbf{l}^T \mathbf{w}_{TP} = 0 \qquad \text{against} \qquad H_1 : \mathbf{l}^T \mathbf{w}_{TP} = \rho \neq 0. \tag{8}$$

in a high dimensional setting. Bodnar and Okhrin (2011) suggested the following test statistics for (8)

$$T = \sqrt{\frac{n(n-k_n)}{n-1}} \frac{\alpha\hat{\theta}}{\sqrt{\mathbf{l}^T \mathbf{S}^{-1} \mathbf{l}} \sqrt{1 + \frac{n}{n-1}\hat{s}}},$$

where $\hat{s} = (\overline{\mathbf{x}} - r_f \mathbf{1}_k)^T \hat{\mathbf{R}}_{\mathbf{l}} (\overline{\mathbf{x}} - r_f \mathbf{1}_k)$ and $\hat{\mathbf{R}}_{\mathbf{l}} = \mathbf{S}^{-1} - \mathbf{S}^{-1}\mathbf{l}\mathbf{l}^T\mathbf{S}^{-1}/\mathbf{l}^T\mathbf{S}^{-1}\mathbf{l}$. Moreover, they delivered the distribution of T both under the null and under alternative hypotheses (see

Bodnar and Okhrin (2011, Theorem 6)). We note that the density function of the statistic T depends on the two parameters $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ only over the quantities $\lambda = \alpha\rho/\sqrt{\mathbf{l}^T\boldsymbol{\Sigma}^{-1}\mathbf{l}}$ and $s = (\boldsymbol{\mu} - r_f\mathbf{1}_k)^T\mathbf{R}_{\mathbf{l}}(\boldsymbol{\mu} - r_f\mathbf{1}_k)$. λ can be viewed as a non-centrality parameter, while s can be treated as the true value of the slope coefficient of the efficient frontier.

It also remarkable that the quantities s and λ relate in the following way

$$\begin{aligned} s &= (\boldsymbol{\mu} - r_f\mathbf{1}_k)^T\mathbf{R}_{\mathbf{l}}(\boldsymbol{\mu} - r_f\mathbf{1}_k) \\ &= (\boldsymbol{\mu} - r_f\mathbf{1}_k)^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r_f\mathbf{1}_k) - ((\boldsymbol{\mu} - r_f\mathbf{1}_k)^T\boldsymbol{\Sigma}^{-1}\mathbf{l})^2/(\mathbf{l}^T\boldsymbol{\Sigma}^{-1}\mathbf{l}) \\ &= (\boldsymbol{\mu} - r_f\mathbf{1}_k)^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r_f\mathbf{1}_k) - (\alpha\rho/\sqrt{\mathbf{l}^T\boldsymbol{\Sigma}^{-1}\mathbf{l}})^2 \\ &= (\boldsymbol{\mu} - r_f\mathbf{1}_k)^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r_f\mathbf{1}_k) - \lambda^2. \end{aligned}$$

Based on the relation between s and λ, one can see that under the null hypothesis $\lambda = 0$, hence $s = (\boldsymbol{\mu} - r_f\mathbf{1}_k)^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r_f\mathbf{1}_k)$. By construction of the test, value of ρ is defined for technical purposes only, namely to be able to present the power of the test which depends on what is specified under the alternative hypothesis. The quantity λ appears as a part of the non-centrality parameter of the non-central t-distribution and it is proportional to ρ. Both values are important if one would like to assess the values of the power function when the null hypothesis is rejected. To this end, the power of the test based on the statistic T can be considered as a function of two parameters λ and s, that is,

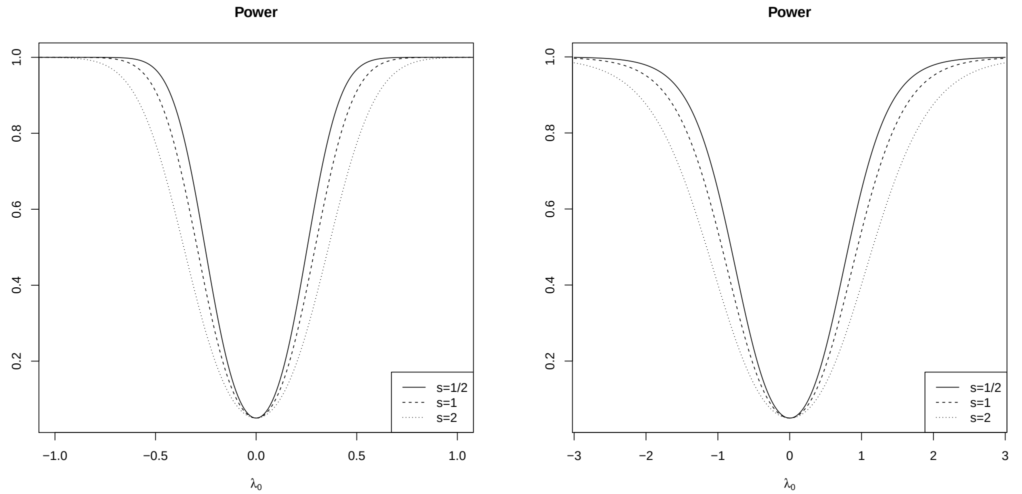
$$\begin{aligned} G_{T,\psi}(\lambda, s) &= 1 - \frac{n(n-k+1)}{(k-1)(n-1)} \int_0^\infty \Big(F_{t_{n-k,v(\lambda,y)}}(t_{n-k;1-\psi/2}) - F_{t_{n-k,v(\lambda,y)}}(t_{n-k;\psi/2}) \Big) \\ &\quad \times f_{\mathcal{F}_{k-1,n-k+1,ns}} \left(\frac{n(n-k+1)}{(k-1)(n-1)} y \right) dy, \end{aligned}$$

where ψ denotes the size of the test, $t_{n-k,v(\lambda,y)}$ stands for a non-central t-distribution with $n-k$ degrees of freedom and non-centrality parameter $v(\lambda, y) = \lambda/\sqrt{1/n + y/(n-1)}$, while $t_{n-k,\psi}$ denotes the ψ-quantile of the central t-distribution with $n-k$ degrees of freedom.

In Figure 2, we illustrate the behaviour of the power of the test statistic T as a function of λ for fixed values of $s \in \{1/2, 1, 2\}$. We consider the sample size to be $n = 100$ and portfolio size to be $k \in \{10, 90\}$. We observe that the power of the test increase as s decreases. We also note that the test rejects the null hypothesis for small values of λ.

The theorem below gives us the distribution of the test statistics T in a high dimensional setting.

Theorem 2. *Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ be identically and independently distributed random vectors with $\mathbf{x}_1 \sim \mathcal{N}_k(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, $k < n$. Let $c_n := k_n/n \to c \in (0, 1)$ as $n \to \infty$. Also, let $\mathbf{l}$ be a k-dimensional vector of constants. Then, under (A1), it holds that the asymptotic distribution of T is given by*



(a) $k = 10$, $n = 100$

(b) $k = 90$, $n = 100$

Figure 2: Power of the test statistic T as a function of λ for $s \in \{1/2, 1, 2\}$, $n = 100$ and $k \in \{10, 90\}$.

(a)

$$\left(T - \frac{\sqrt{n}\alpha\rho}{\sqrt{\mathbf{l}^T \mathbf{\Sigma}^{-1}\mathbf{l}\left(1 + \frac{k_n - 1}{n - k_n + 1}\left(1 + \frac{n}{k_n - 1}s\right)\right)}} \right) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma_T^2)$$

where

$$\sigma_T^2 = 1 + \frac{(\alpha\rho)^2}{\mathbf{l}^T \mathbf{\Sigma}^{-1}\mathbf{l}(1+s)} \left(\frac{1}{2} + \frac{s^2 + c + 2s}{2(1+s)^2} \right)$$

with $s = (\boldsymbol{\mu} - r_f \mathbf{1}_k)^T \mathbf{R}_\mathbf{l} (\boldsymbol{\mu} - r_f \mathbf{1}_k)$ *and* $\mathbf{R}_\mathbf{l} = \mathbf{\Sigma}^{-1} - \mathbf{\Sigma}^{-1}\mathbf{l}\mathbf{l}^T\mathbf{\Sigma}^{-1}/\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l}$*;*

(b) under the null hypothesis it holds that $T \xrightarrow{\mathcal{D}} \mathcal{N}(0, 1)$*.*

Proof. From the proof of Proposition 1 of Bodnar and Schmid (2009), we know that the conditional distribution of T is given by

$$\begin{aligned} T|\hat{s} = y &\sim t_{n-k_n, \upsilon(y)} \\ \frac{n(n-k_n+1)}{(k_n-1)(n-1)}\hat{s} &\sim \mathcal{F}_{k_n-1, n-k_n+1, ns} \end{aligned}$$

with $\upsilon(y) = \frac{\sqrt{n}\alpha\rho}{\sqrt{\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l}\left(1 + \frac{n}{n-1}y\right)}}$.

Using the stochastic representation of a non-central t-distribution, we obtain

$$
\begin{aligned}
T|\hat{s}=y &\stackrel{d}{=} \frac{\frac{\sqrt{n}\alpha\rho}{\sqrt{\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l}\left(1+\frac{n}{n-1}y\right)}}+z_0}{\sqrt{\frac{\xi}{n-k_n}}}\\
&= \sqrt{\frac{n-k_n}{\xi}}\left(\frac{\sqrt{n}\alpha\rho}{\sqrt{\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l}\left(1+\frac{k_n-1}{n-k_n+1}u\right)}}+z_0\right)
\end{aligned}
$$

where $z_0\sim\mathcal{N}(0,1)$, $\xi\sim\chi^2_{n-k_n}$ and $u=\frac{n(n-k_n+1)}{(k_n-1)(n-1)}\hat{s}\sim\mathcal{F}_{k_n-1,n-k_n+1,ns}$; moreover, z_0, ξ and u are independent.

We then have

$$
\begin{aligned}
&T-\frac{\sqrt{n}\alpha\rho}{\sqrt{\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l}\left(1+\frac{k_n-1}{n-k_n+1}\left(1+\frac{n}{k_n-1}s\right)\right)}}\\
&= \sqrt{\frac{n-k_n}{\xi}}\left(\frac{\sqrt{n}\alpha\rho}{\sqrt{\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l}\left(1+\frac{k_n-1}{n-k_n+1}u\right)}}+z_0\right)-\frac{\sqrt{n}\alpha\rho}{\sqrt{\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l}\left(1+\frac{k_n-1}{n-k_n+1}\left(1+\frac{n}{k_n-1}s\right)\right)}}\\
&= \sqrt{\frac{n-k_n}{\xi}}z_0+\frac{\alpha\rho}{\sqrt{\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l}\left(1+\frac{k_n-1}{n-k_n+1}u\right)}}\\
&\times\left(\sqrt{n}\left(\sqrt{\frac{n-k_n}{\xi}}-1\right)+\sqrt{n}\left(1-\frac{\sqrt{1+\frac{k_n-1}{n-k_n+1}u}}{\sqrt{\left(1+\frac{k_n-1}{n-k_n+1}\left(1+\frac{n}{k_n-1}s\right)\right)}}\right)\right).
\end{aligned}
$$

Putting the third term on a common denominator, we get

$$
1-\frac{\sqrt{1+\frac{k_n-1}{n-k_n+1}u}}{\sqrt{1+\frac{k_n-1}{n-k_n+1}\left(1+\frac{n}{k_n-1}s\right)}}=\frac{\sqrt{1+\frac{k_n-1}{n-k_n+1}\left(1+\frac{n}{k_n-1}s\right)}-\sqrt{1+\frac{k_n-1}{n-k_n+1}u}}{\sqrt{1+\frac{k_n-1}{n-k_n+1}\left(1+\frac{n}{k_n-1}s\right)}}.
$$

Multiplying the numerator of the last expression by its conjugate, we obtain

$$
\begin{aligned}
1-\frac{\sqrt{1+\frac{k_n-1}{n-k_n+1}u}}{\sqrt{1+\frac{k_n-1}{n-k_n+1}\left(1+\frac{n}{k_n-1}s\right)}} &= \frac{\frac{k_n-1}{n-k_n+1}\left(1+\frac{n}{k_n-1}s-u\right)}{\sqrt{1+\frac{k_n-1}{n-k_n+1}\left(1+\frac{n}{k_n-1}s\right)}+\sqrt{1+\frac{k_n-1}{n-k_n+1}u}}\\
&\times\frac{1}{\sqrt{1+\frac{k_n-1}{n-k_n+1}\left(1+\frac{n}{k_n-1}s\right)}}.
\end{aligned}
$$

Applying results from the proof of Theorem 1, we have that

$$u \overset{a.s.}{\to} 1+\frac{s}{c} \Rightarrow \sqrt{1+\frac{k_n-1}{n-k_n+1}u} \overset{a.s.}{\to} \sqrt{\frac{1+s}{1-c}} \tag{9}$$

and

$$\sqrt{1+\frac{k_n-1}{n-k_n+1}\left(1+\frac{n}{k_n-1}s\right)} \overset{a.s.}{\to} \sqrt{\frac{1+s}{1-c}}. \tag{10}$$

Using (9) and (10), the denominator becomes

$$\begin{aligned} &\sqrt{1+\frac{k_n-1}{n-k_n+1}\left(1+\frac{n}{k_n-1}s\right)}+\sqrt{1+\frac{k_n-1}{n-k_n+1}u}\frac{1}{\sqrt{1+\frac{k_n-1}{n-k_n+1}\left(1+\frac{n}{k_n-1}s\right)}} \\ &\overset{a.s.}{\to} 2\frac{1+s}{1-c} \end{aligned} \tag{11}$$

for $k_n/n \to c \in (0,1)$ as $n \to \infty$.

Using the stochastic representation of a non-central $\mathcal{F}$ distribution

$$u \overset{d}{=} \frac{\eta_1/(k_n-1)}{\eta_2/(n-k_n+1)}$$

with independent random variables $\eta_1 \sim \chi^2_{k_n-1,ns}$ and $\eta_2 \sim \chi^2_{n-k_n+1}$, we have that

$$\begin{aligned} 1+\frac{ns}{k_n-1}-u &\overset{d}{=} 1+\frac{ns}{k_n-1}-\frac{\eta_1/(k_n-1)}{\eta_2/(n-k_n+1)} \\ &= \frac{1}{\eta_2/(n-k_n+1)}\left[\left(\frac{\eta_2}{n-k_n+1}-1\right)\left(1+\frac{ns}{k_n-1}\right)\right. \\ &\quad \left.-\left(\frac{\eta_1}{k_n-1}-1-\frac{ns}{k_n-1}\right)\right]. \end{aligned}$$

Applying Lemma 3 in Bodnar and Reiß (2016), we obtain

$$\frac{\eta_2}{n-k_n+1} \overset{a.s.}{\to} 1 \qquad \text{and} \qquad \frac{\eta_1}{k_n-1}-1-\frac{n}{k_n-1}s \overset{a.s.}{\to} 0;$$

moreover, it holds that

$$\begin{aligned} \sqrt{n}\left(\frac{\eta_2}{n-k_n+1}-1\right) &\overset{\mathcal{D}}{\to} \mathcal{N}\left(0,\frac{2}{1-c}\right), \\ \sqrt{n}\left(\frac{\eta_1}{k_n-1}-1-\frac{ns}{k_n-1}\right) &\overset{\mathcal{D}}{\to} \mathcal{N}\left(0,\frac{2}{c}\left(1+2\frac{s}{c}\right)\right), \end{aligned}$$

for $k_n/n \to c \in (0,1)$ as $n \to \infty$.

The application of Slutsky's lemma leads to

$$\sqrt{n}\left(1+\frac{ns}{k_n-1}-u\right) \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \frac{2}{c}\left(1+2\frac{s}{c}\right)+\frac{2}{1-c}\left(1+\frac{s}{c}\right)^2\right). \tag{12}$$

Hence, since $\xi \sim \chi^2_{n-k_n}$ and the usage of (4) we have that

$$\sqrt{n}\left(\sqrt{\frac{n-k_n}{\xi}}-1\right) = \frac{\sqrt{n}\left(1-\frac{\xi}{n-k}\right)\frac{n-k}{\xi}}{\sqrt{\frac{n-k}{\xi}}+1} \xrightarrow{\mathcal{D}} \frac{1}{2}\tilde{z}_1 \tag{13}$$

where $\tilde{z}_1 \sim \mathcal{N}(0, \frac{2}{1-c})$.

Using (11) and (12), we obtain

$$\sqrt{n}\left(1-\frac{\sqrt{1+\frac{k_n-1}{n-k_n+1}u}}{\sqrt{\left(1+\frac{k_n-1}{n-k_n+1}\left(1+\frac{n}{k_n-1}s\right)\right)}}\right) \xrightarrow{\mathcal{D}} \frac{\frac{c}{1-c}}{2\frac{1+s}{1-c}}\tilde{z}_2 \tag{14}$$

where $\tilde{z}_2 \sim \mathcal{N}(0, \sigma_0^2)$ with $\sigma_0^2 = \frac{2}{c}\left(1+2\frac{s}{c}\right)+\frac{2}{1-c}\left(1+\frac{s}{c}\right)^2$.

Putting everything together and taking into account (6), (13) and (14) we obtain

$$T-\frac{\sqrt{n}\alpha\rho}{\sqrt{\mathbf{l}^T\mathbf{\Sigma}^{+}\mathbf{l}\left(1+\frac{k_n-1}{n-k_n+1}\left(1+\frac{n}{k_n-1}s\right)\right)}} \xrightarrow{\mathcal{D}} \tilde{z}_0+\frac{\alpha\rho}{\sqrt{\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l}\frac{1+s}{1-c}}}\left(\frac{1}{2}\tilde{z}_1+\frac{\frac{c}{1-c}}{2\frac{1+s}{1-c}}\tilde{z}_2\right)$$

where $\tilde{z}_0 \sim \mathcal{N}(0,1)$ and $\tilde{z}_1$ and $\tilde{z}_2$ are defined in (13) and (14), respectively. Moreover, $\tilde{z}_0$, $\tilde{z}_1$ and $\tilde{z}_2$ are independent.

Finally, the application of the properties of normal random variables leads to

$$\left(T-\frac{\sqrt{n}\alpha\rho}{\sqrt{\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l}\left(1+\frac{k-1}{n-k+1}\left(1+\frac{n}{k-1}s\right)\right)}}\right) \xrightarrow{\mathcal{D}} \mathcal{N}(0,\sigma_T^2)$$

with

$$\begin{aligned}
\sigma_T^2 &= 1+\frac{(\alpha\rho)^2}{\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l}(1+s)}(1-c)\left(\frac{1}{2(1-c)}+\frac{c^2}{4(1+s)^2}\sigma_u^2\right)\\
&= 1+\frac{(\alpha\rho)^2}{\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l}(1+s)}\left(\frac{1}{2}+\frac{s^2+2s+c}{2(1+s)^2}\right).
\end{aligned}$$

The statement of Theorem 2(b) follows by setting $\rho = 0$ under the null hypothesis. The theorem is proved. $\square$

3 Simulation and Empirical Studies

3.1 Simulation Study

In this subsection we present the results of a Monte Carlo simulation study. We investigate the performance of the high-dimensional asymptotic distribution of a linear combination of the TP weights derived in Theorem 1, and the power function of the high-dimensional asymptotic test that is obtained in Theorem 2.

We set $\alpha = 3$, $r_f = 0.005$, and $\mathbf{l} = (1, 0, \ldots, 0)^T$. Each element of $\boldsymbol{\mu}$ is generated from the uniform distribution on $[-0.1, 0.1]$. The population covariance matrix is drawn as follow:

- k non-zero eigenvalues of $\boldsymbol{\Sigma}$ are generated from the uniform distribution on (0,1);
- the eigenvectors are generated from the Haar distribution by simulating a Wishart matrix with 30 degrees of freedom and identity covariance, and calculating its eigenvectors.

Both the mean vector and the population covariance matrix obtained in this manner satisfy assumption (A1), they are then used in all simulation runs.

First, we evaluate the high-asymptotic distribution of $\hat{\theta} = \mathbf{l}^T \hat{\mathbf{w}}_{TP}$ with the corresponding finite-sample one obtained by applying the stochastic representation obtained in Proposition 1. We consider different sample size $n \in \{50, 120, 250, 500\}$ which roughly corresponds to the length of one-year, two-years, five-years and ten-years of weekly financial data. The results are compared for different values of concentration coefficients $c \in \{0.1, 0.4, 0.7, 0.9\}$ and it is based on $N = 10^5$ independent realisations of $\hat{\theta}$ generated from the finite-sample distribution. Lastly, the corresponding kernel density estimator of the finite sample density is computed with Epanechnikov kernel. The following algorithm is used in drawing the finite-sample density

a) generate $\hat{\theta}$ by using the stochastic representation given in Proposition 1

$$\hat{\theta} \stackrel{d}{=} \frac{n-1}{\xi} \left(\theta + \alpha^{-1} \sqrt{\left(\frac{1}{n} + \frac{k_n - 1}{n(n-k_n+1)} u \right) \mathbf{l}^T \boldsymbol{\Sigma}^{-1} \mathbf{l}} z_0 \right)$$

where $\xi \sim \chi^2_{n-k_n}$, $z_0 \sim \mathcal{N}(0,1)$ and $u \sim \mathcal{F}(k_n - 1, n - k_n + 1, ns)$ with $s = (\boldsymbol{\mu} - r_f \mathbf{1}_k)^T \mathbf{R}_{\mathbf{l}} (\boldsymbol{\mu} - r_f \mathbf{1}_k)$ and $\mathbf{R}_{\mathbf{l}} = \boldsymbol{\Sigma}^{-1} - \boldsymbol{\Sigma}^{-1} \mathbf{l}\mathbf{l}^T \boldsymbol{\Sigma}^{-1} / \mathbf{l}^T \boldsymbol{\Sigma}^{-1} \mathbf{l}$; moreover, the random variables ξ, z_0 and u are mutually independently distributed;

b) compute

$$\sqrt{n-k_n}\sigma^{-1}\left(\hat{\theta}-\frac{n-1}{n-k_n}\theta\right) \tag{15}$$

with

$$\sigma^2=\frac{\alpha^{-2}}{(1-c_n)^2}\left[\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l}+(\alpha\theta)^2+\mathbf{l}^T\mathbf{\Sigma}^{-1}\mathbf{l}(\boldsymbol{\mu}-r_f\mathbf{1}_k)^T\mathbf{\Sigma}^{-1}(\boldsymbol{\mu}-r_f\mathbf{1}_k)\right];$$

c) repeat a)-b) N times.

In Figures 3-6, we present the results of the simulation study for $c \in \{0.1, 0.4, 0.7, 0.9\}$, respectively. The finite-sample distribution of (15) is shown as dashed black lines, while the asymptotic distribution (standard normal) is shown as solid black lines. We observe that all obtained results show a good performance of the asymptotic approximation except for $c = 0.9$ and $n = 50$ where the approximation performs badly. It is noticed that even for $n = 50$ and $c \in \{0.1, 0.4, 0.7\}$ our asymptotic results seem to provide a reasonable approximation.

From Theorem 1, we have noticed that the sample estimator $\hat{\theta}$ is biased estimate of θ. In Table 1, we study behaviour of the relative bias for different values of the sample size $n \in \{50, 120, 250, 500\}$ and concentration ratio $c \in \{0.1, 0.4, 0.7, 0.9\}$. It can be observed that the relative bias of the sample estimator of θ grows as the concentration coefficient c get larger and is also independent of n. The simulations suggest that the relative bias converges to $c/(1-c)$ as it should be.

n	$c = 0.1$	$c = 0.4$	$c = 0.7$	$c = 0.9$
50	0.1472	0.7377	2.7638	14.7060
120	0.1240	0.6933	2.5012	11.1048
250	0.1173	0.6849	2.4068	9.8687
500	0.1140	0.6746	2.3723	9.3402

Table 1: Relative bias of $\hat{\theta}$ for different values of the sample size $n \in \{50, 120, 250, 500\}$ and concentration ratio $c \in \{0.1, 0.4, 0.7, 0.9\}$.

The second part of our simulation study compares the exact test with the high-dimensional asymptotic test thatis derived in Theorem 2.

In Figures 7-10, we summarize the results of that comparison for $c \in \{0.1, 0.4, 0.7, 0.9\}$, respectively, with $s = 1$ and $\psi = 5\%$.The dashed black line shows the power function of the exact test, while the power function of the high-dimensional test is plotted as a solid black line. In a similar way to the asymptotic distribution of the linear combination of the estimated TP weights, the power of the asymptotic test is indistinguishable from the power obtained for the finite-sample test. But for lower sample size and large concentration coefficient, i.e. $n = 50$ and $c = 0.9$, the power of the asymptotic test does not show a good performance.

n	50				120				250				500			
c	0.1	0.4	0.7	0.9	0.1	0.4	0.7	0.9	0.1	0.4	0.7	0.9	0.1	0.4	0.7	0.9
Exact	0.0512	0.0504	0.0508	0.0505	0.0503	0.0502	0.0502	0.0496	0.0504	0.0505	0.0503	0.0508	0.0497	0.0506	0.0510	0.0504
Asymptotic	0.0576	0.0596	0.0700	0.1083	0.0531	0.0540	0.0580	0.0734	0.0517	0.0524	0.054	0.0618	0.0503	0.0514	0.0529	0.0556

Table 2: Empirical size of the exact and asymptotic tests for $n \in \{50, 120, 250, 500\}$ and different values of concentration ratio $c \in \{0.1, 04, 0.7, 0.9\}$.

Table 2 reports the empirical size of the exact and asymptotic tests for considered values of n and different values of the concentration coefficient c. We observe that the asymptotic test is oversized for large c and small n. Otherwise the results are quite similar.

3.2 Empirical study

In this part, we present the results of an empirical study in which we show how the theoretical results obtained in Section 2 can be applied to real data. We consider the weekly averages of the daily log returns data from S&P 500 of 270 stocks for the period from January 3, 2007, to December 27, 2017, making a total of 574 observations. We use the weekly returns of the three-moths US treasury bill as the risk free-rate, and the risk aversion coefficient is chosen to be 3. We choose to use weekly logarithmic returns because they can be well approximated by Gaussian distribution (see Fama (1976), Tu and Zhou (2004)).

In Figures 11-14, we present the dynamic behaviour of the p-values obtained from the exact and the asymptotic tests on the hypotheses (8), specifically testing the hypothesis that the weight of one stock is zero with $\mathbf{l}$ a vector of zeros except for one element set to one, by using a rolling window estimator with an estimation window of 300 weeks, i.e. $n = 300$. We analyze portfolios with different number of assets such that $c \in \{0.1, 0.4, 0.7, 0.9\}$, i.e. $k \in \{30, 120, 210, 270\}$. The figures present the results for four stocks: Abbott Laboratories, Affiliated Managers Group Inc, Alphabet Inc Class A, and 3M Company. First of all, we would note that the p-values obtained from both tests are indistinguishable indicating that the high-dimensional asymptotic test performs well. Next, we can observe that in most cases the obtained p-values are relatively large resulting in the conclusion that the null hypothesis (8) cannot be rejected. However, the TP weights are significantly different from zero for all considered stocks from the end of 2012 until the middle of 2014 for the small portfolio size $k = 30$. For Abbott Laboratories and Affiliated Managers Group Inc the TP weights are also significant well into 2017 and 2016, respectively. For larger k and hence larger investment universes we find few occasions with significant TP weights. This is hardly surprising for two reasons. With larger k and fixed sample size c

increases and we can expect lower power. In addition as the investment universe increases the true TP weights will, on average across stocks, be smaller.

4 Conclusion

This paper discussed the statistical properties of the TP weights in high dimension. In particular, we delivered the high-dimensional asymptotic distribution of the weights as well as the high-dimensional asymptotic test on the weights. All theoretical results are obtained under the assumption of normality and they can be extended to the more general case which deserves a separate study. In particular, we are planning to develop new techniques in random matrix theory that will be used for delivering a high-dimensional theory on the weights with more general distributional assumptions. In future research, we would also extend our results to the case when $c > 1$. This case is more complicated since the weights will depend on the inverse of the singular sample covariance matrix.

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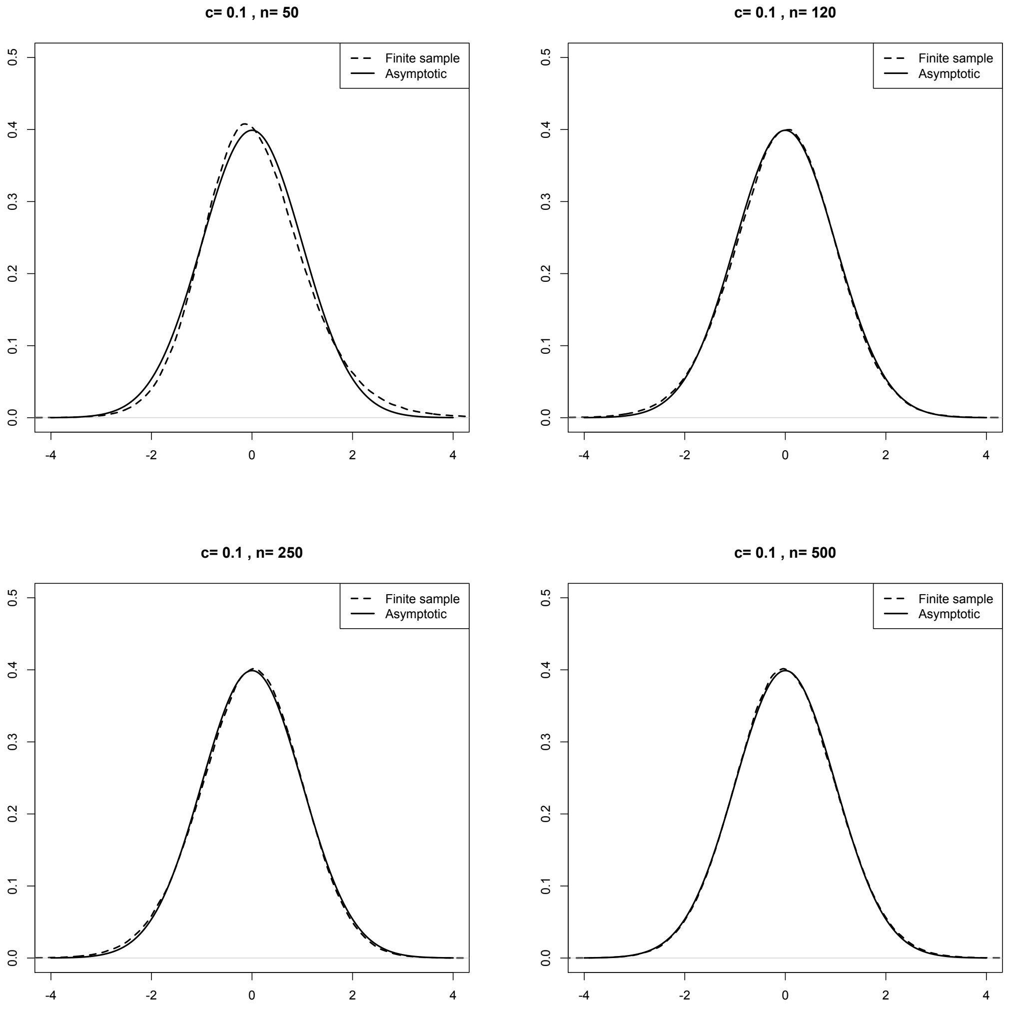


Figure 3: Asymptotic distribution and the kernel density estimator of the finite sample distribution of standardized $\hat{\theta}$ for $c = 0.1$.

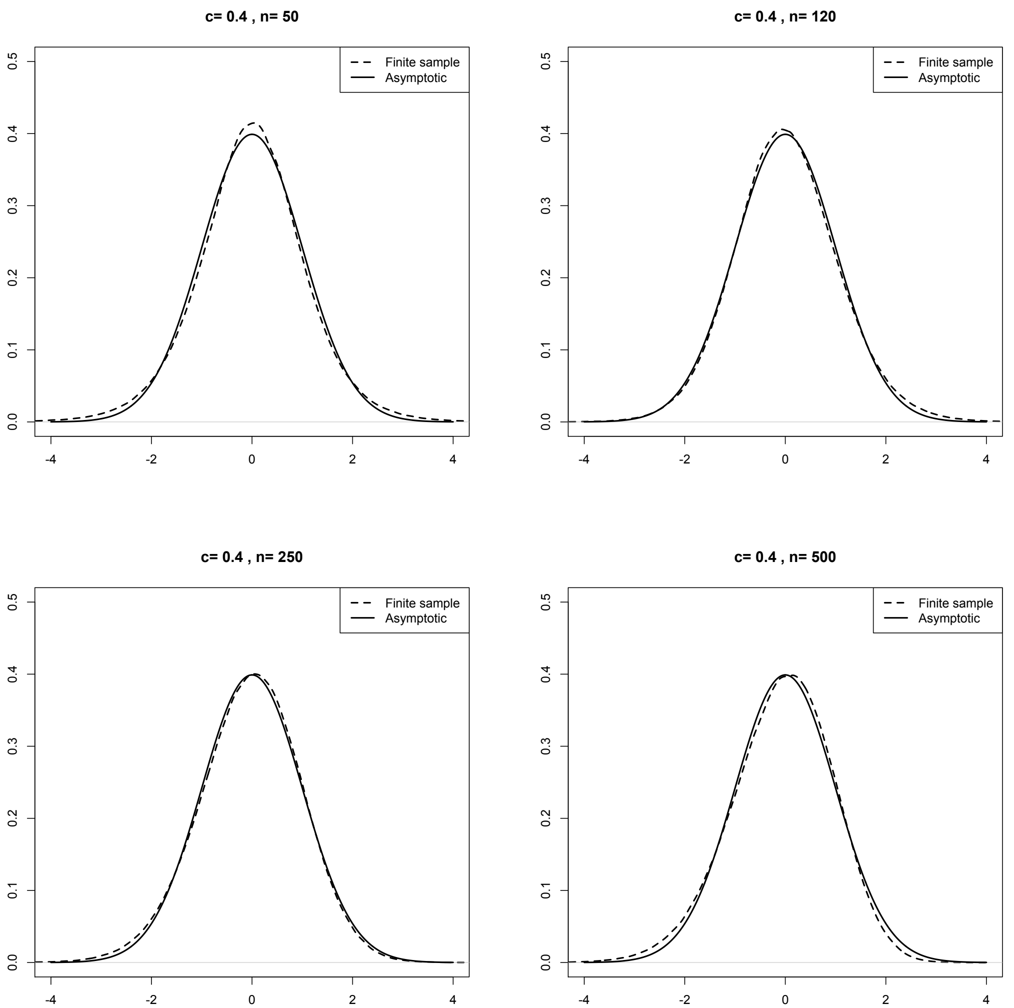


Figure 4: Asymptotic distribution and the kernel density estimator of the finite sample distribution of standardized $\hat{\theta}$ for $c = 0.4$.

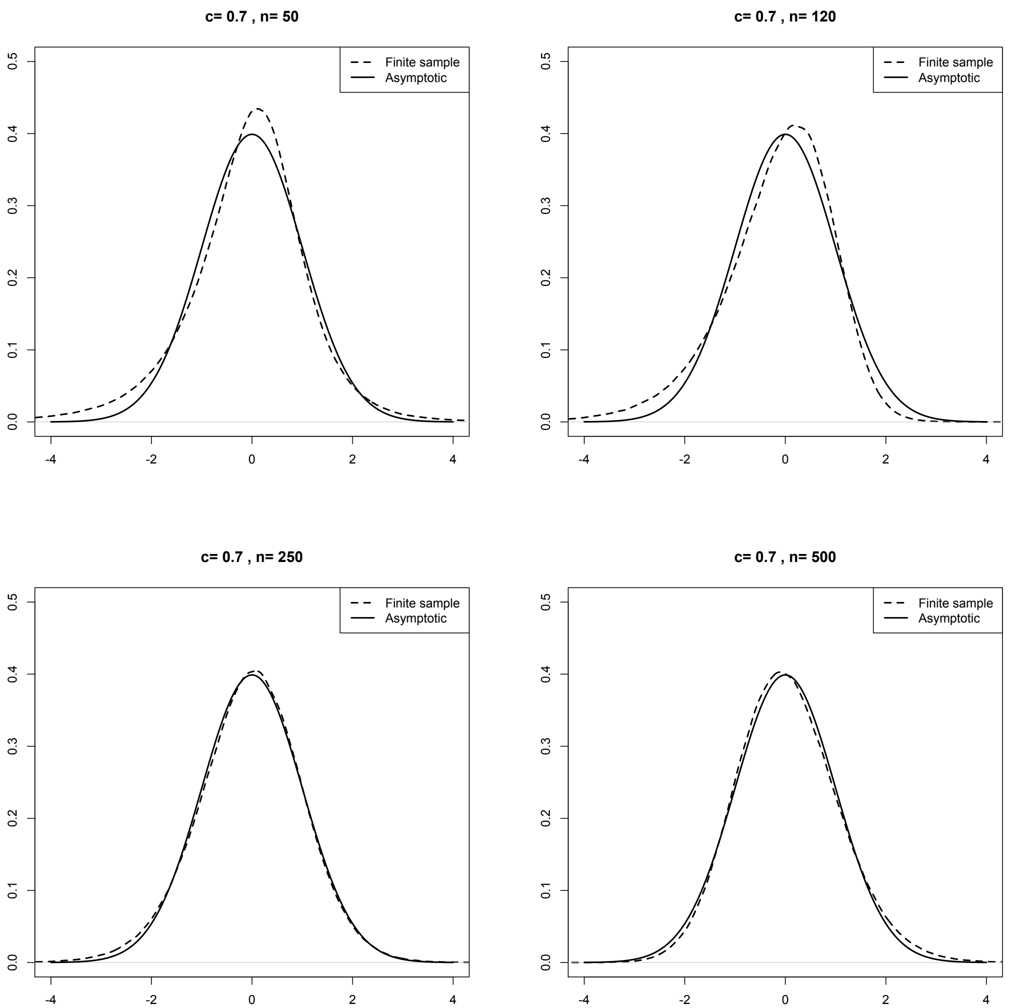


Figure 5: Asymptotic distribution and the kernel density estimator of the finite sample distribution of standardized $\hat{\theta}$ for $c = 0.7$.

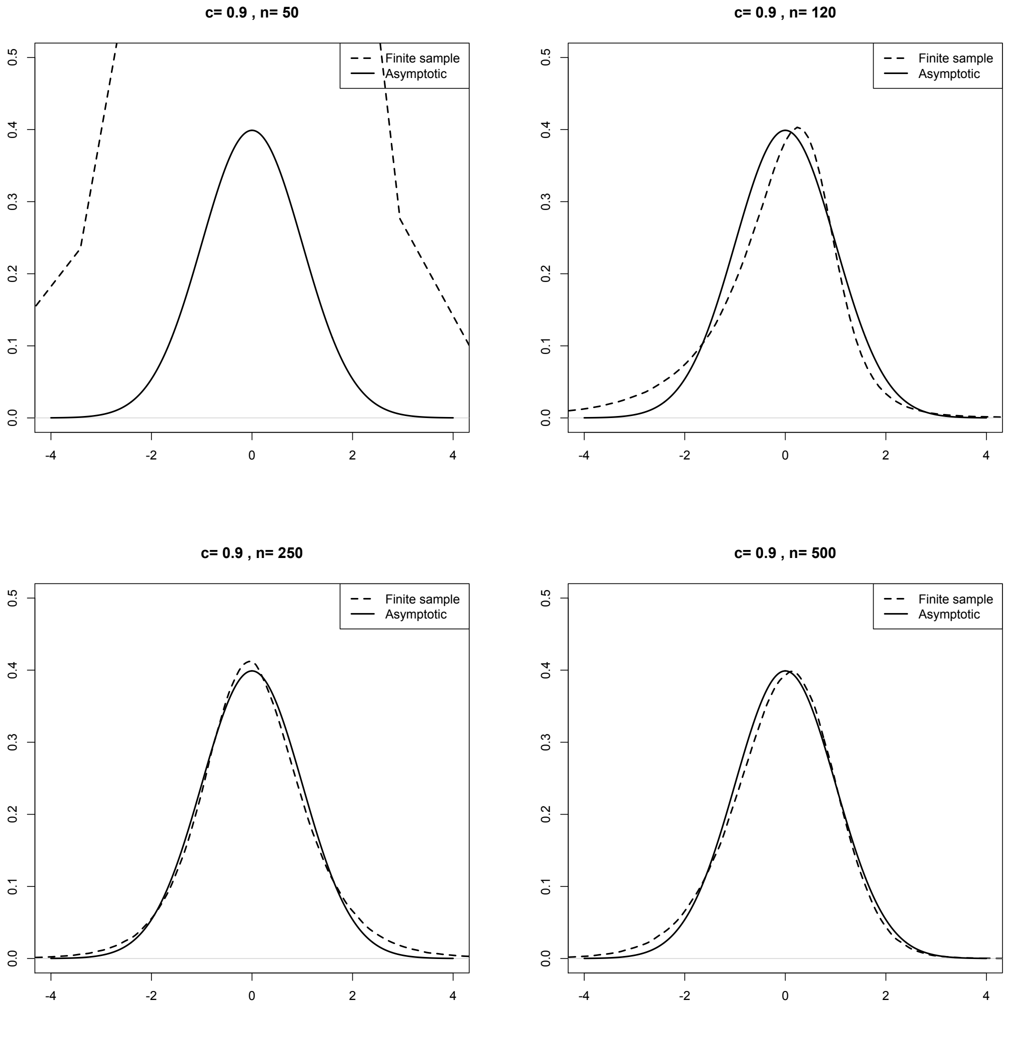


Figure 6: Asymptotic distribution and the kernel density estimator of the finite sample distribution of standardized $\hat{\theta}$ for $c = 0.9$.

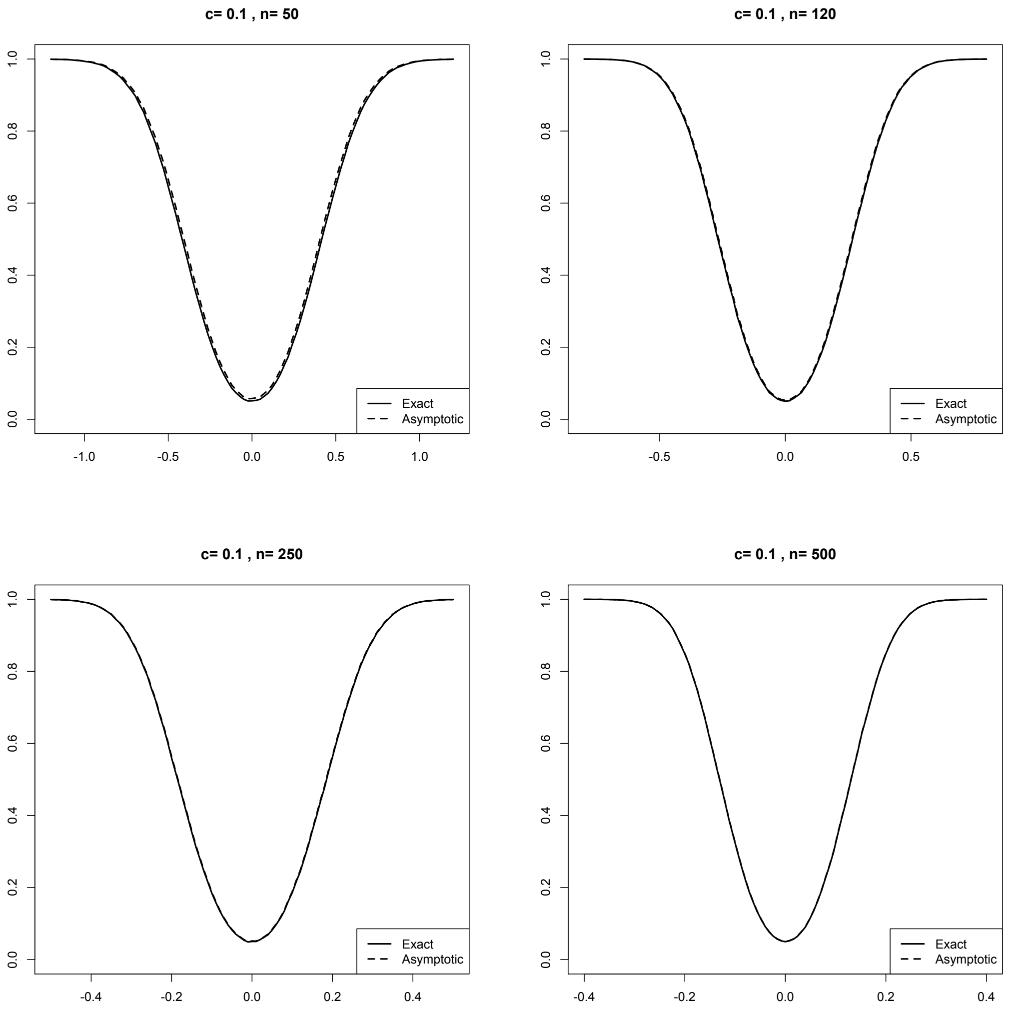


Figure 7: Powers of the exact test and of the high-dimensional asymptotic test based on the statistic T for $c = 0.1$ with $s = 1$ and $\psi = 5\%$.

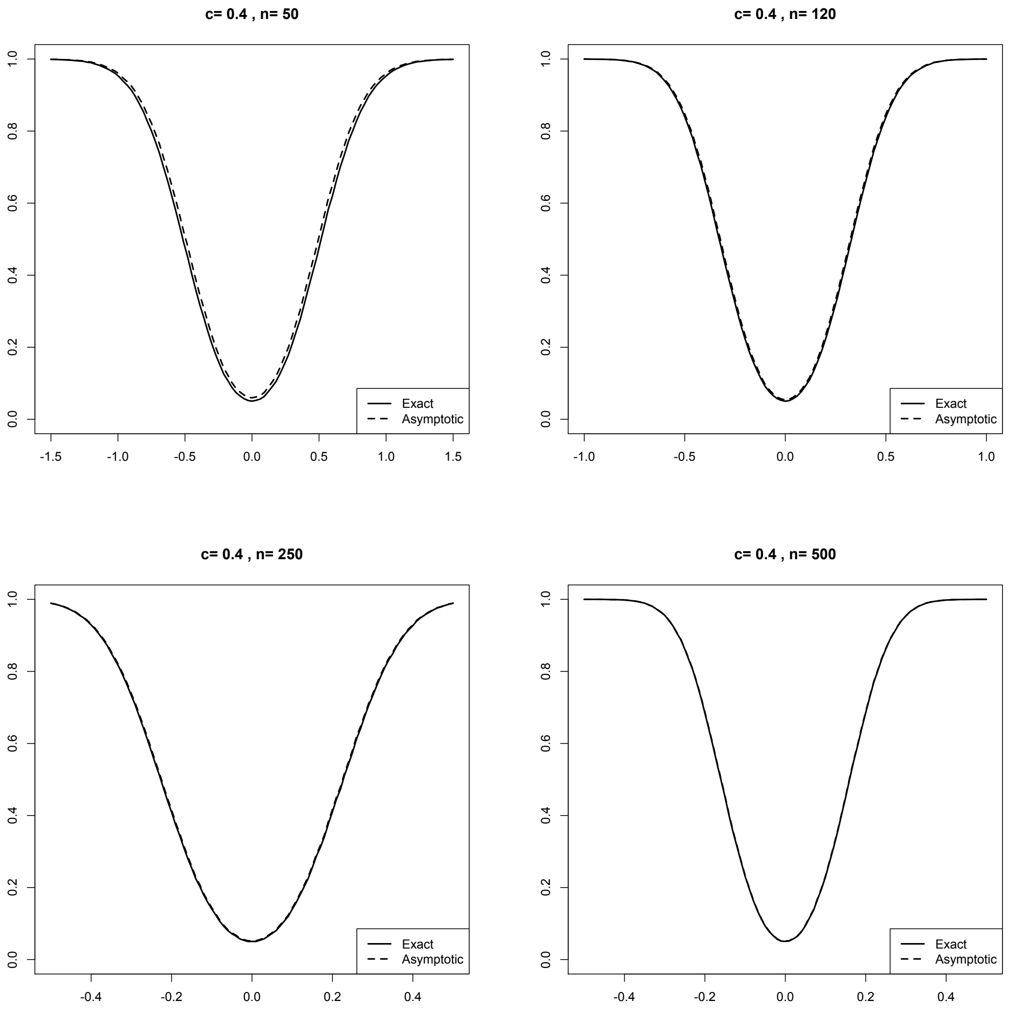


Figure 8: Powers of the exact test and of the high-dimensional asymptotic test based on the statistic T for $c = 0.4$ with $s = 1$ and $\psi = 5\%$.

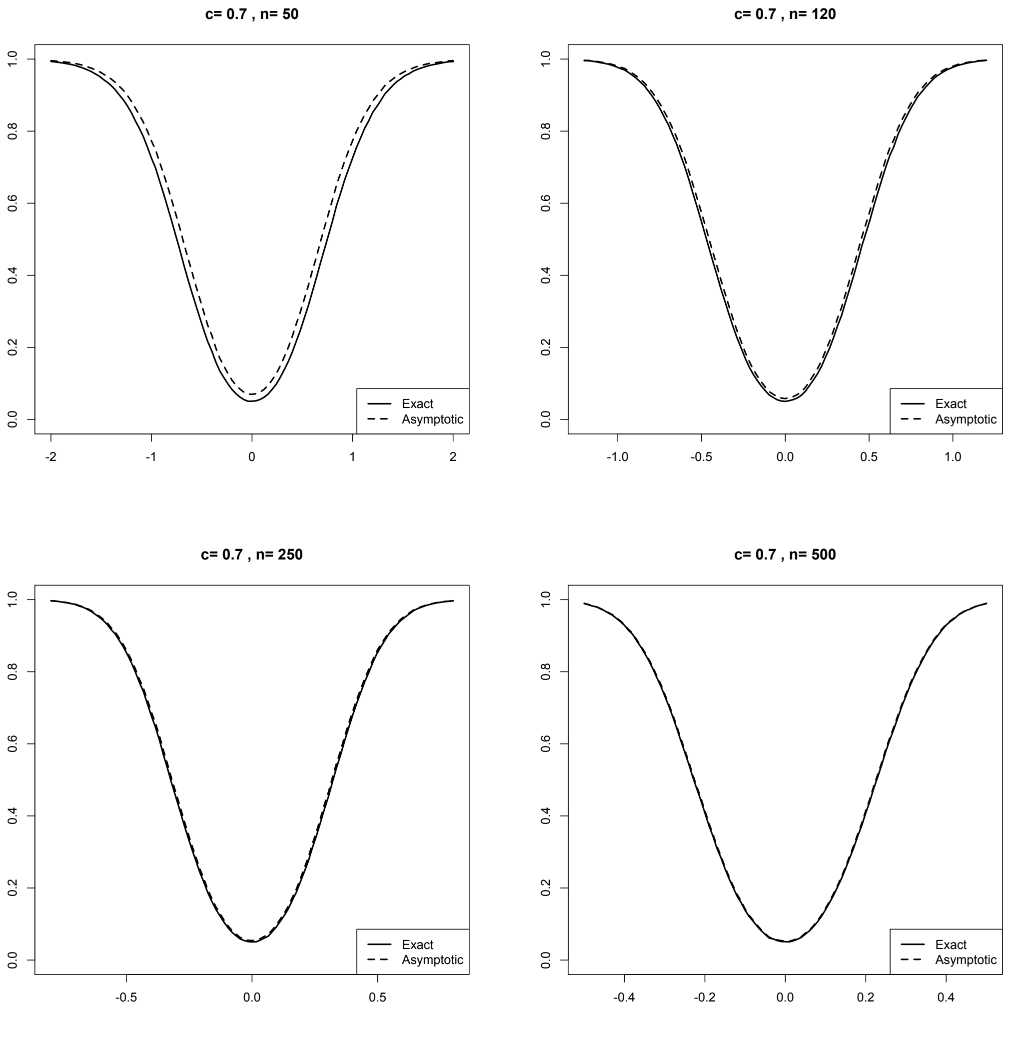


Figure 9: Powers of the exact test and of the high-dimensional asymptotic test based on the statistic T for $c = 0.7$ with $s = 1$ and $\psi = 5\%$.

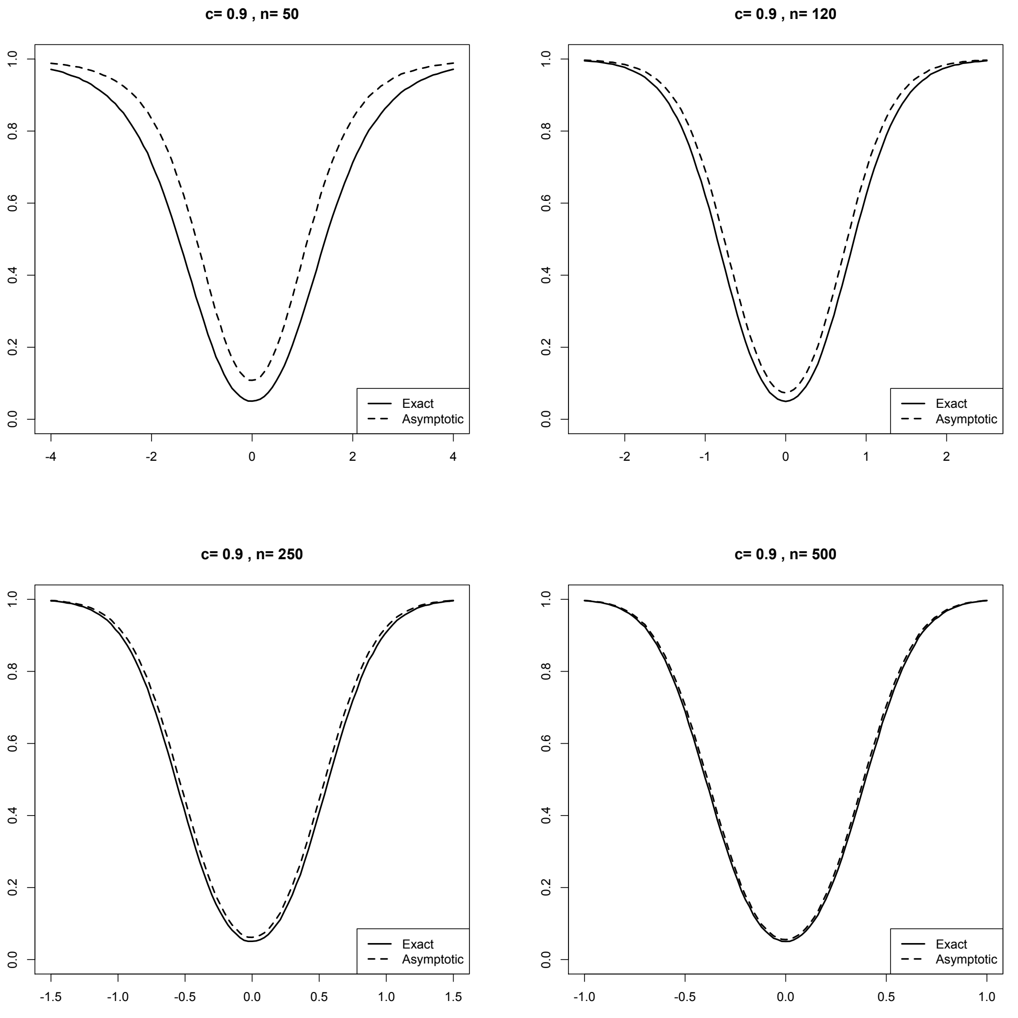
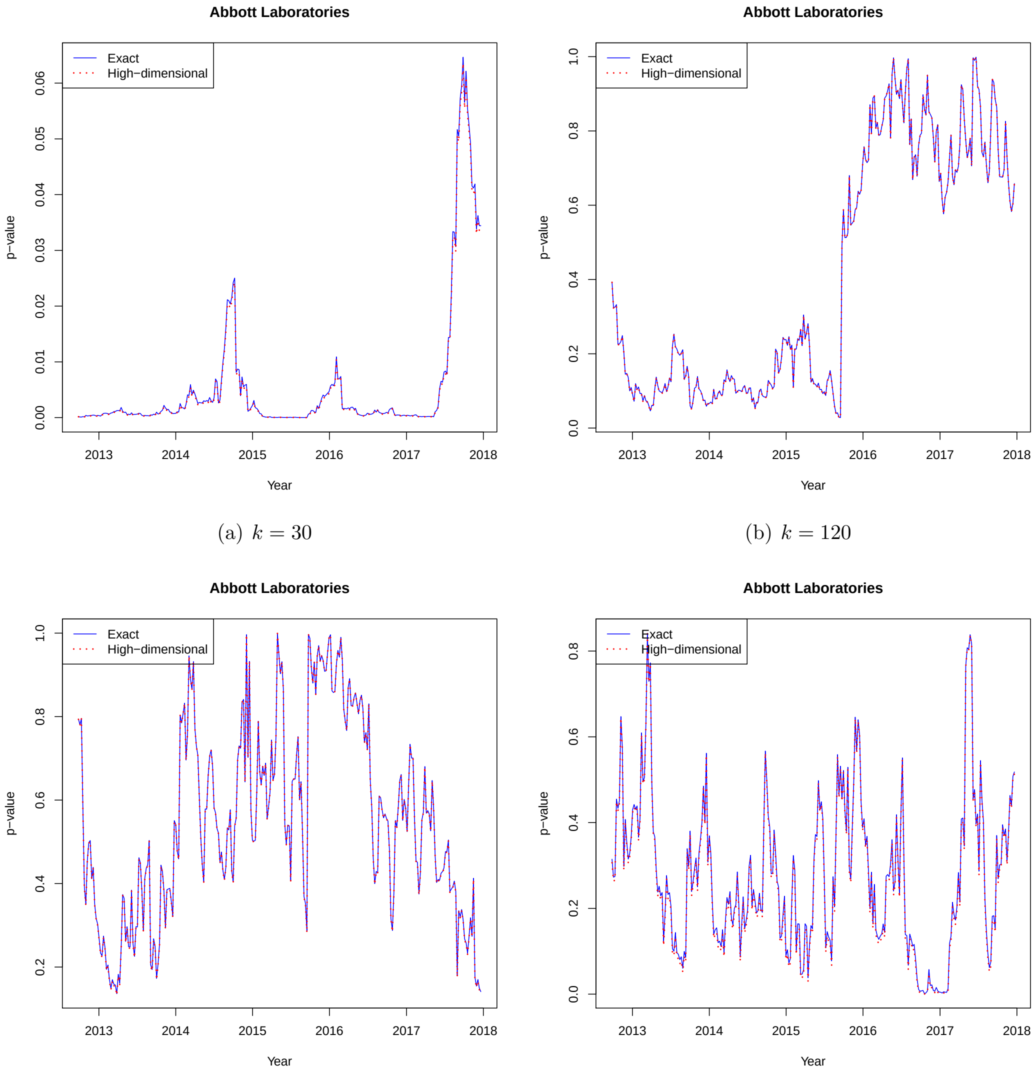


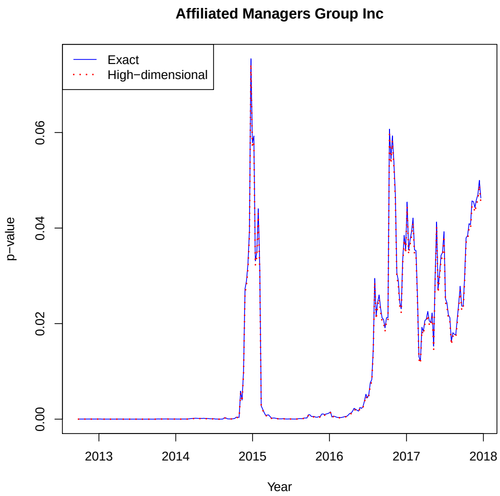
Figure 10: Powers of the exact test and of the high-dimensional asymptotic test based on the statistic T for $c = 0.9$ with $s = 1$ and $\psi = 5\%$.



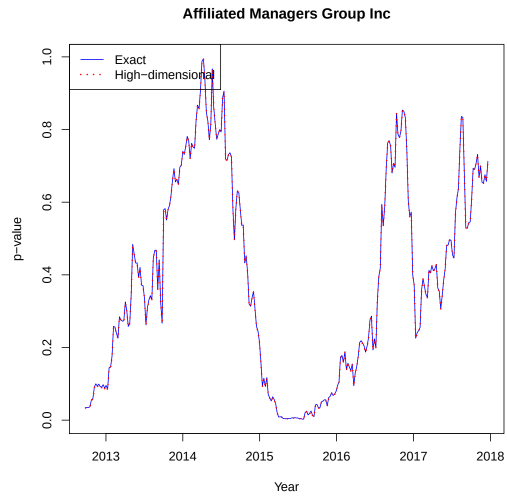
(a) $k = 30$ (b) $k = 120$

(c) $k = 210$ (d) $k = 270$

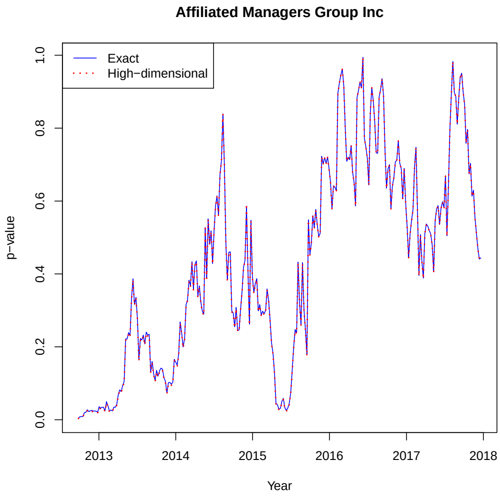
Figure 11: p-values of the exact and the asymptotic tests on the tangency portfolio weights of Abbott Laboratories.



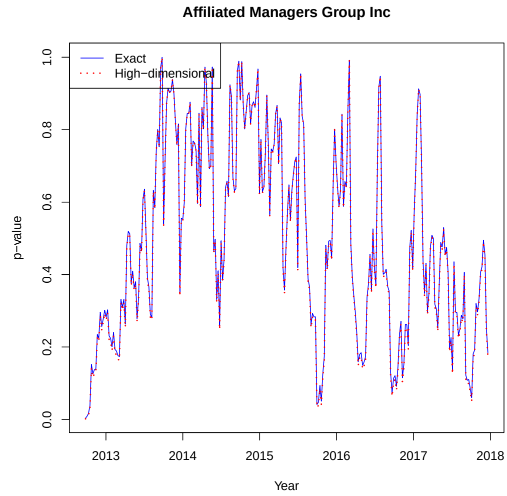
(a) $k = 30$



(b) $k = 120$

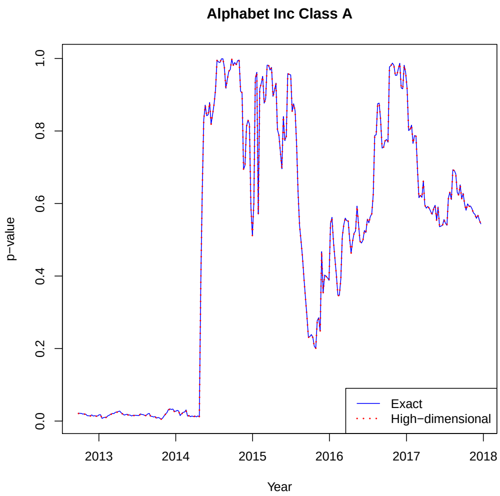


(c) $k = 210$

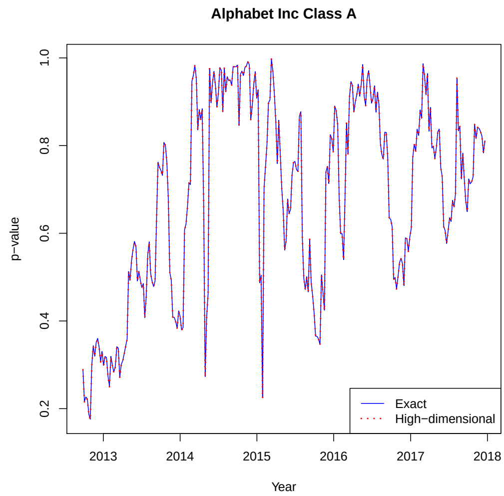


(d) $k = 270$

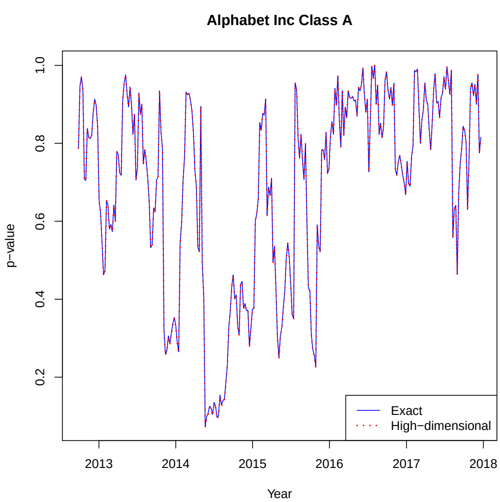
Figure 12: p-values of the exact and the asymptotic tests on the tangency portfolio weights of Affiliated Managers Group Inc.



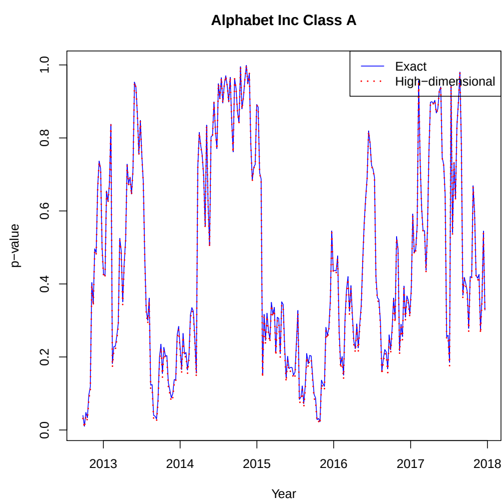
(a) $k = 30$



(b) $k = 120$

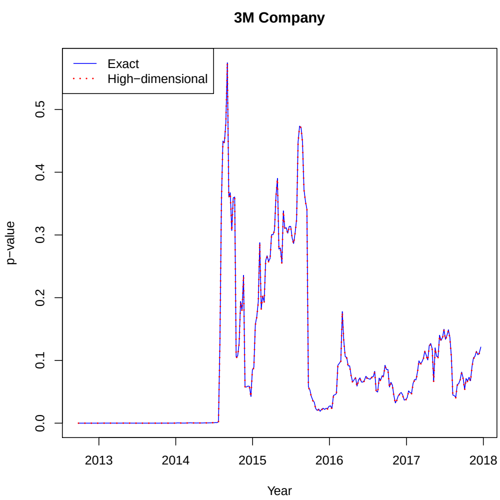


(c) $k = 210$

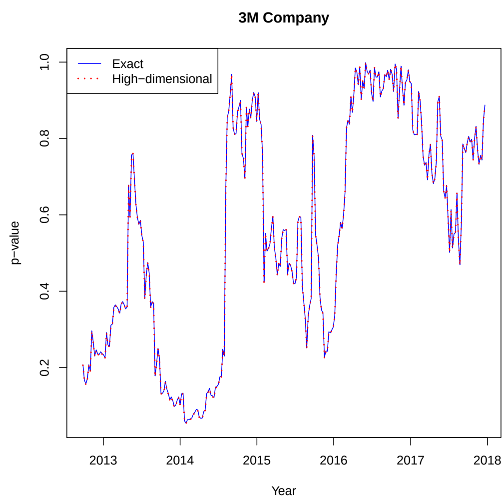


(d) $k = 270$

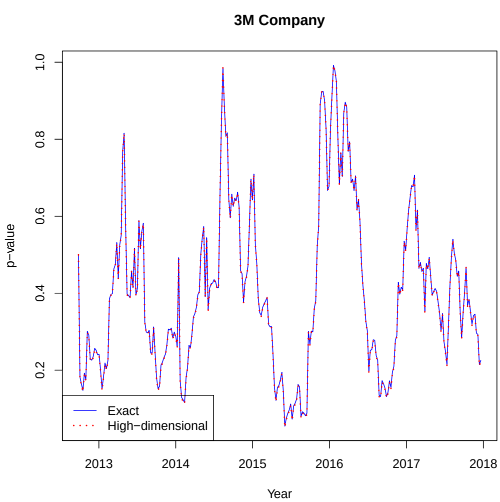
Figure 13: p-values of the exact and the asymptotic tests on the tangency portfolio weights for Alphabet Inc Class A.



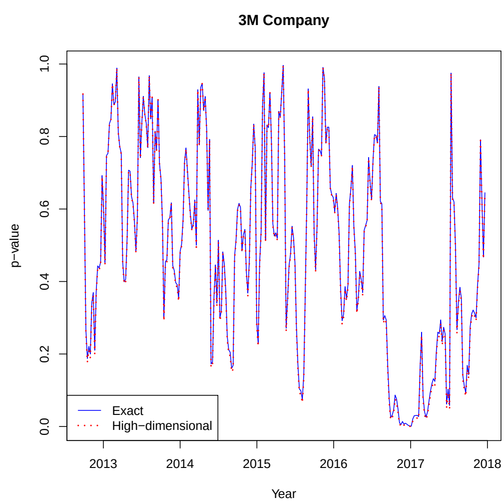
(a) $k = 30$



(b) $k = 120$



(c) $k = 210$



(d) $k = 270$

Figure 14: p-values of the exact and the asymptotic tests on the tangency portfolio weights of 3M Company.

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ON THE PRODUCT OF A SINGULAR WISHART MATRIX AND A SINGULAR GAUSSIAN VECTOR IN HIGH DIMENSION

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Abstract. In this paper we consider the product of a singular Wishart random matrix and a singular normal random vector. A very useful stochastic representation of this product is derived, using which its characteristic function and asymptotic distribution under the double asymptotic regime are established. We further document a good finite sample performance of the obtained high-dimensional asymptotic distribution via an extensive Monte Carlo study.

Key words and phrases. Singular Wishart distribution, singular normal distribution, stochastic representation, high-dimensional asymptotics.

2010 *Mathematics Subject Classification.* Primary: 60E05, 60E10, 60F05, 62H10, 62E20.

1. Introduction

The multivariate normal distribution is one of the basic distributions in probability theory and a building block in multivariate statistical analysis. It is also used as a standard assumption in many applications where the normal distribution is usually accompanied by the Wishart distribution. For instance, when we consider a sample of size n from a k-dimensional normal distribution, then the unbiased estimators for the mean vector and covariance matrix have a k-dimensional normal distribution and a k-dimensional Wishart distribution, respectively. Moreover, they are independent (see e. g. [16, Chapter 3]).

A number of papers deal either with the properties of the sample mean vector or with the properties of the sample covariance matrix, although these two random objects often appear together in the expressions of statistics. Consequently, a question arises how the distributions of functions involving both a Wishart matrix and a normal vector can be characterised. Recently, this topic has attracted a lot of attention in the literature from both the theoretical perspectives (cf. [3, 6]) and the applications (see e. g. [2, 12, 13]). While [6, 15] derived the exact distribution and the approximative distribution of the product of an inverse Wishart matrix and a normal vector, [3] presented similar results for the product of a Wishart matrix and a normal vector. The product of an inverse Wishart matrix and a normal vector has direct applications in discriminant analysis (cf. [19]) and in portfolio theory (see e. g. [7]), whereas the product of a Wishart matrix and a normal vector arises in Bayesian statistics when the aim is to infer the coefficients of the discriminant function or the optimal portfolio weights by employing the inverse Wishart - normal prior which is a conjugate prior for the mean vector and the covariance matrix under normality (see e. g. [1]).

Singular covariance matrix is present in practical applications as well, especially when data generating process is large-dimensional. For example, the construction of an optimal portfolio with a singular covariance matrix has become an important topic in finance (see e. g. [4, 17]). While the normal distribution with the singular covariance matrix is known as the singular normal distribution in statistical literature, there is no unique definition in the case of the Wishart distribution. The singular Wishart distribution introduced by [14] and [20] deals with the case when the number of degrees of freedom is smaller than the process dimension. Its practical relevance was discussed in [22], while some theoretical findings were derived in [5, 21]. Another type of the singular Wishart distributions, the so-called pseudo-Wishart distribution, was defined in [8] where a model with a singular covariance matrix was proposed. The latter stochastic model is considered in the present paper.

We contribute to the existent literature by deriving a stochastic representation for the product of a singular Wishart matrix and a normal vector, which provides an elegant way of characterising the finite sample distribution of the product. Also, it appears to be very useful in the derivation of the asymptotic distribution under the high-dimensional asymptotic regime, i.e. when both the sample size and the process dimension become very large.

The rest of the paper is structured as follows. Section 2 contains several distributional properties of the singular Wishart distribution which are used as a tool to prove the main results of the paper presented in Section 3. Here, the distribution of the product of a singular Wishart matrix and a singular normal random vector is derived in terms of a stochastic representation from which we also obtain the characteristic function of the product. Furthermore, we prove the asymptotic normality of the product under the high-dimensional asymptotic regime. The finite sample performance of the obtained asymptotic results is discussed in Section 4, while Section 5 presents the summary.

2. Preliminary results

We start this section with the formal definition of the singular normal distribution and singular Wishart distribution.

Definition 1. A random vector $\mathbf{z}$ is said to have a singular normal distribution with mean vector μ and covariance matrix $\mathbf{\Sigma}$ if its characteristic function is given by

$$\varphi_{\mathbf{z}}(\mathbf{u}) = \exp\left(i\mu^T\mathbf{u} - \frac{1}{2}\mathbf{u}^T\mathbf{\Sigma}\mathbf{u}\right),$$

where $\mathbf{\Sigma}$ is a positive semi-definite matrix with $\operatorname{rank}(\mathbf{\Sigma}) = r < k$. We denote this distribution by $\mathbf{z} \sim \mathcal{N}_k(\mu, \mathbf{\Sigma})$.

Definition 2. Let $\mathbf{z}_1, ..., \mathbf{z}_n$ be independent and identically distributed where $\mathbf{z}_i$ is singular normal with zero mean vector and covariance matrix $\mathbf{\Sigma}$, $\operatorname{rank}(\mathbf{\Sigma}) = r < k$, and let $\mathbf{Z} = [\mathbf{z}_1, ..., \mathbf{z}_n]$. Then the random matrix $\mathbf{A} = \mathbf{Z}\mathbf{Z}^T$ has a singular Wishart distribution with n degrees of freedom and covariance matrix $\mathbf{\Sigma}$. We denote this distribution by $\mathbf{A} \sim \mathcal{W}_k(n, \mathbf{\Sigma})$.

Throughout the paper, no assumption is made about the relationship between the degrees of freedom n and the dimension k. The results are valid in both cases $n \geq k$ (the Wishart distribution with positive semi-definite covariance matrix $\mathbf{\Sigma}$) and $k < n$ (the singular Wishart distribution with positive semi-definite covariance matrix $\mathbf{\Sigma}$). Also, we use the symbol $\mathbf{I}_k$ to denote the $k \times k$ identity matrix, $\otimes$ is the Kronecker product, and the symbol $\stackrel{d}{=}$ stands for the equality in distribution.

Next, we present several distributional properties of the singular Wishart distribution which are used in proving the main results of the paper. In Proposition 1, we derive the distribution of a linear symmetric transformation of the singular Wishart random matrix.

Proposition 1. *Let* $\mathbf{A} \sim \mathcal{W}_k(n, \mathbf{\Sigma})$ *with* $\operatorname{rank}(\mathbf{\Sigma}) = r < k$ *and let* $\mathbf{M} : p \times k$ *be a matrix of constants with* $\operatorname{rank}(\mathbf{M}) = p$ *such that* $\mathbf{M}\mathbf{\Sigma} \neq \mathbf{0}$. *Then*

$$\mathbf{M}\mathbf{A}\mathbf{M}^T \sim \mathcal{W}_p(n, \mathbf{M}\mathbf{\Sigma}\mathbf{M}^T).$$

Moreover, if $\operatorname{rank}(\mathbf{M}\mathbf{\Sigma}) = p \leq r$, *then* $\mathbf{M}\mathbf{A}\mathbf{M}^T$ *and* $\mathbf{M}\mathbf{\Sigma}\mathbf{M}^T$ *are of the full rank* p.

Proof. From Theorem 5.2 of [21], we have that the stochastic representation of $\mathbf{A}$ is given by

$$\mathbf{A} \stackrel{d}{=} \mathbf{X}\mathbf{X}^T \quad \text{with} \quad \mathbf{X} \sim \mathcal{N}_{k,n}(\mathbf{0}, \mathbf{\Sigma} \otimes \mathbf{I}_n).$$

Then, using Theorem 2.4.2 of [10], we get

$$\mathbf{M}\mathbf{A}\mathbf{M}^T \stackrel{d}{=} \mathbf{M}\mathbf{X}\mathbf{X}^T\mathbf{M}^T \stackrel{d}{=} \mathbf{Y}\mathbf{Y}^T,$$

where $\mathbf{Y} \sim \mathcal{N}_{p,n}(\mathbf{0}, (\mathbf{M}\mathbf{\Sigma}\mathbf{M}^T) \otimes \mathbf{I}_n)$. This completes the proof of the proposition. □

An application of Proposition 1 leads to the following result summarized in Proposition 2.

Proposition 2. *Let $\mathbf{A} \sim \mathcal{W}_k(n, \mathbf{\Sigma})$ with $\operatorname{rank}(\mathbf{\Sigma}) = r < k$ and let $\mathbf{W} : p \times k$ be a random matrix which is independent of $\mathbf{A}$ such that $\operatorname{rank}(\mathbf{W}\mathbf{\Sigma}) = p \leq r \leq n$ with probability one. Then*

$$(\mathbf{W}\mathbf{\Sigma}\mathbf{W}^T)^{-1/2}(\mathbf{W}\mathbf{A}\mathbf{W}^T)(\mathbf{W}\mathbf{\Sigma}\mathbf{W}^T)^{-1/2} \sim \mathcal{W}_p(n, \mathbf{I}_p)$$

and is independent of $\mathbf{W}$.

Proof. Using the fact that $\mathbf{W}$ and $\mathbf{A}$ are independently distributed, we obtain that the conditional distribution of $\mathbf{W}\mathbf{A}\mathbf{W}^T|(\mathbf{W} = \mathbf{W}_0)$ is equal to the distribution of $\mathbf{W}_0\mathbf{A}\mathbf{W}_0^T$. Then, applying Proposition 1, we obtain

$$(\mathbf{W}_0\mathbf{\Sigma}\mathbf{W}_0^T)^{-1/2}(\mathbf{W}_0\mathbf{A}\mathbf{W}_0^T)(\mathbf{W}_0\mathbf{\Sigma}\mathbf{W}_0^T)^{-1/2} \sim \mathcal{W}_p(n, \mathbf{I}_p).$$

Since this distribution does not depend on $\mathbf{W}$, it is also the unconditional distribution of $(\mathbf{W}\mathbf{\Sigma}\mathbf{W}^T)^{-1/2}(\mathbf{W}\mathbf{A}\mathbf{W}^T)(\mathbf{W}\mathbf{\Sigma}\mathbf{W}^T)^{-1/2}$. The proposition is proved. □

In the next corollary, we consider a special case of Proposition 2 with $p = 1$.

Corollary 1. *Let $\mathbf{A} \sim \mathcal{W}_k(n, \mathbf{\Sigma})$ with $\operatorname{rank}(\mathbf{\Sigma}) = r \leq k$ and let $\mathbf{w}$ be a k-dimensional vector which is independent of $\mathbf{A}$ with $P(\mathbf{w}^T\mathbf{\Sigma} = \mathbf{0}) = 0$. Then*

$$\frac{\mathbf{w}^T\mathbf{A}\mathbf{w}}{\mathbf{w}^T\mathbf{\Sigma}\mathbf{w}} \sim \chi^2_n,$$

and is independent of $\mathbf{w}$.

3. Main results

In this section, we present the main results of the paper which are complementary to the ones obtained in [3] to the case of high-dimensional data and singular covariance matrix.

3.1. **Finite sample results.** Let $\mathbf{z}$ be a k-dimensional singular normally distributed random vector with mean vector μ and covariance matrix $\kappa\mathbf{\Sigma}$, $\kappa > 0$, such that $\operatorname{rank}(\mathbf{\Sigma}) = r < k$, i. e. $\mathbf{z} \sim \mathcal{N}_k(\mu, \kappa\mathbf{\Sigma})$. Also, let $\mathbf{M}$ be a $p \times k$ matrix of constants with $\operatorname{rank}(\mathbf{M}) = p \leq r \leq \min\{n, k\}$ such that $\mathbf{M}\mathbf{\Sigma} \neq \mathbf{0}$. We are interested in the distribution of $\mathbf{M}\mathbf{A}\mathbf{z}$, when $\mathbf{A}$ and $\mathbf{z}$ are independently distributed where $\mathbf{A}$ has a singular Wishart distribution as defined in Section 2.

In Theorem 1, we derive a stochastic representation for $\mathbf{M}\mathbf{A}\mathbf{z}$. The stochastic representation is a tool in the theory of multivariate statistics and it is frequently used in Monte Carlo simulations (cf. [9]). Its importance in the theory of elliptically contoured distributions is well described by [11].

Theorem 1. *Let* $\mathbf{A} \sim \mathcal{W}_k(n, \mathbf{\Sigma})$ *with* $\operatorname{rank}(\mathbf{\Sigma}) = r < k$ *and let* $\mathbf{z} \sim \mathcal{N}_k(\mu, \kappa\mathbf{\Sigma})$, $\kappa > 0$. *We assume that* $\mathbf{A}$ *and* $\mathbf{z}$ *are independently distributed. Also, let* $\mathbf{M} : p \times k$ *be a matrix of constants of rank* $p < r \le n$ *and denote* $\mathbf{Q} = \mathbf{P}^T\mathbf{P}$ *with* $\mathbf{P} = (\mathbf{M\Sigma M}^T)^{-1/2}\mathbf{M\Sigma}^{1/2}$. *Then the stochastic representation of* $\mathbf{MAz}$ *is given by*

$$\mathbf{MAz} \stackrel{d}{=} \zeta\mathbf{M\Sigma}^{1/2}\mathbf{t} + \sqrt{\zeta}(\mathbf{M\Sigma M}^T)^{1/2}\left[\sqrt{\mathbf{t}^T\mathbf{t}}\mathbf{I}_p - \frac{\sqrt{\mathbf{t}^T\mathbf{t}} - \sqrt{\mathbf{t}^T(\mathbf{I}_k - \mathbf{Q})\mathbf{t}}}{\mathbf{t}^T\mathbf{Q}\mathbf{t}}\mathbf{Ptt}^T\mathbf{P}^T\right]\mathbf{z}_0,$$

where $\zeta \sim \chi^2_n$, $\mathbf{t} \sim \mathcal{N}_k(\mathbf{\Sigma}^{1/2}\mu, \kappa\mathbf{\Sigma}^2)$, *and* $\mathbf{z}_0 \sim \mathcal{N}_p(\mathbf{0}, \mathbf{I}_p)$; ζ, $\mathbf{t}$, *and* $\mathbf{z}_0$ *are mutually independent.*

Proof. Since $\mathbf{A}$ and $\mathbf{z}$ are independently distributed, it holds that the conditional distribution of $\mathbf{MAz}|(\mathbf{z} = \mathbf{z}^*)$ is equal to the distribution of $\mathbf{MAz}^*$.

Let $\widetilde{\mathbf{M}}$ be the matrix which is obtained from $\mathbf{M}$ by adding a row vector $\mathbf{z}^*$, i.e. $\widetilde{\mathbf{M}} = (\mathbf{M}^T, \mathbf{z}^*)^T$. Consider the following two partitioned matrices

$$\widetilde{\mathbf{A}} = \widetilde{\mathbf{M}}\mathbf{A}\widetilde{\mathbf{M}}^T = \begin{pmatrix} \mathbf{MAM}^T & \mathbf{MAz}^* \\ \mathbf{z}^{*T}\mathbf{AM}^T & \mathbf{z}^{*T}\mathbf{Az}^* \end{pmatrix} = \begin{pmatrix} \widetilde{\mathbf{A}}_{11} & \widetilde{\mathbf{A}}_{12} \\ \widetilde{\mathbf{A}}_{21} & \widetilde{A}_{22} \end{pmatrix}$$

and

$$\widetilde{\mathbf{\Sigma}} = \widetilde{\mathbf{M}}\mathbf{\Sigma}\widetilde{\mathbf{M}}^T = \begin{pmatrix} \mathbf{M\Sigma M}^T & \mathbf{M\Sigma z}^* \\ \mathbf{z}^{*T}\mathbf{\Sigma M}^T & \mathbf{z}^{*T}\mathbf{\Sigma z}^* \end{pmatrix} = \begin{pmatrix} \widetilde{\mathbf{\Sigma}}_{11} & \widetilde{\mathbf{\Sigma}}_{12} \\ \widetilde{\mathbf{\Sigma}}_{21} & \widetilde{\Sigma}_{22} \end{pmatrix}.$$

Since $\mathbf{A} \sim \mathcal{W}_k(n, \mathbf{\Sigma})$ and $\operatorname{rank}(\widetilde{\mathbf{M}}) = p + 1 \le r$, following Proposition 1, it holds that $\widetilde{\mathbf{A}} \sim \mathcal{W}_{p+1}(n, \widetilde{\mathbf{\Sigma}})$. Using Theorem 3.2.10 of [16], we get the conditional distribution of $\widetilde{\mathbf{A}}_{12} = \mathbf{MAz}^*$ given $\widetilde{A}_{22}$ can be expressed as

$$\widetilde{\mathbf{A}}_{12}|\widetilde{A}_{22} \sim \mathcal{N}_p\left(\widetilde{\mathbf{\Sigma}}_{12}\widetilde{\Sigma}_{22}^{-1}\widetilde{A}_{22}, \widetilde{\mathbf{\Sigma}}_{11\cdot 2}\widetilde{A}_{22}\right)$$

with $\widetilde{\mathbf{\Sigma}}_{11\cdot 2} = \widetilde{\mathbf{\Sigma}}_{11} - \widetilde{\mathbf{\Sigma}}_{12}\widetilde{\Sigma}_{22}^{-1}\widetilde{\mathbf{\Sigma}}_{21}$.

Let $\zeta = \widetilde{A}_{22}\widetilde{\Sigma}_{22}^{-1}$. Then, from Corollary 1, we get that $\zeta \sim \chi^2_n$, and it is independent of $\mathbf{z}$. Hence,

$$\mathbf{MAz}|\zeta, \mathbf{z} \sim \mathcal{N}_p\left(\zeta\mathbf{M\Sigma z}, \zeta(\mathbf{z}^T\mathbf{\Sigma z M\Sigma M}^T - \mathbf{M\Sigma zz}^T\mathbf{\Sigma M}^T)\right),$$

which leads to the stochastic representation of $\mathbf{MAz}$ given by

$$\mathbf{MAz} \stackrel{d}{=} \zeta\mathbf{M\Sigma z} + \sqrt{\zeta}(\mathbf{z}^T\mathbf{\Sigma z M\Sigma M}^T - \mathbf{M\Sigma zz}^T\mathbf{\Sigma M}^T)^{1/2}\mathbf{z}_0, \tag{1}$$

where $\zeta \sim \chi^2_n$, $\mathbf{z} \sim \mathcal{N}_k(\mu, \kappa\mathbf{\Sigma})$, and $\mathbf{z}_0 \sim \mathcal{N}_p(\mathbf{0}, \mathbf{I}_p)$. Moreover, ζ, $\mathbf{z}$, and $\mathbf{z}_0$ are mutually independent.

Next, we calculate the square root of $(\mathbf{z}^T\mathbf{\Sigma z M\Sigma M}^T - \mathbf{M\Sigma zz}^T\mathbf{\Sigma M}^T)$ using the following equality

$$(\mathbf{D} - \mathbf{bb}^T)^{1/2} = \mathbf{D}^{1/2}(\mathbf{I}_p - c\mathbf{D}^{-1/2}\mathbf{bb}^T\mathbf{D}^{-1/2})$$

with $c = \frac{1 - \sqrt{1 - \mathbf{b}^T\mathbf{D}^{-1}\mathbf{b}}}{\mathbf{b}^T\mathbf{D}^{-1}\mathbf{b}}$, $\mathbf{b} = \mathbf{M\Sigma z}$, and $\mathbf{D} = \mathbf{z}^T\mathbf{\Sigma z M\Sigma M}^T$ that leads to

$$\mathbf{MAz} \stackrel{d}{=} \zeta\mathbf{M\Sigma z} + \sqrt{\zeta}(\mathbf{M\Sigma M}^T)^{1/2} \times$$
$$\times \left[\sqrt{\mathbf{z}^T\mathbf{\Sigma z}}\mathbf{I}_p - \frac{\sqrt{\mathbf{z}^T\mathbf{\Sigma z}} - \sqrt{\mathbf{z}^T(\mathbf{\Sigma} - \mathbf{\Sigma}^{1/2}\mathbf{Q}\mathbf{\Sigma}^{1/2})\mathbf{z}}}{\mathbf{z}^T\mathbf{\Sigma}^{1/2}\mathbf{Q}\mathbf{\Sigma}^{1/2}\mathbf{z}}\mathbf{P\Sigma}^{1/2}\mathbf{zz}^T\mathbf{\Sigma}^{1/2}\mathbf{P}^T\right]\mathbf{z}_0,$$

where $\mathbf{P} = (\mathbf{M\Sigma M}^T)^{-1/2}\mathbf{M\Sigma}^{1/2}$ and $\mathbf{Q} = \mathbf{P}^T\mathbf{P}$.

Finally, making the transformation $\mathbf{t} = \mathbf{\Sigma}^{1/2}\mathbf{z} \sim \mathcal{N}_k(\mathbf{\Sigma}^{1/2}\mu, \kappa\mathbf{\Sigma}^2)$, we obtain the statement of the theorem. □

Next, we consider the special case of Theorem 1 when $p = 1$ and $\mathbf{M} = \mathbf{m}^T$.

Corollary 2. *Let* $\mathbf{A} \sim \mathcal{W}_k(n, \mathbf{\Sigma})$ *with* $\operatorname{rank}(\mathbf{\Sigma}) = r < k$ *and let* $\mathbf{z} \sim \mathcal{N}_k(\mu, \kappa\mathbf{\Sigma})$, $\kappa > 0$. *We assume that* $\mathbf{A}$ *and* $\mathbf{z}$ *are independently distributed. Let* $\mathbf{m}$ *be a* k*-dimensional vector of constants such that* $\mathbf{m}^T\mathbf{\Sigma}\mathbf{m} > 0$. *Then the stochastic representation of* $\mathbf{m}^T\mathbf{A}\mathbf{z}$ *is given by*

$$\mathbf{m}^T\mathbf{A}\mathbf{z} \stackrel{d}{=} \zeta\mathbf{m}^T\mathbf{\Sigma}\mathbf{z} + \sqrt{\zeta}\big[\mathbf{z}^T\mathbf{\Sigma}\mathbf{z}\cdot\mathbf{m}^T\mathbf{\Sigma}\mathbf{m} - (\mathbf{m}^T\mathbf{\Sigma}\mathbf{z})^2\big]^{1/2} z_0, \tag{2}$$

where $\zeta \sim \chi^2_n$ *and* $z_0 \sim \mathcal{N}(0,1)$; ζ, z_0, *and* $\mathbf{z}$ *are mutually independent.*

The proof of Corollary 2 follows directly from (1). The result of the corollary is very useful from the viewpoint of computational statistics. Namely, in order to get a realization of $\mathbf{m}^T\mathbf{A}\mathbf{z}$ it is sufficient to simulate two random variables from the standard univariate distributions together with a random vector which has a singular multivariate normal distribution. There is no need to generate a large-dimensional object $\mathbf{A}$ and, as a result, the application of (2) speeds up the simulations where the product of $\mathbf{A}$ and $\mathbf{z}$ is present.

Another application of Corollary 2 leads to the expression of the characteristic function of $\mathbf{A}\mathbf{z}$ presented in the following theorem.

Theorem 2. *Let* $\mathbf{A} \sim \mathcal{W}_k(n, \mathbf{\Sigma})$ *with* $\operatorname{rank}(\mathbf{\Sigma}) = r < k$ *and let* $\mathbf{z} \sim \mathcal{N}_k(\mu, \kappa\mathbf{\Sigma})$. *We assume that* $\mathbf{A}$ *and* $\mathbf{z}$ *are independently distributed. Then the characteristic function of* $\mathbf{A}\mathbf{z}$ *is given by*

$$\varphi_{\mathbf{A}\mathbf{z}}(\mathbf{u}) = \frac{\exp\left(-\frac{\kappa^{-1}}{2}\mu^T\mathbf{R}\mathbf{\Lambda}^{-1}\mathbf{R}^T\mu\right)}{\kappa^{r/2}|\mathbf{\Lambda}|^{1/2}}\int_0^\infty |\mathbf{\Omega}(\zeta)|^{-1/2} f_{\chi^2_n}(\zeta)\times$$
$$\times\exp\left(i\zeta\nu(\zeta)^T\mathbf{\Lambda}\mathbf{R}^T\mathbf{u} - \frac{\zeta^2}{2}\mathbf{u}^T\mathbf{R}\mathbf{\Lambda}\mathbf{\Omega}(\zeta)^{-1}\mathbf{\Lambda}\mathbf{R}^T\mathbf{u} + \frac{1}{2}\nu(\zeta)^T\mathbf{\Omega}(\zeta)\nu(\zeta)\right)d\zeta,$$

where $\nu(\zeta) = \kappa^{-1}\mathbf{\Omega}(\zeta)^{-1}\mathbf{\Lambda}^{-1}\mathbf{R}^T\mu$,

$$\mathbf{\Omega}(\zeta) = \kappa^{-1}\mathbf{\Lambda}^{-1} + \zeta\big[\mathbf{\Lambda}\cdot\mathbf{u}^T\mathbf{\Sigma}\mathbf{u} - \mathbf{\Lambda}\mathbf{R}^T\mathbf{u}\mathbf{u}^T\mathbf{R}\mathbf{\Lambda}\big],$$

and $\mathbf{\Sigma} = \mathbf{R}\mathbf{\Lambda}\mathbf{R}^T$ *is the singular value decomposition of* $\mathbf{\Sigma}$ *with diagonal matrix* $\mathbf{\Lambda}$ *consisting of all* r *non-zero eigenvalues of* $\mathbf{\Sigma}$ *and the* $k \times r$ *matrix* $\mathbf{R}$ *of the corresponding eigenvectors;* $f_{\chi^2_n}$ *denotes the density function of the* χ^2 *distribution with* n *degrees of freedom.*

Proof. From the stochastic representation derived in Corollary 2, we get that

$$\begin{aligned}\varphi_{\mathbf{A}\mathbf{z}}(\mathbf{u}) &= \mathbb{E}\big(\exp(i\mathbf{u}^T\mathbf{A}\mathbf{z})\big) = \\ &= \mathbb{E}\left(\exp\left(i\zeta\mathbf{u}^T\mathbf{\Sigma}\mathbf{z} + i\sqrt{\zeta}\big[\mathbf{z}^T\mathbf{\Sigma}\mathbf{z}\cdot\mathbf{u}^T\mathbf{\Sigma}\mathbf{u} - (\mathbf{u}^T\mathbf{\Sigma}\mathbf{z})^2\big]^{1/2} z_0\right)\right) = \\ &= \mathbb{E}\left(\exp(i\zeta\mathbf{u}^T\mathbf{\Sigma}\mathbf{z})\mathbb{E}\left(\exp\left(i\sqrt{\zeta}\big[\mathbf{z}^T\mathbf{\Sigma}\mathbf{z}\cdot\mathbf{u}^T\mathbf{\Sigma}\mathbf{u} - (\mathbf{u}^T\mathbf{\Sigma}\mathbf{z})^2\big]^{1/2} z_0\right)|\zeta,\mathbf{z}\right)\right) = \\ &= \mathbb{E}\left(\exp(i\zeta\mathbf{u}^T\mathbf{\Sigma}\mathbf{z})\exp\left(-\frac{1}{2}\zeta\big[\mathbf{z}^T\mathbf{\Sigma}\mathbf{z}\cdot\mathbf{u}^T\mathbf{\Sigma}\mathbf{u} - (\mathbf{u}^T\mathbf{\Sigma}\mathbf{z})^2\big]\right)\right) = \\ &= \mathbb{E}\left(\mathbb{E}\left(\exp(i\zeta\mathbf{u}^T\mathbf{\Sigma}\mathbf{z})\exp\left(-\frac{1}{2}\zeta\big[\mathbf{z}^T\mathbf{\Sigma}\mathbf{z}\cdot\mathbf{u}^T\mathbf{\Sigma}\mathbf{u} - (\mathbf{u}^T\mathbf{\Sigma}\mathbf{z})^2\big]\right)|\zeta\right)\right) = \\ &= \mathbb{E}\left(\mathbb{E}\left(\exp(i\zeta\mathbf{v}^T\mathbf{\Lambda}\mathbf{y})\exp\left(-\frac{1}{2}\zeta\big[\mathbf{y}^T\mathbf{\Lambda}\mathbf{y}\cdot\mathbf{v}^T\mathbf{\Lambda}\mathbf{v} - (\mathbf{v}^T\mathbf{\Lambda}\mathbf{y})^2\big]\right)|\zeta\right)\right),\end{aligned}$$

where $\mathbf{v} = \mathbf{R}^T\mathbf{u}$; $\mathbf{\Sigma} = \mathbf{R}\mathbf{\Lambda}\mathbf{R}^T$ is the singular value decomposition of $\mathbf{\Sigma}$; $\mathbf{y} = \mathbf{R}^T\mathbf{z} \sim$ $\sim \mathcal{N}_r(\mathbf{R}^T\mu, \kappa\mathbf{\Lambda})$ has a non-singular multivariate normal distribution.

Hence,

$$\mathbb{E}\left(\exp(i\zeta\mathbf{v}^T\mathbf{\Lambda}\mathbf{y})\exp\left(-\frac{1}{2}\zeta\big[\mathbf{y}^T\mathbf{\Lambda}\mathbf{y}\cdot\mathbf{v}^T\mathbf{\Lambda}\mathbf{v} - (\mathbf{v}^T\mathbf{\Lambda}\mathbf{y})^2\big]\right)|\zeta\right) =$$

$$= \frac{1}{(2\pi\kappa)^{r/2}|\mathbf{\Lambda}|^{1/2}} \int_{\mathbb{R}^r} \exp\big(i\zeta\mathbf{v}^T\mathbf{\Lambda}\mathbf{y}\big) \exp\bigg(-\frac{1}{2}\zeta\big[\mathbf{y}^T\mathbf{\Lambda}\mathbf{y}\cdot\mathbf{v}^T\mathbf{\Lambda}\mathbf{v} - (\mathbf{v}^T\mathbf{\Lambda}\mathbf{y})^2\big]\bigg) \times$$
$$\times \exp\bigg(-\frac{\kappa^{-1}}{2}(\mathbf{y}-\mathbf{R}^T\mu)^T\mathbf{\Lambda}^{-1}(\mathbf{y}-\mathbf{R}^T\mu)\bigg)\mathrm{d}\mathbf{y}$$

where

$$\kappa^{-1}(\mathbf{y}-\mathbf{R}^T\mu)^T\mathbf{\Lambda}^{-1}(\mathbf{y}-\mathbf{R}^T\mu) + \zeta\big[\mathbf{y}^T\mathbf{\Lambda}\mathbf{y}\cdot\mathbf{v}^T\mathbf{\Lambda}\mathbf{v} - (\mathbf{v}^T\mathbf{\Lambda}\mathbf{y})^2\big] =$$
$$= (\mathbf{y}-\nu(\zeta))^T\mathbf{\Omega}(\zeta)(\mathbf{y}-\nu(\zeta)) + d$$

with

$$\mathbf{\Omega}(\zeta) = \kappa^{-1}\mathbf{\Lambda}^{-1} + \zeta\big[\mathbf{\Lambda}\cdot\mathbf{v}^T\mathbf{\Lambda}\mathbf{v} - \mathbf{\Lambda}\mathbf{v}\mathbf{v}^T\mathbf{\Lambda}\big],$$
$$\nu(\zeta) = \kappa^{-1}\mathbf{\Omega}(\zeta)^{-1}\mathbf{\Lambda}^{-1}\mathbf{R}^T\mu,$$
$$d = \kappa^{-1}\mu^T\mathbf{R}\mathbf{\Lambda}^{-1}\mathbf{R}^T\mu - \nu(\zeta)^T\mathbf{\Omega}(\zeta)\nu(\zeta) = \kappa^{-1}\mu^T\mathbf{\Sigma}^+\mu - \nu(\zeta)^T\mathbf{\Omega}(\zeta)\nu(\zeta),$$

and $\mathbf{\Sigma}^+$ the Moore–Penrose inverse.

As a result, we get

$$\varphi_{\mathbf{Az}}(\mathbf{u}) = \frac{\exp\Big(-\frac{\kappa^{-1}}{2}\mu^T\mathbf{\Sigma}^+\mu\Big)}{\kappa^{r/2}|\mathbf{\Lambda}|^{1/2}} \int_0^\infty |\mathbf{\Omega}(\zeta)|^{-1/2} f_{\chi^2_n}(\zeta) \times$$
$$\times \exp\bigg(i\zeta\nu(\zeta)^T\mathbf{\Lambda}\mathbf{v} - \frac{\zeta^2}{2}\mathbf{v}^T\mathbf{\Lambda}\mathbf{\Omega}(\zeta)^{-1}\mathbf{\Lambda}\mathbf{v} + \frac{1}{2}\nu(\zeta)^T\mathbf{\Omega}(\zeta)\nu(\zeta)\bigg)\mathrm{d}\zeta.$$

This completes the proof of the theorem. □

3.2. **Asymptotic distribution under double asymptotic regime.** In this section we derive the asymptotic distribution of $\mathbf{MAz}$ under double asymptotic regime, i.e. when both r and n tend to infinity such that $r/n \to c \in [0, +\infty)$. In the derivation of the asymptotic distribution we rely on the results of Corollary 2.

The following conditions are needed to ensure the validity of the asymptotic results presented in this section.

(A1) Let $(\lambda_i, \boldsymbol{u}_i)$ denote the set of non-zero eigenvalues and eigenvectors of $\mathbf{\Sigma}$. We assume that there exist l_1 and L_1 such that

$$0 < l_1 \le \lambda_1 \le \lambda_2 \le \ldots \le \lambda_r \le L_1 < \infty$$

uniformly on k.

(A2) There exists L_2 such that

$$|\boldsymbol{u}_i^T\mu| \le L_2 \text{ for all } i = 1, \ldots, r \text{ uniformly on } k.$$

It is noted that Assumptions (A1) and (A2) are valid uniformly on k, that is both constants L_1 and L_2 should not depend on k. Later on we also assume that κ increases with r. This condition is needed in order to ensure that the random vector $\mathbf{z}$ is well concentrated around its mean vector in large dimension. For example, fulfilled in the case, when $\mathbf{z}$ is the sample mean computed from the independent normal sample.

Theorem 3. *Let $\mathbf{A} \sim \mathcal{W}_k(n, \mathbf{\Sigma})$ with $\mathrm{rank}(\mathbf{\Sigma}) = r < k$ and let $\mathbf{z} \sim \mathcal{N}_k(\mu, \kappa\mathbf{\Sigma}), \kappa > 0$. Assume $\frac{r}{n} = c + o(n^{-1/2}), c \in [0, +\infty)$ and $\kappa r = O(1)$ as $n \to \infty$. Also, let $\mathbf{m}$ be a k-dimensional vector of constants such that $\mathbf{m}^T\mathbf{\Sigma}\mathbf{m} > 0$ and $|\boldsymbol{u}_i^T\mathbf{m}| \le L_2$ for all $i = 1, \ldots, r$ uniformly on k. Assume that $\mathbf{A}$ and $\mathbf{z}$ are independently distributed. Then, under (A1) and (A2), it holds that the asymptotic distribution of $\mathbf{m}^T\mathbf{Az}$ is given by*

$$\sqrt{n}\sigma^{-1}\left(\frac{1}{n}\mathbf{m}^T\mathbf{Az} - \mathbf{m}^T\mathbf{\Sigma}\mu\right) \xrightarrow{d} \mathcal{N}(0,1) \quad \text{for } r/n \to c \text{ as } n \to \infty,$$

where

$$\sigma^2 = \left(\mathbf{m}^T\mathbf{\Sigma}\mu\right)^2 + \mathbf{m}^T\mathbf{\Sigma}\mathbf{m}\left[\kappa tr(\mathbf{\Sigma}^2) + \mu^T\mathbf{\Sigma}\mu\right] + \frac{\kappa}{c}\mathbf{m}^T\mathbf{\Sigma}^3\mathbf{m}.$$

Proof. From Corollary 2, the stochastic representation of $\mathbf{m}^T\mathbf{A}\mathbf{z}$ is given by

$$\mathbf{m}^T\mathbf{A}\mathbf{z} \overset{d}{=} \zeta\mathbf{m}^T\mathbf{\Sigma}\mathbf{z} + \sqrt{\zeta}\left[\mathbf{z}^T\mathbf{\Sigma}\mathbf{z}\cdot\mathbf{m}^T\mathbf{\Sigma}\mathbf{m} - (\mathbf{m}^T\mathbf{\Sigma}\mathbf{z})^2\right]^{1/2} z_0,$$

with $\zeta \sim \chi^2_n$, $z_0 \sim \mathcal{N}(0,1)$ and $\mathbf{z} \sim \mathcal{N}_k(\mu, \kappa\mathbf{\Sigma})$,$\kappa > 0$; ζ, z_0, and $\mathbf{z}$ are mutually independent.

From the property of χ^2-distribution, we immediately obtain the asymptotic distribution of ζ given by

$$\sqrt{n}\left(\frac{\zeta}{n} - 1\right) \overset{d}{\to} \mathcal{N}(0,2) \text{ as } n \to \infty. \tag{3}$$

Further, it holds that $\sqrt{n}(z_0/\sqrt{n}) \sim \mathcal{N}(0,1)$ for all n, consequently it is its asymptotic distribution.

We next show that $\mathbf{m}^T\mathbf{\Sigma}\mathbf{z}$ and $\mathbf{z}^T\mathbf{\Sigma}\mathbf{z}$ are jointly asymptotically normally distributed under the high-dimensional asymptotic regime. For any $a_1 \in \mathbb{R}$ and $a_2 \in \mathbb{R}$, we consider

$$\begin{aligned} a_1\mathbf{z}^T\mathbf{\Sigma}\mathbf{z} + 2a_2\mathbf{m}^T\mathbf{\Sigma}\mathbf{z} &= a_1\left(\mathbf{z} + \frac{a_2}{a_1}\mathbf{m}\right)^T\mathbf{\Sigma}\left(\mathbf{z} + \frac{a_2}{a_1}\mathbf{m}\right) - \frac{a_2^2}{a_1}\mathbf{m}^T\mathbf{\Sigma}\mathbf{m} = \\ &= a_1\tilde{\mathbf{z}}^T\mathbf{\Sigma}\tilde{\mathbf{z}} - \frac{a_2^2}{a_1}\mathbf{m}^T\mathbf{\Sigma}\mathbf{m}, \end{aligned}$$

where $\tilde{\mathbf{z}} \sim \mathcal{N}_k(\mu_a, \kappa\mathbf{\Sigma})$ with $\mu_a = \mu + \frac{a_2}{a_1}\mathbf{m}$. By [18] the random variable $\tilde{\mathbf{z}}^T\mathbf{\Sigma}\tilde{\mathbf{z}}$ can be expressed as

$$\tilde{\mathbf{z}}^T\mathbf{\Sigma}\tilde{\mathbf{z}} \overset{d}{=} \kappa\sum_{i=1}^r \lambda_i^2\zeta_i \quad \text{with} \quad \zeta_i \overset{d}{\sim} \chi^2_1(\delta_i^2),\ \delta_i^2 = \kappa^{-1}\lambda_i^{-1}\left(\boldsymbol{u}_i^T\mu_a\right)^2,$$

where the symbol $\chi^2_d(\delta)$ denotes the non-central chi-squared distribution with d degrees of freedom and non-centrality parameter δ.

Next, we apply the Lindeberg central limit theorem to the i.i.d. random variables $V_i = \kappa\lambda_i^2\zeta_i$. Let $\sigma_i^2 = \mathbb{V}(V_i)$ and $s_n^2 = \mathbb{V}(\sum_{i=1}^r V_i)$. It holds that

$$\begin{aligned} s_n^2 &= \mathbb{V}\left(\sum_{i=1}^r V_i\right) = \kappa^2\sum_{i=1}^r \lambda_i^4\mathbb{V}(\zeta_i) = \kappa^2\sum_{i=1}^r \lambda_i^4 2(1 + 2\delta_i^2) = \\ &= \kappa^2\sum_{i=1}^r \left(2\lambda_i^4 + 4\kappa^{-1}\lambda_i^3(\boldsymbol{u}_i^T\mu_a)^2\right) = \kappa^2\left[2tr(\mathbf{\Sigma}^4) + 4\kappa^{-1}\mu_a^T\mathbf{\Sigma}^3\mu_a\right]. \end{aligned}$$

In order to verify the Lindeberg condition, we need to check if for any small $\epsilon > 0$ it holds that

$$\lim_{r\to\infty}\frac{1}{s_n^2}\sum_{i=1}^r \mathbb{E}\left[(V_i - \mathbb{E}(V_i))^2\mathbb{1}_{\{|V_i - \mathbb{E}(V_i)| > \epsilon s_n\}}\right] = 0, \tag{4}$$

where

$$\begin{aligned} &\sum_{i=1}^r \mathbb{E}\left[(V_i - \mathbb{E}(V_i))^2\mathbb{1}_{\{|V_i - \mathbb{E}(V_i)| > \epsilon s_n\}}\right] \overset{\text{Cauchy–Schwarz}}{\leq} \\ &\overset{\text{Cauchy–Schwarz}}{\leq} \sum_{i=1}^r \sqrt{\mathbb{E}[(V_i - \mathbb{E}(V_i))^4]}\sqrt{\mathbb{E}\left[\mathbb{1}_{\{|V_i - \mathbb{E}(V_i)| > \epsilon s_n\}}\right]} = \\ &= \sum_{i=1}^r \sqrt{\mathbb{E}[(V_i - \mathbb{E}(V_i))^4]}\sqrt{\mathbb{P}[|V_i - \mathbb{E}(V_i)| > \epsilon s_n]} \overset{\text{Chebychev}}{\leq} \end{aligned}$$

$$\overset{\text{Chebychev}}{\leq} \sum_{i=1}^{r} \sqrt{\mathbb{E}[(V_i - \mathbb{E}(V_i))^4]} \frac{\sigma_i}{\epsilon s_n} =$$
$$= 2\sqrt{3}\frac{\kappa^2}{\epsilon} \sum_{i=1}^{r} \lambda_i^4 \sqrt{(1+2\delta_i^2)^2 + 4(1+4\delta_i^2)} \frac{\sigma_i}{s_n}.$$

By using

$$(1+2\delta_i^2)^2 + 4(1+4\delta_i^2) = (5+2\delta_i^2)^2 - 20 \leq (5+2\delta_i^2)^2,$$

for $\sigma_{\max} = \sup_i \sigma_i$, we get the following inequality

$$\frac{1}{s_n^2}\sum_{i=1}^{r}\mathbb{E}\big[(V_i-\mathbb{E}(V_i))^2\mathbb{1}_{\{|V_i-\mathbb{E}(V_i)|>\epsilon s_n\}}\big] \leq 2\sqrt{3}\frac{\kappa^2}{\epsilon}\frac{\sigma_{\max}}{s_n}\frac{1}{s_n^2}\sum_{i=1}^{r}\lambda_i^4(5+2\delta_i^2) =$$
$$= \frac{\sqrt{3}}{\epsilon}\frac{\sigma_{\max}}{s_n}\frac{5tr(\mathbf{\Sigma}^4)+2\kappa^{-1}\mu_a^T\mathbf{\Sigma}^3\mu_a}{tr(\mathbf{\Sigma}^4)+2\kappa^{-1}\mu_a^T\mathbf{\Sigma}^3\mu_a} \leq \frac{5\sqrt{3}}{\epsilon}\frac{\sigma_{\max}}{s_n}.$$

Using

$$(\boldsymbol{u}_i^T\mu_a)^2 = \left(\boldsymbol{u}_i^T\mu + \frac{a_2}{a_1}\boldsymbol{u}_i^T\boldsymbol{m}\right)^2 = 2(\boldsymbol{u}_i^T\mu)^2 + 2\left(\frac{a_2}{a_1}\boldsymbol{u}_i^T\boldsymbol{m}\right)^2 =$$
$$= 2L_2^2\left(1+\left(\frac{a_2}{a_1}\right)^2\right) < \infty$$

and Assumptions (A1) and (A2), we get

$$\frac{\sigma_{\max}^2}{s_n^2} = \frac{\sup_i(\lambda_i^4(1+2\delta_i^2))}{tr(\mathbf{\Sigma}^4)+2\kappa^{-1}\mu_a^T\mathbf{\Sigma}^3\mu_a} = \frac{\sup_i(\lambda_i^4+2\kappa^{-1}\lambda_i^3(\boldsymbol{u}_i^T\mu_a)^2)}{tr(\mathbf{\Sigma}^4)+2\kappa^{-1}\mu_a^T\mathbf{\Sigma}^3\mu_a} \to 0,$$

which verifies the Lindeberg condition.

Since

$$\sum_{i=1}^{r}\mathbb{E}(V_i) = \kappa\sum_{i=1}^{r}\lambda_i^2\mathbb{E}(\zeta_i) = \kappa\sum_{i=1}^{r}\lambda_i^2\left(1+\delta_i^2\right) = \kappa tr(\mathbf{\Sigma}^2) + \mu_a^T\mathbf{\Sigma}\mu_a$$

we obtain by using the Lindeberg central limit theorem that

$$\sqrt{\frac{1}{\kappa}}\frac{\tilde{\mathbf{z}}^T\mathbf{\Sigma}\tilde{\mathbf{z}} - \kappa tr(\mathbf{\Sigma}^2) - \mu_a^T\mathbf{\Sigma}\mu_a}{\sqrt{2\kappa tr(\mathbf{\Sigma}^4)+4\mu_a^T\mathbf{\Sigma}^3\mu_a}} \xrightarrow{d} \mathcal{N}(0,1).$$

Let $\mathbf{a} = (a_1, 2a_2)^T$. Then the last identity leads to

$$\sqrt{n}\left[\mathbf{a}^T\begin{pmatrix}\mathbf{z}^T\mathbf{\Sigma}\mathbf{z}\\ \mathbf{m}^T\mathbf{\Sigma}\mathbf{z}\end{pmatrix} - \mathbf{a}^T\begin{pmatrix}\kappa tr(\mathbf{\Sigma}^2)+\mu^T\mathbf{\Sigma}\mu\\ \mathbf{m}^T\mathbf{\Sigma}\mu\end{pmatrix}\right] \xrightarrow{d}$$
$$\xrightarrow{d} \mathcal{N}\left(0, \mathbf{a}^T\frac{\kappa}{c}\begin{pmatrix}2\kappa tr(\mathbf{\Sigma}^4)+4\mu\mathbf{\Sigma}^3\mu & 2\mathbf{m}^T\mathbf{\Sigma}^3\mu\\ 2\mathbf{m}^T\mathbf{\Sigma}^3\mu & \mathbf{m}^T\mathbf{\Sigma}^3\mathbf{m}\end{pmatrix}\mathbf{a}\right), \tag{5}$$

which implies that the vector $\left(\mathbf{z}^T\mathbf{\Sigma}\mathbf{z}, \mathbf{m}^T\mathbf{\Sigma}\mathbf{z}\right)^T$ is asymptotically multivariate normally distributed because $\mathbf{a}$ is an arbitrary fixed vector.

Taking into account (3),(5) and the fact that ζ, z_0, and $\mathbf{z}$ are mutually independent, we get the following asymptotic result

$$\sqrt{n}\left[\begin{pmatrix}\frac{\zeta}{n}\\ \mathbf{z}^T\mathbf{\Sigma}\mathbf{z}\\ \mathbf{m}^T\mathbf{\Sigma}\mathbf{z}\\ \frac{z_0}{\sqrt{n}}\end{pmatrix} - \begin{pmatrix}1\\ \kappa tr(\mathbf{\Sigma}^2)+\mu^T\mathbf{\Sigma}\mu\\ \mathbf{m}^T\mathbf{\Sigma}\mu\\ 0\end{pmatrix}\right] \xrightarrow{d}$$

$$\xrightarrow{d} \mathcal{N}\left(0, \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2\frac{\kappa^2}{c}tr(\mathbf{\Sigma}^4)+4\frac{\kappa}{c}\mu^T\mathbf{\Sigma}^3\mu & 2\frac{\kappa}{c}\mathbf{m}^T\mathbf{\Sigma}^3\mu & 0 \\ 0 & 2\frac{\kappa}{c}\mathbf{m}^T\mathbf{\Sigma}^3\mu & \frac{\kappa}{c}\mathbf{m}^T\mathbf{\Sigma}^3\mathbf{m} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right).$$

Finally, the application of the delta method leads to

$$\sqrt{n}\sigma^{-1}\left(\frac{1}{n}\mathbf{m}^T\mathbf{A}\mathbf{z} - \mathbf{m}^T\mathbf{\Sigma}\mu\right) \xrightarrow{d} \mathcal{N}(0,1),$$

where

$$\begin{aligned}
\sigma^2 = & \begin{pmatrix} \mathbf{m}^T\mathbf{\Sigma}\mu & 0 & 1 & \left[\left[\kappa tr(\mathbf{\Sigma}^2)+\mu^T\mathbf{\Sigma}\mu\right]\mathbf{m}^T\mathbf{\Sigma}\mathbf{m} - \left(\mathbf{m}^T\mathbf{\Sigma}\mu\right)^2\right]^{\frac{1}{2}} \end{pmatrix} \times \\
& \times \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2\frac{\kappa^2}{c}tr(\mathbf{\Sigma}^4)+4\frac{\kappa}{c}\mu^T\mathbf{\Sigma}^3\mu & 2\frac{\kappa}{c}\mathbf{m}^T\mathbf{\Sigma}^3\mu & 0 \\ 0 & 2\frac{\kappa}{c}\mathbf{m}^T\mathbf{\Sigma}^3\mu & \frac{\kappa}{c}\mathbf{m}^T\mathbf{\Sigma}^3\mathbf{m} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \times \\
& \times \begin{pmatrix} \mathbf{m}^T\mathbf{\Sigma}\mu \\ 0 \\ 1 \\ \left[\left[\kappa tr(\mathbf{\Sigma}^2)+\mu^T\mathbf{\Sigma}\mu\right]\mathbf{m}^T\mathbf{\Sigma}\mathbf{m} - \left(\mathbf{m}^T\mathbf{\Sigma}\mu\right)^2\right]^{\frac{1}{2}} \end{pmatrix} = \\
= & \left(\mathbf{m}^T\mathbf{\Sigma}\mu\right)^2 + \mathbf{m}^T\mathbf{\Sigma}\mathbf{m}\left[\kappa tr(\mathbf{\Sigma}^2)+\mu^T\mathbf{\Sigma}\mu\right] + \frac{\kappa}{c}\mathbf{m}^T\mathbf{\Sigma}^3\mathbf{m}. \qquad \square
\end{aligned}$$

Finally, we extend the results of Theorem 3 to the case of finite number of linear combinations of the elements of $\mathbf{Az}$. The results are summarised in the following theorem.

Theorem 4. *Let* $\mathbf{A} \sim \mathcal{W}_k(n, \mathbf{\Sigma})$ *with* $\operatorname{rank}(\mathbf{\Sigma}) = r < k$ *and let* $\mathbf{z} \sim \mathcal{N}_k(\mu, \kappa\mathbf{\Sigma})$, $\kappa > 0$. *Assume* $\frac{r}{n} = c + o(n^{-1/2}), c \in [0, +\infty)$ *and* $\kappa r = O(1)$ *as* $n \to \infty$. *Let* $\mathbf{M} = (\mathbf{m}_1, \ldots, \mathbf{m}_p)^T : p \times k$ *be a matrix of constants of rank* $p < r \leq n$ *with probability one and let* $|\mathbf{u}_i^T\mathbf{m}_j| \leq L_2$ *for all* $i = 1, \ldots, r$ *and* $j = 1, \ldots, p$ *uniformly on* k. *Assume that* $\mathbf{A}$ *and* $\mathbf{z}$ *are independently distributed. Then under (A1) and (A2) the asymptotic distribution of* $\mathbf{MAz}$ *under the double asymptotic regime is given by*

$$\sqrt{n}\mathbf{\Omega}^{-1/2}\left(\frac{1}{n}\mathbf{MAz} - \mathbf{M\Sigma z}\right) \xrightarrow{d} \mathcal{N}_p(\mathbf{0}, \mathbf{I}_p) \quad \text{for } r/n \to c \text{ as } n \to \infty,$$

where

$$\mathbf{\Omega} = \mathbf{M\Sigma}\mu\mu^T\mathbf{\Sigma M}^T + \mathbf{M\Sigma M}^T\left[\kappa tr(\mathbf{\Sigma}^2) + \mu^T\mathbf{\Sigma}\mu\right] + \frac{\kappa}{c}\mathbf{M\Sigma}^3\mathbf{M}^T.$$

Proof. For all $\mathbf{l} \in \mathbb{R}^p$-fixed, we consider $\mathbf{l}^T\mathbf{MAz}$. The rest of the proof follows from Theorem 3 with $\mathbf{m} = \mathbf{M}^T\mathbf{l}$ and the fact that $\mathbf{l}$ is an arbitrary vector. $\square$

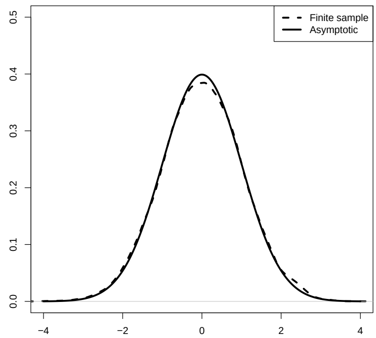
4. Finite sample performance

In this section, we present the results of a Monte Carlo simulation study where the performance of the obtained asymptotic distribution for the product of a singular Wishart matrix and a singular Gaussian vector is investigated.

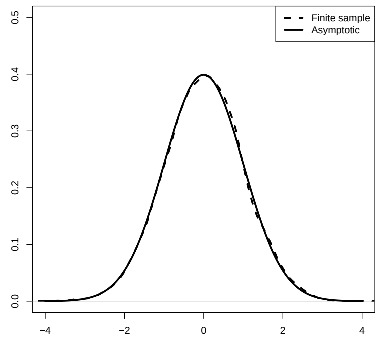
In our simulation, we fix $\mathbf{m} = \mathbf{1}/k$ where $\mathbf{1}$ denotes the k-dimensional vector of ones and generated each element of μ from the uniform distribution on $[-1, 1]$. The population covariance matrix was drawn in the following way:

- r non-zero eigenvalues of $\mathbf{\Sigma}$ were generated from the uniform distribution on $(0, 1)$ and the rest were set to be zero;
- the eigenvectors were generated from the Haar distribution by simulating a Wishart matrix with identity covariance matrix and calculating its eigenvectors.

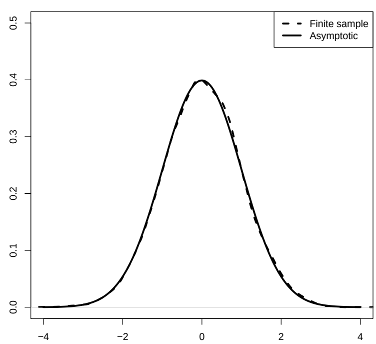
Both the mean vector and the population covariance matrix obtained by such setting satisfy the assumptions (A1) and (A2).



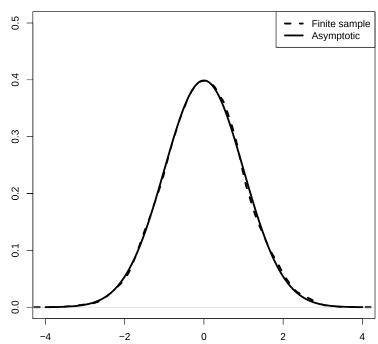
(a) $n = 500, c = 0.1, k = 750$



(b) $n = 500, c = 0.5, k = 750$



(c) $n = 500, c = 0.8, k = 750$



(d) $n = 500, c = 0.95, k = 750$

FIGURE 1. Asymptotic distribution and the kernel density estimator of the finite sample distribution calculated for the product of a singular Wishart matrix and a singular normal vector ($n = 500$)

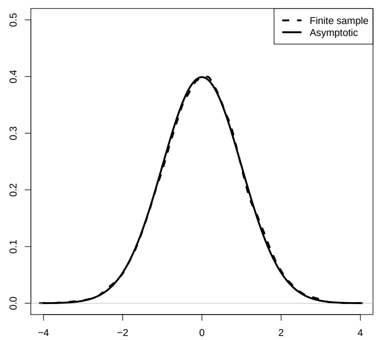
We compare the asymptotic density of the standardized random variable $\mathbf{m}^T\mathbf{A}\mathbf{z}$ with its finite sample one which is obtained by applying the stochastic representation of Corollary 2. More precisely, we draw $N = 10^4$ independent realizations of the standardized random variable $\mathbf{m}^T\mathbf{A}\mathbf{z}$ by using the following algorithm.

a) Generate $\mathbf{m}^T\mathbf{A}\mathbf{z}$ by using stochastic representation (2) of Corollary 2 expressed as
$$\mathbf{m}^T\mathbf{A}\mathbf{z} \stackrel{d}{=} \zeta\mathbf{m}^T\boldsymbol{\Sigma}\mathbf{z} + \sqrt{\zeta}\big[\mathbf{z}^T\boldsymbol{\Sigma}\mathbf{z}\cdot\mathbf{m}^T\boldsymbol{\Sigma}\mathbf{m} - (\mathbf{m}^T\boldsymbol{\Sigma}\mathbf{z})^2\big]^{1/2} z_0,$$
where $\zeta \sim \chi^2_n$, $z_0 \sim \mathcal{N}(0,1)$, $\mathbf{z} \sim \mathcal{N}_k(\mu, \kappa\boldsymbol{\Sigma})$; ζ, z_0, and $\mathbf{z}$ are mutually independent.

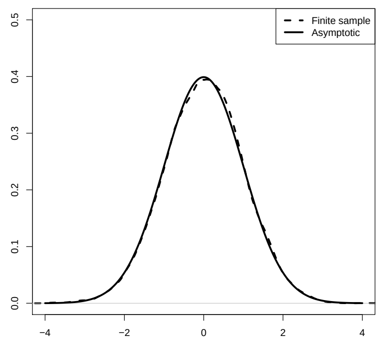
b) Compute
$$\sqrt{n}\sigma^{-1}\left(\frac{1}{n}\mathbf{m}^T\mathbf{A}\mathbf{z}-\mathbf{m}^T\mathbf{\Sigma}\mu\right),$$
where
$$\sigma^2=\left(\mathbf{m}^T\mathbf{\Sigma}\mu\right)^2+\mathbf{m}^T\mathbf{\Sigma}\mathbf{m}\left[\kappa tr(\mathbf{\Sigma}^2)+\mu^T\mathbf{\Sigma}\mu\right]+\frac{\kappa}{c}\mathbf{m}^T\mathbf{\Sigma}^3\mathbf{m}.$$

c) Repeat a)–b) N times.

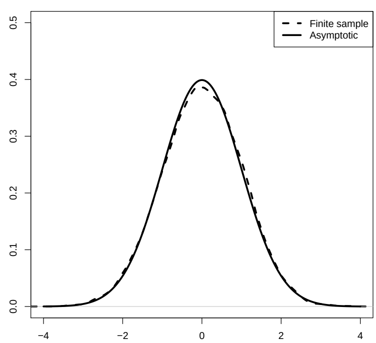
Then, the elements of the generated sample are used to construct a kernel density estimator which is compared to the asymptotic distribution, i.e. to the density of the standard normal distribution. As a kernel, we make use of the Epanechnikov kernel with the bandwidth chosen by applying Silverman's rule of thumb.



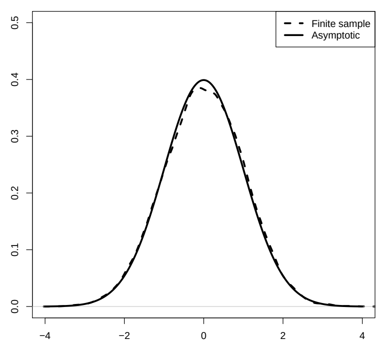
(a) $n=1000, c=0.1, k=750$



(b) $n=1000, c=0.5, k=750$

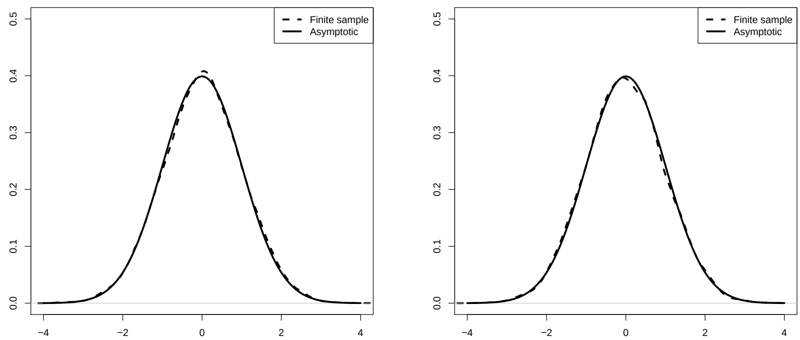


(c) $n=1000, c=0.8, k=990$



(d) $n=1000, c=0.95, k=990$

FIGURE 2. Asymptotic distribution and the kernel density estimator of the finite sample distribution calculated for the product of a singular Wishart matrix and a singular normal vector ($n=1000$)



(a) $n = 500, c = 2, k = 1200$ (b) $n = 1000, c = 2, k = 2100$

FIGURE 3. Asymptotic distribution and the kernel density estimator of the finite sample distribution calculated for the product of a singular Wishart matrix and a singular normal vector ($n \in \{500, 1000\}$)

The results of the simulation study are summarized in Figure 1 for $n = 500$, in Figure 2 for $n = 1000$, and for $n \in \{500, 100\}$ with $c = 2$ in Figure 3. In all cases we set $\kappa = 1/n$. Finally, $k = 750$ is chosen for $n = 500$ and $k \in \{750, 990\}$ for $n = 1000$. For $c = 2$, $k = 1200$ is chosen for $n = 500$ and $k = 2100$ for $n = 1000$. Furthermore, several values of r are considered such that $c = \{0.1, 0.5, 0.8, 0.95\}$ in Figures 1 and 2, while $c = 2$ in Figure 3. The finite sample distributions are shown as dashed lines, while the asymptotic distributions are solid lines. All obtained results show a good performance of the asymptotic approximation which is almost indistinguishable from the corresponding finite sample density. This result remains true even for the values of $c = 0.95$ and $c = 2$.

5. SUMMARY

The Wishart distribution and normal distribution are widely spread in both statistics and probability theory with numerous and useful applications in finance, economics, environmental sciences, biology, etc. Different functions involving a Wishart matrix and a normal vector have been studied in statistical literature recently. However, to the best of our knowledge, combinations of a singular Wishart matrix and a singular normal vector have not been investigated up to now.

In this paper we analyse the product of a singular Wishart matrix and a singular Gaussian vector. A very useful stochastic representation of this product is obtained, which is later used to derive its characteristic function as well as to provide an efficient way how to simulate the elements of the product in practice. With the use of the derived stochastic representation, there is no need in generating a large dimensional Wishart matrix. Its application speeds up simulation studies where the product of a singular Wishart matrix and a singular normal vector is present. Furthermore, we prove the asymptotic normality of the product under the double asymptotic regime. In a numerical study, a good performance of the obtained asymptotic distribution is documented. It is also noted that for the values $c = 0.95$ and $c = 2$, it produces a very good approximation of the corresponding finite sample distribution obtained by applying the derived stochastic representation.

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ДОБУТОК СИНГУЛЯРНОЇ ВИПАДКОВОЇ МАТРИЦІ ВІШАРТА ТА СИНГУЛЯРНОГО НОРМАЛЬНОГО ВИПАДКОВОГО ВЕКТОРА У ВЕЛИКИХ РОЗМІРНОСТЯХ

Т. БОДНАР, С. МАЗУР, С. МУХІНЮЗА, Н. ПАРОЛЯ

Анотація. У статті ми розглядаємо добуток сингулярної випадкової матриці Вішарта та сингулярного нормального випадкового вектора. Отримано дуже корисне стохастичне представлення цього добутку, за допомогою якого виводиться його характеристична функція та асимптотичний розподіл при подвійному асимптотичному режимі. Також, із використанням методу Монте-Карло, показано хороші результати апроксимації, отримані за допомогою виведеного багатовимірного асимптотичного розподілу в умовах скінченної вибірки.

ПРОИЗВЕДЕНИЕ СИНГУЛЯРНОЙ СЛУЧАЙНОЙ МАТРИЦЫ ВИШАРТА И СИНГУЛЯРНОГО НОРМАЛЬНОГО СЛУЧАЙНОГО ВЕКТОРА В БОЛЬШОЙ РАЗМЕРНОСТИ

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Аннотация. В статье мы рассматриваем произведение сингулярной случайной матрицы Вишарта и сингулярного нормального случайного вектора. Получено очень полезное стохастическое представление этого произведения, с помощью которого выводится его характеристическая функция и асимптотическое распределение при двойном асимптотическом режиме. Также, с использованием метода Монте-Карло, показаны хорошие результаты аппроксимации, полученные с помощью выведенного многомерного асимптотического распределения в условиях конечной выборки.

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