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# **Term structure estimation based on a generalized optimization framework**

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## Dedication

To my loving family, Emah Wamuhu Ndengo,  
Bhakita Shiku Ndengo,  
Blaise Mugisha Ndengo,  
Blandine Kedza Ndengo.





# Abstract

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The current work is devoted to estimating the term structure of interest rates based on a generalized optimization framework. To fix the ideas of the subject, we introduce representations of the term structure as they are used in finance: yield curve, discount curve and forward rate curve.

Yield curves are used in empirical research in finance and macroeconomic to support financial decisions made by governments and/or private financial institutions. When governments (or financial corporations) need fundings, they issue to the public (i.e. the market) debt securities (bills, bonds, notes, etc ) which are sold at the discount rate at the settlement date and promise the face value of the security at the redemption date, known as maturity date. Bills, notes and bonds are usually sold with maximum maturity of 1 year, 10 years and 30 years respectively.

Let us assume that the government issues to the market zero-coupon bonds, which provide a single payment at maturity of each bond. To determine the price of the security at time of settlement, a single discount factor is used. Thus, the yield can be defined as the discount rate which makes the present value of the security issued (the zero-coupon bond) equal to its initial price. The yield curve describes the relationship between a particular yield and a bond’s maturity. In general, given a certain number of bonds with different time to maturity, the yield curve will describe the one-to-one relationship between the bond yields and their corresponding time to maturity. For a realistic yield curve, it is important to use only bonds from the same class of issuer or securities having the same degree of liquidity when plotting the yields.

Discount factors, used to price bonds, are functions of the time to maturity. Given that yields are positive, these functions are assumed to be monotonically decreasing as the time to maturity increases. Thus, a discount curve is simply the graph of discount factors for different maturities associated with different securities.

Another useful curve uses the forward rate function which can be deduced from both the discount factor and the yield function. The forward rate is the rate of return for an investment that is agreed upon today but which starts at some time in the future and provides payment at some time in the future as well. When forward rates are used, the resulting curve is referred to as the forward rate curve. Thus, any of these curves, that is, the yield curve, the discount curve or the forward rate curve, can be used to represent what is known as the term structure of interest rate. The shapes that the term structure of interest rates can assume include upward sloping, downward sloping, flatness or humped, depending on the state of the economy. When the expectations of market participants are incorporated in the construction of these curves representing the term structure, their shapes capture and summarize the cost of credit and risks associated with every security traded.

However, constructing these curves and the choice of an appropriate representation of the term structure to use is not a straightforward task. This is due to the complexity of the market data, precisely, the scarcity of zero-coupon bonds which constitutes the backbone of the term structure. The market often provides coupons alongside market security prices for a small number of maturities. This implies that, for the entire maturity spectrum,

yields can not be observed on the market. Based on available market data, yields must be estimated using traditional interpolation methods. To this end, polynomial splines as well as parsimonious functions are the methods mostly used by financial institutions and in research in finance. However, it is observed in literature that these methods suffer from the shape constraints which cause them to produce yield curves that are not realistic with respect to the market observations. Precisely, the yield curves produced by these methods are characterized by unrealistic fit of the market data, either in the short end or in the long end of the term structure of interest rate.

To fill the gap, the current research models the yield curve using a generalized optimization framework. The method is not shape constrained, which implies that it can adapt to any shape the yield curve can take across the entire maturity spectrum. While estimating the yield curve using this method in comparison with traditional methods on the Swedish and US markets, it is shown that any other traditional method used is a special case of the generalized optimization framework. Moreover, it is shown that, for a certain market consistency, the method produces lower variances than any of the traditional methods tested. This implies that the method produces forward rate curve of higher quality compared to the existing traditional methods.

Interest rate derivatives are instruments whose prices depend or are derived from the price of other instruments. Derivatives instruments that are extensively used include the forward rate agreement (FRA) contracts where forward rate is used and the interest rate swap (IRS) where LIBOR rate is used as floating rate. These instruments will only be used to build up the term structure of interest rates. Since the liquidity crisis in 2007, it is observed that discrepancies in basis spread between interest rates applied to different interest rate derivatives have grown so large that a single discount curve is no longer appropriate to use for pricing securities consistently. It has been suggested that the market needs new methods for multiple yield curves estimation to price securities consistently with the market. As a response, the generalized optimization framework is extended to a multiple yield curves estimation. We show that, unlike the cubic spline for instance, which is among the mostly used traditional method, the generalized framework can produce multiple yield curves and tenor premium curves that are altogether smooth and realistic with respect to the market observations.

U.S. Treasury market is, by size and importance, a leading market which is considered as benchmark for most fixed-income securities that are traded worldwide. However, existing U.S. Treasury yield curves that are used in the market are of poor quality since they have been estimated by traditional interpolation methods which are shape constrained. This implies that the market prices they imply contain lots of noise and as such, are not safe to use. In this work, we use the generalized optimization framework to estimate high-quality forward rates for the U.S. Treasury yield curve. Using efficient frontiers, we show that the method can produce low pricing error with low variance as compared to the least squares methods that have been used to estimate U.S. Treasury yield curves.

We finally use the high-quality U.S. Treasury forward rate curve estimated by the generalized optimization framework as input to the essentially affine model to capture the randomness property in interest rates and the time-varying term premium. This premium is simply a compensation that is required for additional risks that investors are exposed to. To determine optimal investment in the U.S. Treasury market, a two-stage stochastic programming model without recourse is proposed, which model borrowing, shorting and proportional transaction cost. It is found that the proposed model can provide growth of wealth in the long run. Moreover, its Sharpe ratio is better than the market index and its

Jensen's alpha is positive. This implies that the Stochastic Programming model proposed can produce portfolios that perform better than the market index.



# Populärvetenskaplig sammanfattning

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Detta arbete handlar om att estimeras räntestrukturen utifrån ett generaliserat optimeringsramverk. För att beskriva arbetet kommer de vanligaste använda representationerna av räntestrukturen att presenteras: nollkupongräntor, diskonteringsfaktorer och terminsräntor.

Räntekurvor används i empirisk forskning inom finans och i makroekonomi för att stödja finansiella beslut som fattas av staten eller andra privata finansiella institutioner. När stater (eller företag) behöver finansiering från marknaden, utfärdar de värdepapper (t.ex. statsskuldsväxlar eller obligationer) som sedan säljs till dess nuvärde, och utbetalar det nominella beloppet på förfallodagen. Statsskuldsväxlar har en löptid på upp till ett år och obligationer har löptider ver 1 år.

Utställaren av en obligation betalar ofta även ut en sekvens av kassaflöden under obligationens löptid, så kallade kuponger, antingen halvårsvis eller en gång per år. På förfallodagen erhåller obligationsinnehavaren både det nominella beloppet och en kupongutbetalning. Varje kassaflöde diskonteras med en diskonteringsränta. När utfärdaren endast betalar det nominella värdet på förfallodagen, kallas obligationen för en nollkupongsobligation. För att bestämma värdet på obligationen används en diskonteringsfaktor, som även kan uttryckas som en diskonteringsränta. Nollkupongräntekurvan beskriver sambandet mellan räntan för olika löptider, och det finns en koppling mellan nollkupongräntekurvan och obligationspriserna. För att nollkupongräntekurvan skall vara realistisk måste obligationerna ha samma emittent och vara lika likvida.

Diskonteringsfaktorer som används för att prissätta obligationer beror på löptiden. Givet att räntorna är positiva, kommer funktionen att vara monotont avtagande för ökande löptider.

En annan användbar räntekurva beskriver terminsräntorna, vilka kan härledas från diskonteringsfaktorer eller nollkupongräntor. Terminsräntan är en idag överenskommen ränta för ett lån som startar vid en framtida tidpunkt och som förfaller vid en senare tidpunkt. När terminsräntor används, kallas räntekurvan för terminsräntekurvan. Därmed kan vilken som helst av diskonteringsfaktorkurvan, nollkupongräntekurvan eller terminsräntekurvan användas för att beskriva räntekurvan. Räntekurvor antar vanligen en form som är uppåt lutande, nedåt lutande, platt eller att den har en puckel beroende på konjunkturen. Räntekurvorna innefattar marknadsaktörernas förväntningar, och fångar därmed premier för exponering mot kreditrisker och andra risker som finns i de handlade tillgångarna.

Att estimeras en räntekurva och att välja en lämplig beskrivning av den är inte enkelt. Det grundläggande problemet är att räntekurvan måste estimeras från ett begränsat antal instrument, och ofta innefattar instrumenten även kupongräntor. Det innebär att de eftersökta räntorna inte är direkt observerbara i marknadspriserna. Baserat på de tillgängliga marknadspriserna, estimeras därför räntekurvorna med traditionella interpolationsmetoder. Vanligtvis används då polynomiska spline funktioner eller funktioner med ett fåtal parametrar av marknadsaktörer och forskare inom finans. Dock har det observerats i forskning att de här metoderna ger upphov till räntekurvor som är orealistiska, de har i allmänhet svårigheter att estimeras antingen korta eller långa räntor.

För att förbättra estimeringen av räntekurvor, används i den här avhandlingen ett nytt generaliserat optimeringsramverk för att estimeras räntekurvorna. Den här metoden är inte begränsad till olika specifika former på räntekurvan, vilket betyder att den kan anta alla möjliga former på alla delar av räntekurvan. I avhandlingen visas att de traditionella metoderna är specialfall till modellen. I utvärderingar av det nya optimeringsbaserade ramverket och de traditionella metoderna på den svenska och amerikanska marknaden framkommer det att, för en viss nivå av konsistens med marknadspriserna så erhålls lägre varians med den nya metoden jämfört med alla traditionella metoder. Det här innebär att metoden producerar mer högkvalitativa räntekurvor än traditionella metoder.

Räntederivat är instrument vars priser beror på priset av en annan tillgång. Derivatinstrument som används mycket inkluderar FRA kontrakt och ränteswappar, som används för att estimeras räntekurvan. Sedan likviditetskrisen, som inleddes 2007, har prissättningen på kontrakt som baseras på LIBOR med olika löptider vuxit så pass mycket att det inte längre räcker med en räntekurva för att prissätta alla de här kontrakten konsistent. Det finns ett behov av nya metoder för att estimeras multipla räntekurvor. Det generaliserade optimeringsramverket har därför vidareutvecklats till ett ramverk för att estimeras multipla räntekurvor. Vi kan då visa att jämfört med en traditionellt vanligt använd metod som kubisk spline, så kan det generaliserade optimeringsramverket estimeras multipla räntekurvor som är jämna och konsistenta med marknadspriserna.

Marknaden för amerikanska statspapper är storleksmässigt en betydande marknad, och fungerar som referens för många andra delar av räntemarknaden. De nuvarande räntekurvorna är av dålig kvalitet, då de estimerats med traditionella metoder som är begränsade till vissa bestämda former. Det innebär att priserna som räntekurvorna implicerar innehåller mycket brus, och inte alltid är tillförlitliga. I den här avhandlingen använder vi det generaliserade optimeringsramverket för att estimeras högkvalitativa terminsräntor för amerikanska statspapper. Vi visar att vi kan bestämma stabila räntekurvor med lägre varians i kombination med lägre prissättningsfel jämfört med traditionella minstakvadratmetoder som använts för att estimeras amerikanska statsräntekurvor.

Vi har slutligen använt de högkvalitativa amerikanska statsräntekurvorna i en huvudsakligen affin modell för att fånga osäkerheten i räntekurvor och den tidsvarierande räntepremien. Den här premien är en kompensation för den risk som investerare på räntemarknaden exponeras mot. För att bestämma optimala investeringar används en stokastisk programmeringsmodell med två tidssteg, där vi modellerar blankning, belåning och proportionella transaktionskostnader. Modellen genererar positiva avkastningar, och vid utvärdering har den bättre Sharpekvot än marknadsportföljen, och Jensens alfa är positiv. Stokastisk programmeringsmodellen kan därmed ta fram optimala beslut, som är bättre än marknadsportföljen.

---

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# List of Symbols

## 1. SYMBOLS AND OPERATORS

$P(t, T)$	Price of a zero-coupon bond
$y(t, T)$	Zero-coupon yield
$D(t, T)$	Discount factor
$B(t)$	Saving or Bank account
$r(t)$	Instantaneous spot rate
$f(t, T)$	Instantaneous forward rate
$\Delta t$	Length of time period
$\mathbb{E}^Q[\cdot \mathcal{F}_t]$	Expectation operator under certain measure $Q$
$\mathcal{F}_t$	Information available up to time $t$
$\gamma_t$	Penalty function used to penalize the slope
$\varphi_t$	Penalty function used to penalize the curvature
$z_e$	Deviations from the market unique price
$z_b$	Deviations from the market bid/ask price
$F_e$	Diagonal matrix which indicates which instrument is allowed to deviate from the market unique price
$F_b$	Diagonal matrix which indicates which instrument is allowed to deviate from the market bid/ask price
$E_e$	Diagonal matrix containing penalties for instruments that deviates from the market unique price
$E_b$	Diagonal matrix containing penalties for instruments that deviates from the market bid/ask price
$g_e(f)$	Vector-valued function that transforms the forward rates into market unique price
$g_b(f)$	Vector-valued function that transforms the forward rates into market bid/ask price
$x_l$	Lower bound from market prices
$x_u$	Upper bound from market prices
$f_l$	Lower bound for forward rate
$\mathcal{M}$	A set of interpolation methods
$P^m(T)$	Theoretical zero-coupon bond price estimated from method $m \in \mathcal{M}$
$\omega_i$	Weight for instrument $i$
$L_\tau(T_i, T_j)$	Libor rate for tenor $\tau$
$\bar{\pi}_\tau(t)$	Compensation associated with tenor $\tau$
$\mathcal{U}(\cdot)$	Utility function
$\Omega$	Universe of interest rate instruments
$\mathcal{U}$	Subset of interest rate instruments that should be consistent with unique price
$\mathcal{B}$	Subset of interest rate instruments that should be consistent with bid/ask spreads
$\mathcal{F}_T$	Set of forward rates which is specific for each interpolation method
$\mathcal{F}_{LS}$	Set of forward rates which is specific for each Least Squares method
$Q^{T_n}$	$T_n$ -forward measure

## 2. ABBREVIATIONS AND ACRONYMS

PCA	Principal Components Analysis
OIS	Overnight Index Swap
IRS	Interest Rate Swap
FRA	Forward Rate Agreement
VRP	Variable Roughness Penalty
IBOR	Inter Bank Offered Rate
IMM	International Monetary Market
LIBOR	London Inter bank Offered Rate
EURIBOR	Europe Inter Bank Offered Rate
STIBOR	Stockholm Inter Bank Offered Rate
TS	Tenor Swap
AD	Adams and Deventer
LS	Least Squares
CD	Certificate of Deposit
ON	Over-night
TN	Tomorrow Next
SEK	Swedish krona
USD	United State Dollar
EONIA	Euro Overnight Index Average
OTC	Over The Counter
CAPM	Capital Asset Pricing Model

# Part I

## Background on estimation of the term structure of interest rates

## 1. INTRODUCTION

The relationship between interest rates and the term to maturity, commonly called the term structure of interest rates, is fundamental for financial institutions. The term structure of interest rates is used for various financial objectives. Given the current interest rate and the implied forward rate curves, the yield curve is used to assess the impact of economic policy over the entire economy. This includes forecasting the yields on long-term securities, supporting monetary policy and debt policy, ensuring reasonableness in derivative pricing and hedging. Thus, appropriate methods for estimating the yield curves need be identified which can be used to support decisions making.

In literature, two main streams of yield curves are discussed. On the one hand, scholars explore extensively yield curve models which focus on the dynamics of the term structure. The need for such models is motivated by the ever growing necessity to price accurately on long term basis interest rate derivatives. To achieve such goal, it requires to model, not only the yield curves but also the volatility of interest rates as they evolve in time. A succinct exposé on short rate models can be found in textbooks such as (Brigo and Mercurio 2006) and the like. These models are deduced from equilibrium condition or/and no-arbitrage condition in assets pricing and will not be part of the current work. On the other hand, scholars develop spline-based models and parametric models of the yield curves whose implementations have been popular for financial institutions. Well known models include Hagan and West (2006) for simple interpolation methods, McCulloch (1971, 1975), and Adams and Deventer (1994) for the spline-based models, Nelson and Siegel (1987), Svensson (1994) for the parsimonious functions. This second stream of yield curve models constitute a building block for the current research.

The current work is composed of four papers covering each one of the following aims: The first aim of the thesis is to show that the generalized optimization framework for estimating the yield curves proposed in Blomvall (2011) produces high-quality yield curves, i.e. yield curve that are smooth, reasonable and consistent with the market prices. We show that, for a certain level of market consistency, the method produces smaller variance than all other interpolation methods. We also show that all traditional methods for estimating yield curves are special cases of the generalized optimization framework. From PCA analysis, we find that the short end rates move independently of the long end. This is supported by the fact that it is the central bank which regulates the short end rate to control and regulate inflation and that longer term rates are affected by the future expectation of the inflation. The second aim is to extend the method to a multiple yield curve estimation in order to satisfy the current market trends where discrepancies are observed between overnight index swap (OIS), forward rate agreement (FRA) and interest rate swap (IRS). These price differences necessitate a new pricing methodology where appropriate discount functions corresponding to each tenor are used for market consistency. The third aim is to use the same framework to estimate the U.S. Treasury yield curve. This is motivated by the fact that the U.S. Treasury yield curves are considered as the benchmark from market and influence the pricing of other debt securities. Finally, we use high-quality yield curves estimated using the generalized optimization framework as input to the essentially affine term structure (Duffee 2002) to capture the time-varying term premium, which is a compensation required for investors who are exposed to the duration risks. These high-quality yields are subsequently used in a two-stage Stochastic Programming model that is proposed to study the long run consequences of Stochastic Programming investments in the U.S. Treasury market.

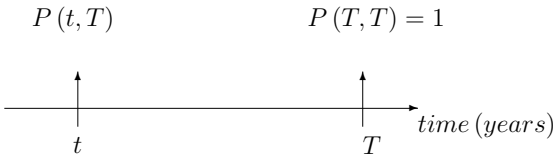
## 2. REPRESENTATIONS OF THE TERM STRUCTURE OF INTEREST RATES

In this section, we introduce basic theoretical constructs that are used to represent the term structure of interest rates and highlight the relationship among them: the yield curve, the forward rate curve and the discount curve.

A discount bond which starts at time  $t$  and matures at time  $T$  is a security with the promise from the bond issuer to pay a unit currency, say *USD 1*, to the bond holder when it matures. Its price at time  $t \leq T$ , denoted by  $P(t, T)$ , attains its maximum at time  $T$ . Thus, by definition, it follows that

$$(1) \quad P(T, T) = 1.$$

To emphasize the single payment embedded in a discount bond, most literature refers to this security as a *zero-coupon* bond. The cash flow of a  $T$ -bond that pays one unit of currency at maturity time  $T$  can be visualized in the figure below



where  $t$  is usually considered as the bond settlement date.

The yield, denoted by  $y(t, T)$ , is regarded as the continuously compounded rate of return for investment which causes the price of a discount bond,  $P(t, T)$ , to increase up to 1 at time  $T$ . Thus, by definition, it holds that

$$(2) \quad P(t, T) e^{y(t, T)(T-t)} = 1$$

which implies

$$(3) \quad P(t, T) = e^{-y(t, T)(T-t)}.$$

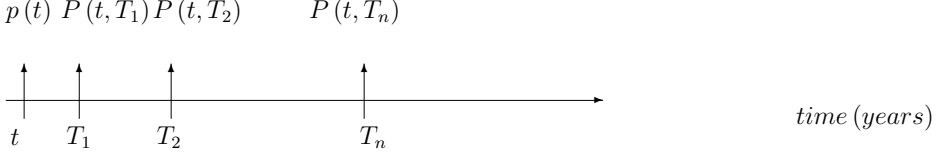
From (3), it follows that

$$(4) \quad y(t, T) = -\frac{\log P(t, T)}{T-t}.$$

A bond that provides multiple payments (or coupons) to the bond holder at regular frequencies is referred to as a *coupon-bearing bond*. These intermediate payments are naturally included in the valuation proceedings of this security. Let  $p(t)$  denote the time  $t$  market value of a fixed coupon bond, having coupon payment dates scheduled as  $T_1 < T_2 < \dots < T_n$  with corresponding coupons,  $c_1, \dots, c_n$  and a nominal investment,  $N$ . Then, using equation (3), the time  $t$  market price of the bond is given by

$$(5) \quad p(t) = \sum_{i=1}^n c_i P(t, T_i) + P(t, T_n) N = \sum_{i=1}^n c_i e^{-y(t, T_i)(T_i-t)} + N e^{-y(t, T_n)(T_n-t)}, \quad t \leq T_1$$

where  $P(t, T_i)$ , depicted in the figure below, are appropriate discount factors associated



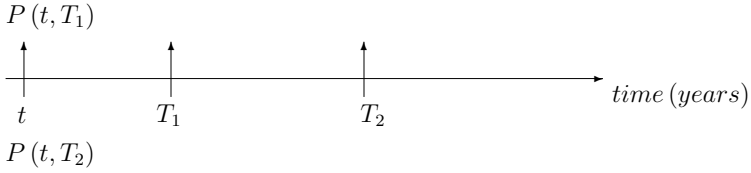
with the coupon payment  $c_i$  and  $y(t, T_i)$ ,  $i = 1, \dots, n$ , is then the continuously compounded yield defined in (4).

The instantaneous spot rate at time  $t$ , denoted by  $r(t)$ , is thought of as the yield on the currently maturing bond. Using equation (4), this rate is given by

$$(6) \quad r(t) = \lim_{T \rightarrow t} y(t, T) = y(t, t).$$

In other words, the instantaneous spot rate is the rate of return that is earned by investors over the next very short interval of time. From equation (4), the yield curve is simply the function  $T \rightarrow y(t, T)$  which, at time  $t$ , describes the relationship between the bonds' yields and their respective time to maturity.

Using the figure below, we now consider the rate of return of an investor who, at time  $t$ , holds a bond with maturity at time  $T_1 > t$  whose price is  $P(t, T_1)$  and decides to roll it over the next equivalent period of time,  $T_2 > T_1$  to a fixed rate which is agreed upon today, denoted by  $f(t, T_1, T_2)$ . This should be equivalent to investing, at time  $t$ , in a bond maturing at time  $T_2$  which trades for the price  $P(t, T_2)$ . This considerations imply that the forward rates are interest rates, or the rate of return, which are locked in today for an investment in a future time period, and most importantly, they are set consistently with the current term structure of discount factors.



Formally, these rates are deduced from the equation

$$(7) \quad P(t, T_1) = P(t, T_2) e^{(T_2 - T_1)f(t, T_1, T_2)},$$

which, must hold for any pair of maturities  $T_i < T_j$ . Solving for  $f(t, T_1, T_2)$ , we obtain the formal definition of the forward rate as

$$(8) \quad f(t, T_1, T_2) = \frac{1}{T_2 - T_1} \log \left[ \frac{P(t, T_1)}{P(t, T_2)} \right].$$

Using the definitions in (3) and (4), equation (8) can be written as

$$(9) \quad f(t, T_1, T_2) = \frac{y(t, T_2)(T_2 - t) - y(t, T_1)(T_1 - t)}{T_2 - T_1}.$$

Equation (9) defines the rate of return for an investment on a forward contract entered at time  $t$  but starting at time  $T_1$  and provides payment at time  $T_2$ .

To define the instantaneous forward rate, denoted by  $f(t, T)$ , we set  $T_1 = T$  and let  $T_2 \rightarrow T$ . This yields

$$(10) \quad f(t, T) = \lim_{T_2 \rightarrow T^+} f(t; T, T_2) = -\frac{\partial \log P(t, T)}{\partial T}.$$

In other words, the instantaneous forward rate can be seen as the overnight forward rate which has only one day after the settlement date. The forward rate curve is thus the function  $T \rightarrow f(t, T)$ , which is the graph of forward rates for all maturities.

Using (4), equation (10), can also be written as

$$(11) \quad f(t, T) = -\frac{\partial}{\partial T} \log(P(t, T)) = \frac{\partial}{\partial T} [y(t, T)(T - t)].$$

Thus, given the values of  $f(t, T)$ , for  $0 \leq t \leq T$ , in (11), we recover the price  $P(t, T)$  in (3) as follows

$$\int_t^T f(t, u) du = -[\log P(t, T) - \log P(t, t)].$$

From  $P(t, t) = 1$ , we have that

$$\int_t^T f(t, u) du = -\log P(t, T).$$

Hence, for  $0 \leq t \leq T$ , it holds that

$$(12) \quad P(t, T) = \exp\left(-\int_t^T f(t, u) du\right).$$

Equating (3) and (12), it then follows that

$$(13) \quad y(t, T) = \frac{1}{T - t} \int_t^T f(t, u) du$$

in which the continuously compounded spot rate is seen as the average of the forward rates prevailing between  $t$  and  $T$ .

To define the discount factor, we introduce the relationship between the saving account and the short rate. According to (12), an investment of *USD* 1 at time  $t = 0$  for period  $(0, \Delta t)$  yields a return given by

$$(14) \quad \frac{1}{P(0, \Delta t)} = \exp\left(\int_0^{\Delta t} f(0, u) du\right) = 1 + r(0) \Delta t + o(\Delta t)$$

where  $o(\Delta t)/\Delta t \rightarrow 0$  as  $\Delta t \rightarrow 0$ . A saving account, or bank account,  $B(t)$ , is an asset growing instantaneously between time  $t$  and  $t + \Delta t$ , at short rate  $r(t)$  and is computed as

$$(15) \quad B(t + \Delta t) = B(t) (1 + r(t) \Delta t).$$

As  $\Delta t \rightarrow 0$ , we obtain

$$(16) \quad dB(t) = r(t) B(t) dt.$$

Since  $B(0) = 1$ , it follows that

$$(17) \quad B(t) = \exp\left\{\int_0^t r(s) ds\right\}.$$

As such,  $B(t)$  is the risk-free asset since the future value in the short interval, from  $t$  to  $t + \Delta t$  is known with certainty. The discount factor, denoted by  $D(t, T)$ , between time  $t$  and  $T$  is thus defined, from equation (17), as

$$(18) \quad D(t, T) = \frac{B(t)}{B(T)} = \frac{\exp \left\{ \int_0^t r(s) ds \right\}}{\exp \left\{ \int_0^T r(s) ds \right\}} = \exp \left\{ - \int_t^T r(s) ds \right\}$$

which is the amount, at time  $t$ , equivalent  $US\ 1\$$  that is paid at time  $T$ .

The difference between  $P(t, T)$  and  $D(t, T)$  lies in the nature of the short rate  $r(t)$ . Following Brigo and Mercurio (2006, p.4),

$$(19) \quad D(t, T) = \begin{cases} P(t, T), & \text{if } r(t) \text{ is deterministic} \\ \text{random variable,} & \text{if } r(t) \text{ is stochastic} \end{cases}$$

in which case it depends upon the evolution of  $r(t)$  from time  $t$  to  $T$ . The link between both quantities is defined by the relation

$$(20) \quad P(t, T) = \mathbb{E}^Q [D(t, T) | \mathcal{F}_t]$$

where  $\mathbb{E}^Q[\cdot]$  is the expectation operator under a certain probability measure  $Q$ , and  $\mathcal{F}_t$  is the information available up to time  $t$ . For now, we assume that the short rate,  $r(t)$  is deterministic and so the discount factor, equivalent then to (12), can be expressed as

$$(21) \quad D(t, T) = \exp \left( - \int_t^T f(t, u) du \right).$$

Thus, the discount curve, denoted by  $T \rightarrow D(t, T)$ , is simply the graph that describes the relationship between discount factors and their associated maturities.

To construct the term structure of interest rates, any of the following representations can be used

- the forward rate curve,  $T \rightarrow f(t, T)$ ,
- the yield curve,  $T \rightarrow y(t, T)$  or
- the discount curve,  $T \rightarrow D(t, T)$ .

since they are all equivalent.

Depending on the state of the economy, the yield curve can take different shapes ranging from ascending, descending, horizontal or humped. For notation, we set the time  $t = 0$  so as we can use a more simpler notation,  $d(T)$ ,  $f(T)$  and  $y(T)$  for discount factor, forward rate and yield respectively.

In practice, yield curves, forward rate curves or discount curves can not be observed because the market provides only bond prices for a limited number of maturities as well as coupon payments. Therefore, yield curves must be estimated from bond prices using adequate interpolation methods. The current research is concerned with methods for estimating yield curve (or forward rate curve). We seek to identify and test, against traditional methods, an estimation method with high-quality yield curves. Yield curves are widely and extensively used by financial institutions to support financial decisions.

### 3. OVERVIEW OF PREVIOUS WORKS AND CURRENT RESEARCH CONTRIBUTION

The main objective each interpolation method for estimating yield curves seeks to achieve is to determine yields, that is,  $y(T)$  for all  $T$ . It is preferable if these yields in



either of the representation of the term structure, (11), (12) and (21) or (13) are smooth, realistic and consistent with the market observations.

**3.1. Previous research on estimation of the term structure.** Pioneer works on interpolation methods for estimating yield curves can be put into three groups. The first group of scholars uses spline functions. Early works in this group include McCulloch (1971, 1975) and McCulloch and Kwon (1993) who model the discount curve with a spline. They found that the fitted discount curve provides poor fit of the yield curves, especially at the longest maturities where the yield curve exhibited flatness behavior. For this group, the forward rate curve,  $T \rightarrow f(T)$ , implied by the method is not smooth and could be even negative, since the slope of the discount curve is not explicitly constrained to be strictly decreasing.

The second group of scholars use the exponential splines. To circumvent discontinuity of the forward rate curve observed with the spline-based functions, Vasicek and Fong (1982) model the discount curve with exponential splines. To ensure that the forward rates and zero-coupon bond yields converge to an asymptote as the maturity tends to infinity, they instead used a negative transformation of maturity. If their model fits the long end as desired, its drawback is that it does not guarantee a positive forward rate since its estimation requires iterative nonlinear optimization where it is tricky to constrain the method to produce positive forward rates always. This makes these methods prone to arbitrage opportunities.

The third group uses a parsimonious functional to enhance varying shapes of the yield curve. Nelson and Siegel (1987) introduces a parsimonious function to model the instantaneous forward rates as a solution to a second-order differential equation with constant coefficients whose characteristic equation has real and equal roots, which later was extended by Svensson (1994) to increase its flexibility. The forward rate curves produced from these methods are smooth but still unable to price accurately instruments at the longest end of the yield curve due to high level of non-convexity of the methods, which can cause large fluctuations in long rates.

**3.2. Contribution of this work.** Given results from previous researches, the main problem of finding a method that produces high-quality yield curve has remained partially unanswered as each method exhibits noticeable drawbacks that prevent the methods to produce yield curve of high-quality. The current research is an attempt to fill this gap.

To this end, we first suggest an approach that uses a generalized optimization method discussed in (Blomvall 2011). Unlike previous approaches, which rely entirely on the functional forms that are used to model any of (11), (12) and (21) or (13), this method is a constrained optimization-based method which produces high-quality yield curves (or forward rate curves). The improvement in quality is validated in tests using actual market data, traditional methods and Kalman filtering. Secondly, the method is extended to a multiple yield curve framework. This is done to respond to the current increasing market demand after the liquidity crisis (2008) for a new methodology to price contracts of different tenors consistently. Thirdly, since available U.S. Treasury yield curves contain lots of noise, the method is used to estimate high-quality forward rates for U.S. Treasury market which can be employed in research as source of high-quality data. Lastly, we use high-quality yield curves estimated using the generalized optimization framework as input to the essentially affine term structure (Duffee 2002) to capture the time-varying term premium, which is a compensation required for investors who are exposed to the duration risks. These high-quality yields are then used in a two-stage Stochastic Programming

model that is proposed to determine the long run consequences of Stochastic Programming investments in the U.S. Treasury market.

**3.3. Description of the model.** In discrete time space, the forward rates (10), at time  $t = 0, 1, 2, \dots, n$ , are given by

$$(22) \quad f_t = \frac{r_{t+1}T_{t+1} - r_t T_t}{\xi_t} ; \quad \xi_t = T_{t+1} - T_t.$$

where  $r_t$  are spot rates. The roughness in the forward rate curve is measured as

$$(23) \quad h(f) = \frac{1}{2} \sum_{t=0}^{n-2} \gamma_t \left( \frac{f_{t+1} - f_t}{\xi_t} \right)^2 \xi_t + \frac{1}{2} \sum_{t=1}^{n-2} \varphi_t \left( \frac{2}{\xi_{t-1} + \xi_t} \left( \frac{f_{t+1} - f_t}{\xi_t} - \frac{f_t - f_{t-1}}{\xi_{t-1}} \right) \right)^2 \frac{\xi_{t-1} + \xi_t}{2}.$$

where  $\gamma_t$  and  $\varphi_t$  are respectively penalty functions.

Denote by  $z_e$  and  $z_b$  deviations from the market prices of unique prices and of bid/ask prices<sup>1</sup>,  $F_e$  and  $F_b$  diagonal matrices indicating instruments that are allowed to deviate from market prices,  $E_e$  and  $E_b$  are both diagonal matrices containing penalties for instruments that deviate from the market price. The functions,  $g_e(f)$  and  $g_b(f)$  are respectively functions that transform the forward rates into the market unique price and bid/ask price,  $x_l$  and  $x_u$  being respectively lower and upper limits market prices.

We solve the optimization problem

$$(24) \quad \begin{aligned} \min_{f, x, z_e, z_b} \quad & h(f) + \frac{1}{2} z_e^T E_e z_e + \frac{1}{2} z_b^T E_b z_b \\ \text{s.t.} \quad & g_e(f) + F_e z_e = \rho \\ & x_l \leq g_b(f) + F_b z_b \leq x_u \\ & f \geq f_l \\ & f \in \mathcal{F}, \end{aligned}$$

where  $f_l$  is a lower bound, often set to zero, and  $\mathcal{F}$  contains additional constraints on the forward rate curve.

The model above is a generalized framework for estimating forward rates. Through setting of parameters, we have shown that the model has ability to capture movements of yields and can provide a high-quality yield curve. We show that traditional methods for estimating yield curves are special cases of (24) with the difference being the formulation of constraints which are adapted to each interpolation method which defines the set  $\mathcal{F}$ . In out-of-sample tests, using Swedish and U.S. market, we show that for the same level of market consistency, the method produces lower variance than all other interpolation methods tested.

#### 4. CRITERIA FOR JUDGING INTERPOLATION METHODS AND EVALUATION MEASURES

To assess the quality of the yield curve interpolation method, appropriate criteria and statistical measures are used. Criteria for assessing the quality of the yield curve interpolation methods are proposed in (Hagan and West 2006, p.91-92).

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<sup>1</sup>By definition, bid price is the highest price that the buyer or bidder is willing to pay for a instruments while ask price is the lowest price the seller is willing to sell the instruments.

**4.1. Criteria for assessing high-quality yield curves.** Let  $\mathcal{M} = \{m_1, \dots, m_\tau\}$  denote the set of available methods for estimating yield curves. Assume that we want to estimate a zero-coupon bond curve,  $T \rightarrow P^m(T)$  from the market quotes  $P = (p_1, \dots, p_n)^T$  by using method  $m \in \mathcal{M}$ .

In general, given the graph,  $T \rightarrow P^m(T)$ , we first examine how "good" the forward rate,  $f_t$ , looks like. In this instance, we seek to determine whether  $f_t \geq 0$  and also whether it is continuous. The first requirement guarantees no-arbitrage opportunity while the second improves the ability to price interest rate derivatives. Recently, some cases have been observed where  $f_t < 0$  due to the extreme conditions on financial markets.

Secondly, we study localness property of the method. In other word, we examine if a perturbation in the input data at some time does affect also points elsewhere over the entire yield curve.

Lastly, stability of the forward rates can be checked. This is measured by considering the maximum basis points<sup>2</sup> change in the forward rate curve that corresponds to a fixed change in one of the inputs. The presence of oscillations in the forward rate curve or yield curve signals instability of the forward rate curve or the yield curve estimated by a method  $m \in \mathcal{M}$ .

To conclude, the best method is the one that produces a smooth and realistic yield curve and also a yield curve that is consistent with the market prices. The latter property can be captured using least squares measures.

**4.2. Criterion for assessing the reasonableness of yield curve: Shimko test.**

This is an out-of-sample test which examines reasonableness of asset prices when the interpolation method,  $m \in \mathcal{M}$  is used. It is described in (Deventer and Imai 1997, p.127, 133) and is considered as the ultimate test of accuracy and realism. The main idea of the test is to remove one asset from the data set, use a method to estimate the yield curve with the remaining of the data to estimate the missing data point, then compute the interpolated value for the missing asset.

The test is used in Adams and Deventer (1994), with the Mean absolute deviation on prices,  $MAD_P$ . The Shimko test is also suitable for both prices and yields. When it is used, it is recommended that all maturities be considered and that sample size be large in order to get a complete picture of how accurate all predictions are compared to all corresponding market observations. Although results from applying Shimko test have been satisfactory, critiques from practitioners point out the risk of inaccurately predicting the price of the asset left out of the estimation because of loss of valuable information.

**4.3. The Least Squares Measures and absolute errors.** To measure consistency with the market data of the yield curve using interpolation method,  $m \in \mathcal{M}$ , least squares measures are used (Tables 1, 2).

It is important to note that each least squares measure listed discloses on the average how far apart the predicted values  $\hat{y}_i$  (or  $\hat{P}_i$ ) are from the observed data point  $y_i$  (or  $P_i$ ) over time.

To capture the observed heteroscedasticity of fitted-price errors (in Table 2) and the theoretical relation between prices and interest rate levels, the duration-based weight,

$$(25) \quad \omega_i = \left( (1/d_i) / \left( \sum_{i=1}^n 1/d_i \right) \right)^2$$

<sup>2</sup>A basis point is equal to 0.01%.

TABLE 1. To measure goodness of fit relative to the yield curve, measures listed in this table are used but the choice of which one to use depends with the objective of the analysis.

Least squares measures	Computation formula	Weights	
		Weighted $\sum_{i=1}^N \omega_i = 1$	Mean $\omega_i = \frac{1}{N}$
Residual Sum of Squares ( $RSS_y$ )	$RSS_y = \sum_{i=1}^N \omega_i (\hat{y}_i - y_i)^2$	$WRSS_y$	$MRSS_y$
Root Residual Sum of Squares ( $RRSS_y$ )	$RRSS_y = \sqrt{\sum_{i=1}^N \omega_i (\hat{y}_i - y_i)^2}$	$WRRSS_y$	$MRRSS_y$
Absolute Deviation ( $AD_y$ )	$AD_y = \sum_{i=1}^N \omega_i  \hat{y}_i - y_i $	$WAD_y$	$MAD_y$

TABLE 2. To measure goodness of fit relative to the price curve, measures listed in this table are used but the choice of which one to use depends with the objective of the analysis.

Least squares measures	Computation formula	Weights	
		Weighted $\sum_{i=1}^N \omega_i = 1$	Mean $\omega_i = \frac{1}{N}$
Residual Sum of Squares ( $RSS_P$ )	$RSS_P = \sum_{i=1}^N \omega_i (\hat{P}_i - P_i)^2$	$WRSS_P$	$MRSS_P$
Root Residual Sum of Squares ( $RRSS_P$ )	$RRSS_P = \sqrt{\sum_{i=1}^N \omega_i (\hat{P}_i - P_i)^2}$	$WRRSS_P$	$MRRSS_P$
Relative Residual Errors ( $RRE_P$ )	$RRE_P = \sum_{i=1}^N \omega_i \left( \frac{\hat{P}_i - P_i}{P_i} \right)^2$	$WRRE_P$	$MRRE_P$
Root Relative Residual Errors ( $RRRE_P$ )	$RRRE_P = \sqrt{\sum_{i=1}^N \omega_i \left( \frac{\hat{P}_i - P_i}{P_i} \right)^2}$	$WRRRE_P$	$MRRRE_P$
Absolute Deviation ( $AD_P$ )	$AD_P = \sum_{i=1}^N \omega_i  \hat{P}_i - P_i $	$WAD_P$	$MAD_P$
Relative Absolute Error ( $RAE_P$ )	$RAE_P = \sum_{i=1}^N \omega_i \frac{ \hat{P}_i - P_i }{P_i}$	$WRAE_P$	$MRAE_P$

is commonly used where  $d_i$  is the Macaulay duration for bond  $i$  and  $N$  is the number of bonds in the valuation (Bliss 1996; Jordan and Mansi 2003).

Observe that the residual sum of squares ( $RSS$ ), the root residual sum of squares ( $RRSS$ ), the relative residual error ( $RRE$ ) and the root relative residual errors ( $RRRE$ ) are sensitive to outliers. To avoid mispricing when comparing the forward rate curves or the yield curves, it is therefore convenient to use absolute deviation ( $AD$ ), a measure which exhibits the magnitude of deviation from the observations or to use the relative absolute error ( $RAE$ ). Another measure that is widely used is the coefficient of determination ( $R_p^2$ ), defined as

$$(26) \quad R_p^2 = 1 - \frac{SSE}{SST}; \quad SSE = \sum_{i=1}^N (\hat{y}_i - y_i)^2; \quad SST = \sum_{i=1}^N (y_i - \bar{y})^2; \quad \bar{y} = \frac{1}{N} \sum_{i=1}^N y_i$$

where  $SSE$  is the sum of squares error,  $SST$  is the total sum of squares residual,  $\bar{y}$  is the mean,  $N$  is the number of observations, and  $p$  is the total number of regressors in the model.  $R_p^2$  measures the proportion of variability in the data set that is accounted for the statistical model.

Notice that there is no guideline as to which error measure is the best to use. Practitioners are left with the task of selecting an error measure which is suitable for the functional form used in modeling of the yield (or price) curves as well as for underlying securities which are subjected to the pricing process.

## 5. ESTIMATING YIELD CURVES USING TRADITIONAL INTERPOLATION METHOD

In this section, some traditional interpolation methods for estimating yield curves are reviewed within the scope of the generalized optimization framework described by (24).

**5.1. Simple Interpolation methods.** These methods are described in (Hagan and West 2006) and are designed to price contracts exactly. Considered as special cases of the optimization model (24), these methods can be written as follows:

$$(27) \quad \begin{aligned} & \min && 0 \\ & \text{s.t.} && g_e(f) = \rho \\ & && f \in \mathcal{F}_{\mathcal{I}}, \end{aligned}$$

where  $\mathcal{F}_{\mathcal{I}}$  is a set of forward rates which is specific for each method. To determine the constraints  $g_e(f)$ , it is necessary to discount cash flows, which can be done with  $r(T)T$  through (11) and (21) where  $r(T)$  is continuously compounded spot rate. Independent of which traditional method is used, as long as  $r(T)T$  is known, the value of the function  $g_e(f)$  can be computed.

Let  $\{d_i\}_{i=0}^n$  and  $\{r_i\}_{i=0}^n$  with  $r_i = r(T_i)$ , denote respectively discount factors and spot rates defined on a discrete time space,  $0 = T_0 < T_1 < \dots < T_n$ .

**5.1.1. Linear interpolation on the discount factors.** By definition, the discount factor is given by

$$(28) \quad d(T) = e^{-r(T)T}.$$

where  $r(T)$  is the continuously compounded spot rate. In discrete space, we have (28)  $d_i$  and  $d_{i+1}$  defined respectively at time  $T_i$  and  $T_{i+1}$ . Linear interpolation on (28) yields

$$(29) \quad d(T) = \frac{T_{i+1} - T}{T_{i+1} - T_i} d_i + \frac{T - T_i}{T_{i+1} - T_i} d_{i+1}; \quad T_i \leq T < T_{i+1}.$$

Using (28) and (29), the spot rates can be computed as

$$(30) \quad r(T) = -\frac{1}{T} \ln \left[ \frac{T_{i+1} - T}{T_{i+1} - T_i} d_i + \frac{T - T_i}{T_{i+1} - T_i} d_{i+1} \right].$$

In view of (11) and (30), the forward rates can also be computed as

$$(31) \quad f(T) = -\frac{-\frac{1}{T_{i+1}-T_i}d_i + \frac{1}{T_{i+1}-T_i}d_{i+1}}{\frac{T_{i+1}-T}{T_{i+1}-T_i}d_i + \frac{T-T_i}{T_{i+1}-T_i}d_{i+1}} = \frac{d_i - d_{i+1}}{(T_{i+1} - T)d_i + (T - T_i)d_{i+1}}$$

To implement this method, equation (30) is used. To determine  $r(T)T$ , with the help of (31), we have

$$(32) \quad r(T)T = \int_0^T f(t)dt = r(T_i)T_i + \int_{T_i}^T f(t)dt = r(T_i)T_i + \int_{T_i}^T \frac{d_i - d_{i+1}}{(T_{i+1} - t)d_i + (t - T_i)d_{i+1}} dt.$$

The drawbacks of the method are that the function defined in (29) may not be necessarily a decreasing function and the forward rate function in (31) is not continuous since the information held at time  $T_i$  is still active by the time  $T$  reaches  $T_{i+1}$ . As consequence,

the curve,  $T \rightarrow f(T)$  produces jumps at each node. However, since the method is local, it has the advantage that noise in one yield affects only two intervals of the spline.

5.1.2. *Raw Interpolation.* The method is referred to as linear on the logarithm of discount factors and uses the function

$$(33) \quad \ln d(T) = \frac{T_{i+1} - T}{T_{i+1} - T_i} \ln d_i + \frac{T - T_i}{T_{i+1} - T_i} \ln d_{i+1}.$$

For each interval  $T \in [T_i, T_{i+1}]$ , constant instantaneous forward rates are given by

$$(34) \quad f_i = \frac{r_{i+1}T_{i+1} - r_iT_i}{T_{i+1} - T_i}.$$

Analogously to (32) and using equation (34), we have that

$$(35) \quad \begin{aligned} r(T)T &= r_iT_i + \int_{T_i}^T f_i dt = r_iT_i + \frac{r_{i+1}T_{i+1} - r_iT_i}{T_{i+1} - T_i} (T - T_i) \\ &= \frac{T_{i+1} - T_i}{T_{i+1} - T_i} r_iT_i + \frac{T - T_i}{T_{i+1} - T_i} r_{i+1}T_{i+1} - \frac{T - T_i}{T_{i+1} - T_i} r_iT_i \\ &= \frac{T_{i+1} - T}{T_{i+1} - T_i} r_iT_i + \frac{T - T_i}{T_{i+1} - T_i} r_{i+1}T_{i+1}. \end{aligned}$$

The method is also known as the exponential interpolation, because it involves exponential interpolation of the discount factor. As such, one can write

$$(36) \quad d(T) = e^{-r(T)T} \stackrel{(35)}{=} e^{-T_{i+1}r_{i+1} \frac{T - T_i}{T_{i+1} - T_i}} e^{-T_i r_i \frac{T_{i+1} - T}{T_{i+1} - T_i}} = d_i^{\frac{T - T_i}{T_{i+1} - T_i}} d_{i+1}^{\frac{T_{i+1} - T}{T_{i+1} - T_i}}$$

which, in terms of the forward rates, is equivalent to the linear interpolation of the logarithm of the discount factors. To show this, we use equation (36) and write

$$(37) \quad f(T) = -\frac{\partial \ln d(T)}{\partial T} = -\frac{\partial \frac{T - T_i}{T_{i+1} - T_i} \ln d_{i+1} + \frac{T_{i+1} - T}{T_{i+1} - T_i} \ln d_i}{\partial T} = \frac{\ln d_i - \ln d_{i+1}}{T_{i+1} - T_i}.$$

The implied spot rate can be determined by dividing (35) through, which yields

$$(38) \quad r(T) = \frac{r_iT_i}{T} + \frac{1}{T} \left[ \frac{r_{i+1}T_{i+1} - r_iT_i}{T_{i+1} - T_i} \right] (T - T_i) = \frac{1}{T} \left( \frac{T - T_i}{T_{i+1} - T_i} r_{i+1}T_{i+1} + \frac{T_{i+1} - T}{T_{i+1} - T_i} r_iT_i \right).$$

Observe that, for each  $T_i, i = 1, 2, \dots, n$ , the instantaneous forward rate, defined in equation (10), is not defined. As consequence, the forward rate curve,  $T \rightarrow f(T)$ , has jumps at each node, an effect which indicates clearly that the curve is not continuous. An advantage of this method is its localness, but the method does not guarantee positive forward rates.

5.1.3. *Linear interpolation on the spot rates.* Given the spot rates  $\{r_i\}_{i=0}^n$  with  $r_i = r(T_i)$ , for  $T_i \leq T < T_{i+1}$ , the rates at time  $T$  are interpolated as

$$(39) \quad r(T) = \frac{T_{i+1} - T}{T_{i+1} - T_i} r_i + \frac{T - T_i}{T_{i+1} - T_i} r_{i+1}.$$

Note that the spot rate,  $r_0$ , for time point  $T_0 = 0$  can not be observed in the market. Using (11) and (39), the instantaneous forward rates can be written as

$$(40) \quad f(T) = \frac{2T - T_i}{T_{i+1} - T_i} r_{i+1} + \frac{T_{i+1} - 2T}{T_{i+1} - T_i} r_i.$$

To determine the prices, we compute

$$(41) \quad r(T)T = \frac{T - T_i}{T_{i+1} - T_i} r_{i+1}T + \frac{T_{i+1} - T}{T_{i+1} - T_i} r_iT.$$

Observe that  $r(T)T$  is not differentiable at  $T_i$  since  $r(T)$  is piecewise linear. Consequently,  $f(T)$  will not be defined for each  $T_i$ ,  $i = 1, \dots, n$  and so the resulting forward rate curve will not be continuous. Moreover, the method does not guarantee positive forward rates but it is local.

5.1.4. *Interpolation on the Logarithm of rates.* This is the log-linear interpolation defined by the expression

$$(42) \quad \ln r(T) = \frac{T_{i+1} - T}{T_{i+1} - T_i} \ln r_i + \frac{T - T_i}{T_{i+1} - T_i} \ln r_{i+1}.$$

It is also referred to as the exponential interpolation, because, for every index  $i$ , the spot rates can be written as

$$(43) \quad r(T) = r_{i+1}^{\frac{T-T_i}{T_{i+1}-T_i}} r_i^{\frac{T_{i+1}-T}{T_{i+1}-T_i}}; T \in [T_i, T_{i+1}].$$

Adding  $\ln T$  to equation (42) above, yields

$$(44) \quad \ln(r(T)T) = \frac{T - T_i}{T_{i+1} - T_i} \ln r_{i+1} + \frac{T_{i+1} - T}{T_{i+1} - T_i} \ln r_i + \ln T.$$

Expressing the instantaneous forward rate, defined in (11), using (44), one writes

$$(45) \quad f(T) = \frac{\partial r(T)T}{\partial T} = \frac{\partial e^{\ln r(T)T}}{\partial T} = e^{\ln r(T)T} \frac{\partial \ln r(T)T}{\partial T} = r(T)T \frac{\partial \ln r(T)T}{\partial T}.$$

It then follows that

$$(46) \quad \frac{f(T)}{r(T)T} = \frac{\partial \ln(r(T)T)}{\partial T} = \frac{1}{T_{i+1} - T_i} \ln \frac{r_{i+1}}{r_i} + \frac{1}{T}.$$

Hence

$$(47) \quad f(T) = r(T) \left[ \frac{T}{T_{i+1} - T_i} \ln \frac{r_{i+1}}{r_i} + 1 \right] = r_{i+1}^{\frac{T-T_i}{T_{i+1}-T_i}} r_i^{\frac{T_{i+1}-T}{T_{i+1}-T_i}} \left[ \frac{T}{T_{i+1} - T_i} \ln \frac{r_{i+1}}{r_i} + 1 \right].$$

Prices are determined using (43) as

$$(48) \quad r(T)T = r_{i+1}^{\frac{T-T_i}{T_{i+1}-T_i}} r_i^{\frac{T_{i+1}-T}{T_{i+1}-T_i}} T.$$

Drawbacks of the method is that it produces discontinuous local forward rate curves. Besides, the method does not guarantee a positive forward rate and when used, the discount function is not always decreasing.

5.2. **Other interpolation methods.** To circumvent discontinuity of the yield curve observed with simple interpolation methods, polynomial functions are used because they are continuously differentiable functions. A family of functions that is of great interest in our subject are the splines.

5.2.1. *Cubic splines.* The method is described in, for instance, (Hagan and West 2006). For  $1 \leq i \leq n - 1$ , the discrete spot rate,  $r_i(T)$ , is modeled by a cubic polynomial as

$$(49) \quad r_i(T) = a_i(T - T_i)^3 + b_i(T - T_i)^2 + c_i(T - T_i) + d_i, T \in [T_i, T_{i+1}],$$

where  $a_i$ ,  $b_i$ ,  $c_i$  and  $d_i$  are parameters to be determined for all  $n - 1$  intervals. Thus a total of  $4(n - 1)$  constraints are required to solve (49). To determine these parameters, the equations must fit the observable data (knot) points, their first and second derivatives must be equal at  $n - 2$  knot points. Formally, we solve (49) subject to  $r_i(T_i) = r_{i+1}(T_i)$ ,  $n - 1$  equations of the form,  $r'_i(T_i) = r'_{i+1}(T_i)$  and  $n - 1$  equations of the form,  $r''_i(T_i) = r''_{i+1}(T_i)$ .

If the splines are natural, then at  $T_0$  and  $T_n$ , the constraints  $r_0''(T_0) = 0$  and  $r_n''(T_n) = 0$  apply. This makes the yield curve straight at the beginning as well as at the longest maturity. However, if the financial cubic spline (see Adams and Deventer (1994)) is used, then the constraints  $r_0''(T_0) = 0$  and  $r_n'(T_n) = 0$  are added. The latter constraint ensures there is a horizontal rate asymptotic to the yield curve. This signifies that the rate can be extrapolated beyond the longest maturity. The table below gives a summary of all constraints that are needed for a continuous yield curve. The drawback of this method

	Number of equations	Formulation of constraints	Description of the constraints
1	n-1	$r_i(T_i)$	Equation (49) fits all $n$ data points: $T_0, \dots, T_{n-1}$
2	n-1	$r_i(T_{i+1})$	Equation (49) fit the $n$ data points: $T_1, \dots, T_n$
3	n-2	$r_i'(T_i) = r_{i+1}'(T_i)$	Continuity of first derivative at the knot points
4	n-2	$r_i''(T_i) = r_{i+1}''(T_i)$	Continuity of second derivative at the knot points
5	1	$r_0''(T_0) = 0$ ;	The yield curve is linear at the left hand side of the yield curve
6	1	$r_n''(T_n) = 0$ ;	The yield curve is linear at the longest maturity of the yield curve
Total	4(n-1)		

TABLE 3. Summary of constraints that are used when the cubic spline is implemented to model yield curves.

when forward rate curve is used is that the second derivative of the forward rate,  $f(T)$ , is not continuous at the knot points because it depends on the third derivative of (49), which is not restricted to be continuous at the knot points.

As consequence, the forward rate curve will not be twice differentiable, therefore not smooth. Moreover, the cubic spline based on the yield (or price) curve produces an implausible forward rate curve (Deventer et al. 2005, p.147). To remedy the unsmoothness of the yield curve by the cubic splines, the Adam-Deventer method can be used.

5.2.2. *The Adams-Deventer method.* The Quartic spline is discussed in Adams and Deventer (1994) and Deventer et al. (2005). For this method, the forward rate function (11) is modeled by the quartic polynomial

$$(50) \quad f_i(T) = a_i(T - T_i)^4 + b_i(T - T_i)^3 + c_i(T - T_i)^2 + d_i(T - T_i) + e_i,$$

where  $a_i$ ,  $b_i$ ,  $c_i$ ,  $d_i$  and  $e_i$  are parameters to be estimated for each interval  $[T_i, T_{i+1}]$ , for  $i = 0, \dots, n - 1$ . To obtain a smooth forward rate curve, a total of  $5n$  constraints, summarized in Table (4) are imposed on (50).

	Number of constraints	Formulation of constraints	Description of the constraints
1	n-1	$f_i(T_i) = f_i(T_{i+1})$	the forward rates are equal at each knot point
2	n-1	$f_i'(T_i) = f_i'(T_{i+1})$	the first derivative of (50) are equal at each knot point
3	n-1	$f_i''(T_i) = f_i''(T_{i+1})$	the second derivative of (50) are equal at the knot point
4	n-1	$f_i'''(T_i) = f_i'''(T_{i+1})$	the third derivative of (50) are equal at the knot point
5	n	$\int_{T_i}^{T_{i+1}} f(t) dt = -\ln \left[ \frac{P(T_{i+1})}{P(T_i)} \right]$	the forward rate curve should be consistent with observable data
6	1	$f_0(T_0) = r_0(T_0)$	the forward rate curve is consistent with an observable short rate
7	1	$f_{n-1}(T_n) = 0$	the slope of (50) at the rhs of the yield curve is zero
8	1	$f_0''(T_0) = 0$	the second derivative of (50) at the lhs of the yield curve is zero
9	1	$f_{n-1}''(T_n) = 0$	the second derivative of (50) at the rhs of the yield curve is zero
Total	5n		

TABLE 4. The table gives a summary of constraints that are needed when the quartic polynomial is used to model the yield curve. The abbreviations lhs and rhs stand respectively for left hand side and right hand side.



For zero coupon bonds, equation (50) has been shown to be the optimal solution to

$$(51) \quad \begin{aligned} \min \quad & \int_0^{T_n} [f''(t)]^2 dt \\ \text{s.t.} \quad & \int_0^{T_i} f(t) dt = -\ln P_i, \quad i = 1, 2, \dots, n \end{aligned}$$

where  $P_i$  is the price of a zero-coupon bond with maturity  $T_i$ . However (Deventer and Imai 1997, p.160-163) has replaced the constraint that the third order derivative should be equal to zero at  $T_0$  and  $T_n$  with constraint 6 and 7 in Table (4) to improve the quality of the yield curves. The technique outperforms cubic spline and linear smoothing techniques considerably and is more accurate in modeling true market yields according to (Adams and Deventer 1994, p.160).

**5.3. The Least squares methods.** The least squares methods do not require to reprice contracts exactly. To model the yield curve, the methods admit pricing errors to improve smoothness of the yield curve while remaining consistent with the market data. Viewed as special case of (24), one solves the optimization problem

$$(52) \quad \begin{aligned} \min_{f,z} \quad & \frac{1}{2} z_e^T E_e z_e \\ \text{s.t.} \quad & g_e(f) + F_e z_e = \rho \\ & f \in \mathcal{F}_{LS} \end{aligned}$$

where  $\mathcal{F}_{LS}$  indicates a set of feasible forward rate functions and other parameters are as in (24). In this section, we consider a few methods which belong to this family.

5.3.1. *McCulloch quadratic splines (1971)*. To fit the observed market data for US Treasury yield curve, McCulloch (1971) uses the discount function defined by

$$(53) \quad d(T) = 1 + \sum_{i=1}^n a_i h_i(T); \quad h_i(0) = 0; \quad a_0 = 1$$

where  $\{a_i\}_{i=1}^n$  are parameters to be estimated and  $h_i(T)$ ,  $i = 1, \dots, n$ , is a piecewise quadratic basic function of the form,

$$(54) \quad h_1(T) = \begin{cases} T - \frac{1}{2T_2} T^2 & \text{if } 0 \leq T \leq T_2, \\ \frac{1}{2} T_2 & \text{if } T_2 < T \leq T_n, \end{cases}$$

for  $i = 2, 3, \dots, n-1$ , the quadratic polynomial takes the form

$$(55) \quad h_i(T) = \begin{cases} 0 & \text{for } 0 \leq T \leq T_{i-1} \\ \frac{(T-T_{i-1})^2}{2(T_i-T_{i-1})} & \text{if } T_{i-1} < T \leq T_i \\ \frac{1}{2}(T_i - T_{i-1}) + (T - T_i) - \frac{(T-T_i)^2}{2(T_{i+1}-T_i)} & \text{if } T_i < T \leq T_{i+1} \\ \frac{1}{2}(T_{i+1} - T_{i-1}) & \text{if } T_{i+1} < T \leq T_n \end{cases}$$

and for  $i = n$

$$(56) \quad h_n(T) = \begin{cases} 0, & \text{if } 0 \leq T \leq T_{n-1} \\ \frac{(T-T_{n-1})^2}{2(T_n-T_{n-1})} & \text{if } T_{n-1} < T \leq T_n. \end{cases}$$

To ensure that (53) is continuously differentiable, the choice of the basic function is made such that, for adjacent intervals,  $(T_{i-1}, T_i)$  and  $(T_i, T_{i+1})$ , the function has the same slope and the same value at  $T_i$ . Given that (53) is linear in the discount function, ordinary least squares regression methods is used to estimates all the parameters of the model.

*Selection of knot points.* Given  $n_c$ , the total number of securities alongside their respective maturities,  $M_j, j = 1, \dots, n_c$ , such that  $M_j < M_{j+1}$ , choose  $n$ , the number of intervals as  $n = \lfloor \sqrt{n_c} + \frac{1}{2} \rfloor$ . The interval  $[0, T_n]$  is divided into  $(n - 1)$  subintervals with  $T_1 = 0, T_n = M_{n_c}$ , and  $T_i = (1 - \theta_i) M_{L_i} + \theta_i M_{L_i+1}$ , where  $L_i = \lfloor \frac{(i-1)n_c}{n-1} \rfloor$  and  $\theta_i = \frac{(i-1)n_c}{n-1} - L_i$ . This selection of knots ensures greater resolution of the yield curve in any part of the interval  $[0, T_n]$  with greater number of securities.

The main drawback of this method is that it provides forward rate curve with discontinuous first derivative. This affects the forward rate curve with kinks at some knot points. The drawback of the method is that the equivalent forward rate curve is not smooth because it has knuckles corresponding to the knot points. To improve smoothness of the forward rate curve, a cubic splines is proposed in (McCulloch 1975).

5.3.2. *The McCulloch splines (1975).* The method uses the discount function described in (53) with the difference that the basic function is now a cubic polynomial defined as follows. For  $i = 1$ ,

$$(57) \quad h_1(T) = \begin{cases} \frac{T^2}{2} - \frac{T^3}{6T_2} & \text{if } 0 \leq T < T_2 \\ T_2 \left( \frac{T_2}{3} + \frac{(T-T_2)}{2} \right) & \text{if } T_2 \leq T. \end{cases}$$

For  $i = 2$ ,

$$(58) \quad h_2(T) = \begin{cases} \frac{T^3}{6T_2} & \text{if } 0 \leq T \leq T_2 \\ \frac{T_2^3}{6} + \frac{T_2(T-T_2)}{2} + \frac{(T-T_2)^2}{2} - \frac{(T-T_2)^3}{6(T_3-T_2)} & \text{if } T_2 < T \leq T_3 \\ T_3 \left( \frac{2T_3-T_2}{6} + \frac{T-T_3}{2} \right) & \text{if } T_3 \leq T \end{cases}$$

For  $i = 3, \dots, n - 2$

$$(59) \quad h_i(T) = \begin{cases} 0 & \text{if } 0 \leq T \leq T_{i-1} \\ \frac{(T-T_{i-1})^3}{6(T_i-T_{i-1})} & \text{if } T_{i-1} < T \leq T_i \\ \frac{(T_i-T_{i-1})^3}{6} + \frac{(T_i-T_{i-1})(T-T_i)}{2} + \frac{(T-T_i)^2}{2} - \frac{(T-T_i)^3}{6(T_{i+1}-T_i)} & \text{if } T_i < T \leq T_{i+1} \\ (T_{i+1} - T_{i-1}) \left[ \frac{2T_{i+1}-T_i-T_{i-1}}{6} + \frac{T-T_{i+1}}{2} \right] & \text{if } T_{i+1} \leq T \end{cases}$$

For  $i = n - 1$ ,

$$(60) \quad h_{n-1}(T) = \begin{cases} 0 & \text{if } 0 \leq T \leq T_{n-2} \\ \frac{(T-T_{n-2})^3}{6(T_{n-1}-T_{n-2})} & \text{if } T_{n-2} < T \leq T_{n-1} \end{cases}$$

Finally, for  $i = n$ ,

$$(61) \quad h_n(T) = T.$$

*Selection of knots points.* The selection of knots is mainly the same as in section (5.3.1) except that  $T_0 = T_1 = 0$  in the interval  $[0, T_n]$ . Analogously to the quadratic splines, the regression techniques are used to estimate the parameters of the model. For this method, it requires that at every knot, adjacent cubic spline curve have same value, same first order derivative and second order derivative which preserves continuity and smoothness of the resulting curve. As noted by Shea (1984), the method still produces yield curve having kinks corresponding to the knots points.

5.3.3. *The Nelson-Siegel (1994) method.* Nelson and Siegel (1987) introduce a parsimonious function to model the instantaneous forward rates as a solution to a second-order differential equation with real and equal roots

$$(62) \quad f(T) = \beta_0 + \beta_1 e^{-\frac{T}{\tau}} + \beta_2 \frac{T}{\tau} e^{-\frac{T}{\tau}}$$

where  $\beta_0$ ,  $\beta_1$  and  $\beta_2$  are interpreted respectively as the level factor, the slope factor and the curvature factor and  $\tau$  is the rate of exponential decay. Using (11), the spot rate can be deduced as

$$(63) \quad r(T) = \beta_0 + \beta_1 \left( \frac{1 - e^{-\frac{T}{\tau}}}{\frac{T}{\tau}} \right) + \beta_2 \left( \frac{1 - e^{-\frac{T}{\tau}}}{\frac{T}{\tau}} - e^{-\frac{T}{\tau}} \right).$$

From (63), observe that as  $T \rightarrow 0$ ,  $r(T) \rightarrow \beta_0 + \beta_1$ , and so does  $f(T)$  and as  $T \rightarrow \infty$ ,  $r(T) \rightarrow \beta_0$  and so does  $f(T)$ . Both Nelson and Siegel (1987) and Diebold and Li (2006) consider the parameters,  $\beta_0$ ,  $\beta_1$  and  $\beta_2$  as the three latent dynamic factors for (62) or (63). As such, they observed that the weight on  $\beta_0$  is a constant for all  $T$ . It is therefore considered as the long-term factor. The weight on  $\beta_1$  is  $\left( \frac{1 - e^{-\frac{T}{\tau}}}{\frac{T}{\tau}} \right)$  which starts at 1 but decays monotonically (or increases, if  $\beta_1 < 0$ ) to 0. Hence, it is the short-term factor. The weight on  $\beta_2$  is  $\beta_2 \left( \frac{1 - e^{-\frac{T}{\tau}}}{\frac{T}{\tau}} - e^{-\frac{T}{\tau}} \right)$  which starts at 0 then increases and at last decreases to 0. Given this behavior, it is seen as a medium-term factor. These factors coincide with shift, twist and butterfly for (62) or (63) curves. The assumption that  $f(T) > 0$  implies that  $\beta_0 > 0$  and  $\beta_0 + \beta_1 > 0$ . The parameter  $\tau$  measures the location of the hump. Given the nature of the optimization model which is non-convex, all the parameters in the model must be estimated using nonlinear optimization methods.

Concerning the performance of the model, with only a few parameters, the method is able to capture the underlying relationship between yield and term to maturity. A desirable property of the model is that the term structures built from (62) and (63) are both smooth and the specification of model parameters can produce a wide range of curve shape, including U-shape, S-shape, monotonic or humped. The drawback of this method is that it is not flexible enough to produce as many shapes as is required for yield curves.

5.3.4. *The Extended Nelson-Siegel (1994) method.* To increase the flexibility of the Nelson-Siegel (1994) method and improve its fitting performance, Svensson (1994) adds the term  $\beta_3 \left( \frac{T}{\tau_2} e^{-\frac{T}{\tau_2}} \right)$  to (62) to have the instantaneous forward rate as

$$(64) \quad f(T) = \beta_0 + \beta_1 e^{-\frac{T}{\tau_1}} + \beta_2 \left( \frac{T}{\tau_1} e^{-\frac{T}{\tau_1}} \right) + \beta_3 \left( \frac{T}{\tau_2} e^{-\frac{T}{\tau_2}} \right)$$

where  $\beta_0, \beta_1, \beta_2, \beta_3 \in \mathcal{R}$ ,  $\tau_1, \tau_2 > 0$  are parameters to be estimated using nonlinear constrained optimization methods. Using (11), the spot rates can be expressed as

$$(65) \quad r(T) = \beta_0 + \beta_1 \left( \frac{1 - e^{-\frac{T}{\tau_1}}}{\frac{T}{\tau_1}} \right) + \beta_2 \left( \frac{1 - e^{-\frac{T}{\tau_1}}}{\frac{T}{\tau_1}} - e^{-\frac{T}{\tau_1}} \right) + \beta_3 \left( \frac{1 - e^{-\frac{T}{\tau_2}}}{\frac{T}{\tau_2}} - e^{-\frac{T}{\tau_2}} \right)$$

Analogously to (62),  $\beta_0$  is asymptotic to  $r(T)$  for  $\lim_{T \rightarrow \infty} r(T)$  which implies  $\beta_0 > 0$ .  $\beta_1$  determine the rate of convergence with which  $r(T)$  approaches its long-term interest rate. The slope is negative if  $\beta_1 > 0$ .  $\beta_2$  determines the size and the form of the hump. When  $\beta_2 > 0$ , the hump is located at  $\tau_1$  and whereas for  $\beta_2 < 0$ , the curve is in U shape.  $\tau_1$  specifies the location of the first hump or the U-shape of the curve.  $\beta_3$  determine the size

and shape of the second hump and  $\tau_2$ , its location. To get a meaningful financial interpretation of the yield curve, it requires that  $\tau_1, \tau_2 > 0$ . The estimation of all parameters in the model is carried out via nonlinear optimization methods because the problem at hand is non-convex in which case it has multiple local optima. This property implies that the solution to the problem depends heavily on the numerical solution of the starting values.

5.3.5. *Penalized Least Squares measure.* To improve smoothing in the spline methods discussed in (McCulloch 1971, 1975), Fisher's method and the variable roughness penalty (VRP) method proposed in (Waggoner 1997) are used.

In *Fisher's method*, the forward rate curve is the cubic spline that minimizes the objective function

$$(66) \quad \min \sum_{i=1}^n \left( P_i - \hat{P}_i(f) \right)^2 + \lambda \int_0^{T_n} \left( f''(u) \right)^2 du; T_0 < T_1 < \dots < T_n$$

where  $P_i$  are the observed bond prices,  $\hat{P}_i$ , the estimated bond prices. The parameter  $\lambda$  controls the trade-off between smoothness, as measured by  $\int_0^{T_n} \left( f''(u) \right)^2 du$  and goodness of fit, measured by the term  $\sum_{i=1}^n \left( P_i - \hat{P}_i(f) \right)^2$ . According to Fisher, the value of  $\lambda$  is selected by minimizing the function

$$(67) \quad \min \frac{\sum_{i=1}^n \left( P_i - \hat{P}_i(f) \right)^2}{(n - \theta ep(\lambda))^2}$$

where  $n$  is the number of bonds,  $ep(\lambda)$  is the effective number of parameters which depends on the choice of  $\lambda$  and  $\theta$  is the cost or tuning parameter<sup>3</sup>. The larger  $\theta$ , the smoother the forward rate curve and the poorer the goodness of fit (Bank for International Settlements 2005, p .17).

In evaluating the effect of different values of  $\lambda$ , it is observed that if  $\lambda$  increases, the cubic spline, whose flexibility depends on the spacing of the nodes and magnitude of  $\lambda$ , tends to be linear function. When  $\lambda$  grows larger, the spacing of node becomes unimportant. This implies that the flexibility of the spline remains invariant across all regions for large values of  $\lambda$ . To improve this effect, the VRP method is used (Waggoner 1997).

In the *VRP method*, which is still a smoothing spline method, one solves the optimization problem

$$(68) \quad \min \sum_{i=1}^N \left( P_i - \hat{P}_i^f(f) \right)^2 + \int_0^{T_N} \lambda(u) \left( f''(u) \right)^2 du$$

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<sup>3</sup>In relation to Spline function, a larger value of  $\theta$  tends to produce stiff spline.

where  $\lambda(u)$ , is no longer a constant<sup>4</sup> but assumes values that reflect the segmentation of the market. Applied on US market, Waggoner (1997) assigns to  $\lambda(u)$ , the values

$$(69) \quad \lambda(u) = \begin{cases} 0.1 & 0 \leq u \leq 1 \\ 100 & 1 \leq u \leq 10 \\ 100000 & u > 10 \end{cases}$$

where  $u$  is measured in years. Waggoner (1997)'s choices of different values for  $\lambda$  reflects effectively the segmentation structure of the US market for fixed-income securities: bills, notes and bonds. Applied on other markets embedded with different segmentation for fixed-income instruments, adaptive values for  $\lambda$  can be used. Since the roughness penalty,  $\lambda$ , can take different values across maturities, oscillations can be damped on the long end while retaining flexibility on the short end to accommodate instruments of shorter maturity.

## 6. OVERVIEW OF PAPERS

The problems embedded in the traditional interpolation methods for estimating yield curves can be considered from two aspects, mainly on the behavior of functional forms that the method uses and also the inability to adapt to the complexity of the market data.

On the one hand, the functional forms used by the methods may fail to be continuous, differentiable and to provide positive forward rates, i.e.  $f_t > 0$ , as in the case of simple interpolation methods. These defects alone compromise the ability of these methods to produce smooth yield curves. Moreover, they expose the methods to arbitrage opportunities. On the other hand, even when continuity or differentiability of the forward rate is guaranteed, as in the case of least squares methods, the mathematical properties of the polynomial forms or the parsimonious functions used may not be flexible enough to price accurately securities at both the short end and at the long end of the yield curves.

The current research, summarized in the following papers attempt to fill in the gap by proposing to use the generalized optimization framework for estimating the yield curve discussed in Blomvall (2011). The method encompasses traditional interpolation methods as special cases and produces high-quality yield curves.

**6.1. Paper 1: High-quality yield curve from a generalized optimization framework.** In this paper, it is shown that traditional methods (i.e. simple interpolation methods and least squares methods) are indeed special cases of the generalized optimization framework. Using the Shimko test<sup>5</sup> (i.e. an out-of-sample test) in the Swedish and U.S. interest rate swap markets, it is shown that the framework dominates or is close to dominating all other methods by first order stochastic dominance, except simple interpolation methods for the U.S. market when the short end of the yield curve includes shorter tenors.

When measuring risks which are implied by the traditional methods, it is shown that, with the same level of market consistency, the framework incurs lower variance than the traditional methods (Figure 1).

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<sup>4</sup>A constant smoothing parameter  $\lambda$  is not ideal because theoretically, short term rate should fluctuate more than long term rate. That also justifies the distributions allotted to  $\lambda$  in which shorter term instruments are less penalized than longer term instruments.

<sup>5</sup>For this test, we considered instruments that could not be used in the in-sample set of data.

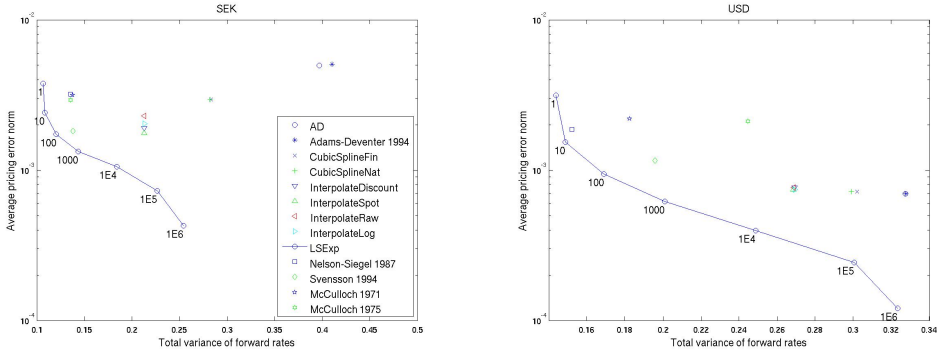


FIGURE 1. The method implied interest rate risk and consistency with market prices. Given a level of consistency with market prices, lower interest rate risk measure can be obtained by using the LSExp methods instead of the standard methods

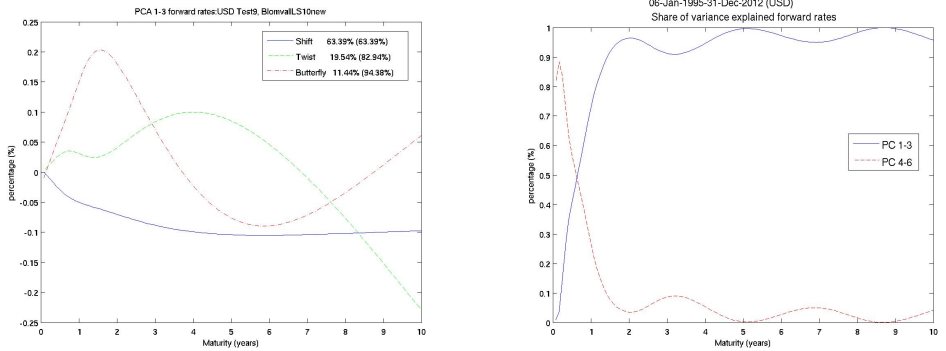


FIGURE 2. The left panels display the first three factor loadings for the forward rate innovations obtained using the LSExp 10 method. We see that the shift, twist and butterfly all converge to zero. The right panel depicts share of variance explained forward rate between the first three and the next three factor loadings for the same method. In this panel, we see clearly that the PC 1-3 do not explain movements in the short end but rather explain uncorrelated movements in the long end. In the same line of thoughts, it can be seen also that subsequent loadings (i.e. PC 4-6) explain uncorrelated movements of the forward rate innovations in the short end.

In studying the movements of the yield curve produced by the generalized optimization framework via principal component analysis (PCA) of innovations in forward rates, we find that the short end rates move independently with the long term rates (right panel of Figure 2). This is consistent with the fact that it is the central bank which regulates the short term rate to control inflation while long term rates are affected by future expectation of the inflation. We also find that the first three factor loadings explain the movement of the forward rates from 2 years onwards and the subsequent PC explain movement of the forward rates in the short end (left panel of Figure 2). Theses findings are validated using traditional methods, market prices and Kalman filtering.

**6.2. Paper 2: Multiple yield curve estimation using the generalized optimization framework.** Since the liquidity crisis which began in 2007, discrepancies between market quotes on FRA, LIBOR and IRS have reached levels where they can no longer be neglected. To adapt to these new market conditions, traditional interpolation methods are extended to a multiple yield curve framework (Mercurio 2008), (Ametrano and Bianchetti 2009), where the instability of these methods become obvious. To provide to the market with smooth multiple yield curves as well as smooth spread curves, the generalized optimization framework described in section (6.1) is extended to a multiple yield curves estimation framework.

To model the FRA, the IRS and Tenor Swap (TS) contracts, the tenor premium is considered. For each tenor  $\tau$ , the risk neutral expectation of the LIBOR rate,  $L_\tau(T_i, T_j)$  with time,  $\Delta t_i$ , being measured with a day count convention, is defined by

$$(70) \quad E_i^{T_j} [L_\tau(T_i, T_j)] = \frac{\exp \left\{ \int_{T_i}^{T_j} f_\tau(t) dt \right\} - 1}{\Delta t_i}$$

where  $f_\tau(t) = f_0(t) + \bar{\pi}_\tau(t)$ , which is the sum of the forward rate described in (11) plus a compensation,  $\bar{\pi}_\tau(t)$ , for being exposed to risks of tenor  $\tau$  at time  $t$ . Equation (70) is important to the valuation of securities because it guarantees that the increase in risks for each tenor  $\tau$  leads to an increase in the risk premium.

We compare the performance of the generalized optimization framework with the financial cubic spline method, also extended to a multiple yield curve estimation framework. To achieve this, parameters in the generalized optimization framework are set in such a way that, on the one hand, the method prices instruments exactly, as does any exact traditional interpolation method. We refer to the resulting generalized optimization framework as the E model. On the other hand, parameters in the generalized optimization framework are also set to retain its least squares property. In this instance, the method is referred to as the LS model.

We use snapshots data from ICAP retrieved with Thomson Reuters 3000 Xtra, and examine the smoothness of the multiple yield curves as well as tenor premium curves associated with each method tested on twenty-seven dates. For the date when the financial cubic spline performs the worst, we find that the cubic spline and the E model are both consistent with respect to the market prices. However, due to the shape constraints, the former method exhibits non smooth multiple yield curves (Figure 3). To remedy to the instability that is observed with the financial cubic spline, the E model can be used instead (Figure 5). However, it has been observed that, due to noise in the input data, the E model does not provide entirely smooth tenor premium curves. To obtain, both smooth multiple yield curves as well as smooth tenor premium curves, the LS method can be used (Figure 6). We also show that, when the market data are adjusted by the LS model's residuals, the multiple yield curve produced by the financial cubic spline improve quite remarkably, even though its instability still persists (Figure 4).

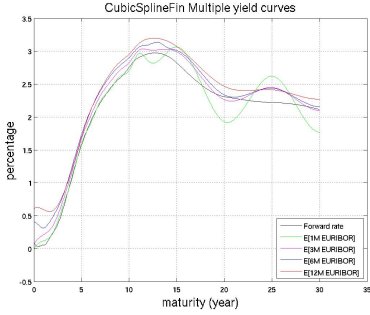


FIGURE 3. 14 Nov 2012 at 12:15. The current window depicts forward rate and  $Q^T$ -expected 1m, 3m, 6m and 12m EURIBOR curves estimated sequentially using the financial cubic spline. This is the date when the cubic spline produced the worst performance among all 27 snapshots.

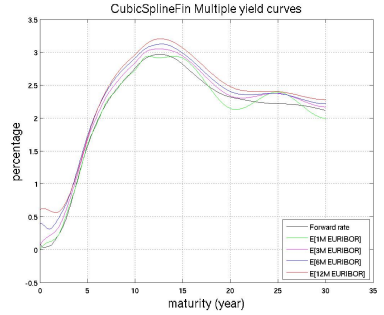


FIGURE 4. The 14th of November 2012 at 12:15: The current panel depicts also forward rates alongside  $Q^T$ -expected 1-month, 3-month, 6-month and 12-month EURIBOR curves estimated sequentially using the financial cubic spline for the modified market data.

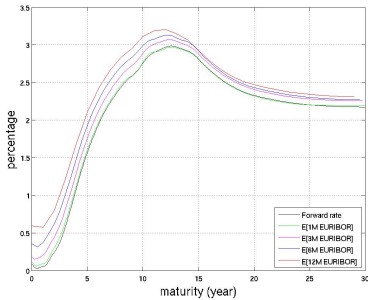


FIGURE 5. 14 Nov 2012 at 12:15. The figure displays the forward rate,  $Q^T$ -expected 1m, 3m, 6m and 12m EURIBOR curves estimated simultaneously using the generalized optimization framework. The results in the left panel is for the E model, which reprice the contracts exactly.

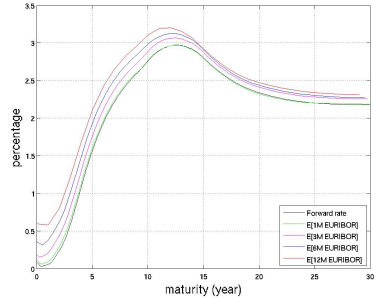


FIGURE 6. 14 Nov 2012 at 12:15. The figure displays the forward rate,  $Q^T$ -expected 1m, 3m, 6m and 12m EURIBOR curves estimated simultaneously using the generalized optimization framework. The results in the left panel is for the LS model, which allows deviations from the market prices.

**6.3. Paper 3: Estimating U.S. Treasury yield curves using a generalized optimization framework.** The U.S. Treasury yield curve is the most used yield curve in many financial applications. As such, it is required to be of higher quality to avoid unrealistic output which might distort the results. The most frequently used data sets relative to U.S. Treasury yields are provided in McCulloch and Kwon (1993), Bliss (1996) and Gurkaynak et al. (2007). Using Principal component analysis, we show that traditional data sets for the U.S. Treasury yield curves contain lots of noise. Consequently, they all produce yield curves of poor quality.

The objective of this paper is twofold. We first estimate high-quality U.S. Treasury yield curves using the generalized optimization framework, then seek to validate the performance of the method against the performance of some traditional least squares methods that are used in such context. These traditional methods include the McCulloch quadratic, the McCulloch cubic splines, the Nelson-Siegel and the Svensson methods. To



this end, we examine the efficient frontier attributed to the generalized estimation method for in-sample and out-of-sample and assess the performance of each method implemented, especially for the out-of-sample test (Figure 7).

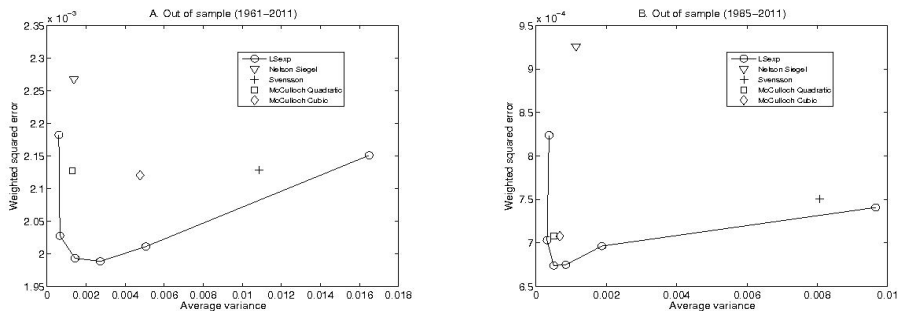


FIGURE 7. Efficient frontier for out-of-sample performance for the McCulloch quadratic, the McCulloch cubic, the Nelson-Siegel and the Svensson methods as compared to the LSExp method. In terms of weighted price error and average variance, both panels show that the LSExp method is more efficient than all the methods implemented.

In both evaluation periods, we clearly see that the LSExp parameter setting, which is a version of the generalized optimization framework, is more efficient than other methods. In term of weighted price error and average variance associated with each methods, Figure 7 shows clearly that, for certain level of market consistency, the LSExp method can produce smaller variance as compared to other methods.

Secondly, we also seek to confirm the behavior of factor loadings for forward rate changes obtained in Blomvall and Ndengo (2013), according to which the short end rates move independently from the long end rates. This is achieved by studying PCA (Figure 8)

Findings from PCA (left panel of Figure 8) show that factor loadings for forward rate innovations where the short end of forward rates moves independently of the long end, which validates the findings in Blomvall and Ndengo (2013). These findings are consistent with markets where the central bank governs the short rate and where longer rates are governed by the term premium and the markets expectations about e.g. inflation. Hence, the U.S. Treasury yield curves produced by this method is better to use, due to its high quality and its consistency with respect to the market prices.

**6.4. Paper 4: Optimal Investment in the fixed-income market with focus in the term premium.** Stochastic programming models are used in financial industry, precisely in areas including asset and liability management (Kusy and Ziemba 1986) and

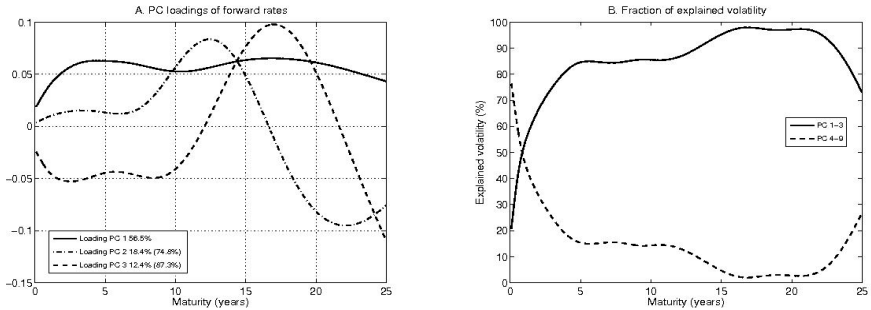


FIGURE 8. The left and right panels display respectively the first three factor loadings for LSEXP method and explained variance attributed to PC 1-3 and PC 4-9.

(Carino et al. 1994). It is well known that there exists a time-varying term premium on the interest rate market for bonds with longer maturities. This is a compensation for additional duration risks investors are exposed to. But to our knowledge, there is no Stochastic Programming model which has included the time-varying term premium.

The objective of this paper is twofold. We first use the high-quality U.S. Treasury yield curve estimated by the generalized optimization framework as input to the essentially affine model (Duffee 2002) to capture the randomness in interest rate and the time-varying term premium as well. Secondly, to determine the optimal investment in the U.S. Treasury market, we propose the following two-stage Stochastic Programming model without recourse, which model borrowing, shorting and proportional transaction cost and makes use of the power utility function

$$(71) \quad \mathcal{U}(x) = \begin{cases} \frac{1}{\gamma} x^\gamma & \text{if } 0 < \gamma \leq 1; \gamma < 0. \\ \ln x & \text{if } \gamma = 0. \end{cases},$$

Given a set of assets,  $A$ , such that asset  $a \in A$ , a set of scenarios,  $\mathcal{I}$ , we solve the optimization problem

$$(72) \quad \begin{aligned} \max \quad & \sum_{i \in \mathcal{I}} p_i \mathcal{U} \left( c + \sum_{a \in A} P_{i,a} x_a \right) \\ \text{s.t.} \quad & x_a = h_a + (b_a - s_a), \quad a \in A \\ & c = \left( h_c - \sum_{a \in A} P_a (b_a - s_a) - T \sum_{a \in A} P_a (b_a + s_a) \right) e^{rt} \\ & b_a \geq 0, \quad s_a \geq 0, \quad a \in A \end{aligned}$$

where  $p_i \in (0, 1)$ , is the probability of scenario  $i \in \mathcal{I}$ ,  $\mathcal{U}(\cdot)$  is the utility function,  $c$ , the amount of cash flow owned,  $P_{i,a}$ , the price of asset  $a \in A$  in scenario  $i$ ,  $T$ , the transaction cost,  $r$ , the risk-free continuously compounded interest rate,  $t$  is the time step of one period,  $x_a$  is the total amount of asset  $a \in A$ ,  $h_a$  is the initial holding of assets  $a \in A$ ,  $b_a$  and  $s_a$  are respectively the amount of assets purchased and sold, and finally  $P_a$  the price of asset  $a \in A$ .

We use Monte Carlo simulation to generate sample of bond prices of size 10000 for some degree of risk aversion  $\gamma$ . The overall assessment of the model is that, in the long run, the model provides increase in wealth which corresponds to the degree of exposure to risks.

The model is validated by examining the Sharpe ratio and the Jensen's alpha for each portfolio the method generates against the market portfolio. Findings reveal that the proposed Stochastic Programming model can produce portfolios that perform better than the equity index.

This is the first Stochastic Programming model which captures this compensation for risk. As such, it can make adequate and well-balanced decision about the exposure to interest rate risk and be used for optimal investment in the U.S. Treasury market.

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