EXACT MINIMIZERS IN REAL INTERPOLATION

CHARACTERIZATION AND APPLICATIONS

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For people who have inspired me in science

and in life.

Abstract

The main idea of the thesis is to develop new connections between the theory of real interpolation and applications. Near and exact minimizers for E–, K– and L– functionals of the theory of real interpolation are very important in applications connected to regularization of inverse problems such as image processing. The problem which appears is how to characterize and construct these minimizers. These exact minimizers referred to as optimal decompositions in the thesis, have certain extremal properties that we completely express and characterize in terms of duality. Our characterization generalizes known characterization for a particular Banach couple. The characterization presented in the thesis also makes it possible to understand the geometrical meaning of optimal decomposition for some important particular cases and gives a possibility to construct them. One of the most famous models in image processing is the total variation regularization published by Rudin, Osher and Fatemi. We propose a new fast algorithm to find the exact minimizer for this model. Optimal decompositions mentioned have some connections to optimization problems which are also pointed out.
The thesis is based on results that have been presented in international conferences and have been published in five papers.

In Paper 1, we characterize optimal decomposition for the E–, K– and L_{p_0,p_1}– functional. We also present a geometrical interpretation of optimal decomposition for the $L_{p,1}$–functional for the couple (ℓ^p, X) on $\mathbb{R}^n$. The characterization presented is useful in the sense that it gives insights into the construction of these minimizers.

The characterization mentioned in Paper 1 is based on optimal decomposition for infimal convolution. The operation of infimal convolution is a very important and non–trivial tool in functional analysis and is also very well–known within the context of convex analysis. The L–, K– and E– functionals can be regarded as an infimal convolution of two well–defined functions. Unfortunately tools from convex analysis can not be applied in a straightforward way in this context of couples of spaces. The most important requirement that an infimal convolution would satisfy for a decomposition to be optimal is subdifferentiability.
In Paper 2, we have used an approach based on the famous Attouch–Brezis theorem to prove subdifferentiability of infimal convolution on Banach couples.

In Paper 3, we apply result from Paper 1 to the well—known Rudin–Osher– Fatemi (ROF) image denoising model on a general finite directed graph. We define the space BV of functions of bounded variation on the graph and show that the unit ball of its dual space can be described as the image of the unit ball of the space ℓ^∞ on the graph by a divergence operator. Based on this result, we propose a new fast algorithm to find the exact minimizer for the ROF model. Proof of convergence of the algorithm is presented and its performance on image denoising test examples is illustrated.

In Paper 4, we present some extensions of results presented in Paper 1 and Paper 2. First we extend the results from Banach couples to Banach triples. Then we prove that our approach can apply when complex spaces are considered instead of real spaces. Finally we compare the performance of the algorithm that was proposed in Paper 3 with the Split Bregman algorithm which is one of the benchmark algorithms known for the ROF model. We find out that in most cases both algorithms behave in a similar way and that in some cases our algorithm decreases the error faster with the number of iterations.

In Paper 5, we point out some connections between optimal decompositions mentioned in the thesis and optimization problems. We apply the approach used in Paper 2 to two well–known optimization problems, namely convex and linear programming to investigate connections with standard results in the framework of these problems. It is shown that we can derive proofs for duality theorems for these problems under the assumptions of our approach.

List of Papers

The thesis is based on the following appended papers, which are referred to in the text by their Arabic numerals.

1. Natan Kruglyak, **Japhet Niyobuhungiro, "Characterization of optimal decompositions in real interpolation"**, *Journal of Approximation Theory*, **185**, 1–11, (2014)

2. Natan Kruglyak, **Japhet Niyobuhungiro**, **"Subdifferentiability of infimal convolution on Banach Couples"**, *Funct. Approx. Comment. Math.*, (*to appear*)

3. **Japhet Niyobuhungiro**, Eric Setterqvist, **"A new reiterative algorithm for the Rudin–Osher–Fatemi denoising model on the graph"**, *Proceedings of The 2nd International Conference on Intelligent Systems and Image Processing 2014, ICISIP2014*, Kitakyushu, Japan, September 26–29, (2014), doi:10.12792/icisip2014.018

4. **Japhet Niyobuhungiro, "Exact minimizers in real interpolation. Some additional results"**, *Preprint (2015)*

5. **Japhet Niyobuhungiro, "Optimal decomposition for infimal convolution on Banach couples. Some connections to linear and convex programming"**, Part of: *Tech. Rep. LiTH-MAT-R-2013/07-SE*, Department of Mathematics, Division of Mathematics and Applied Mathematics, Linköping University (2013)

Parts of this thesis have been presented at the following international conferences:

1. ***First Kenyatta University Mathematics Conference***, *Nairobi, Kenya*, June 8-11, (2011)

2. ***Conference on Inverse Problems and Applications***, *Linköping, Sweden*, April 2-6, (2013)

3. ***Joint Meeting of the German Mathematical Society (DMV) and the Polish Mathematical Society (PTM)***, *Pznań, Poland*, September 17–20, (2014)

4. ***The 2nd International Conference on Intelligent Systems and Image Processing 2014, ICISIP2014***, *Nishinippon Institute of Technology, Kitakyushu, Japan*, September 26–29, (2014)

Populärvetenskaplig sammanfattning

Avhandlingens övergripande idé är att undersöka och finna nya samband mellan reell interpolationsteori och tillämpningar. Approximativa och exakta minimerare för E–, K– och L– funktionalerna från den reella interpolationsteorin är av stor betydelse i tillämpningar där regularisering av inversa problem förekommer. Ett naturligt problem som uppkommer är att karakterisera och konstruera dessa minimerare. De exakta minimerarna, som benämns optimala uppdelningar i avhandlingen, har särskilda extremalegenskaper. Vi erhåller en fullständig karakterisering av dessa egenskaper genom dualitet. Vår karakterisering generaliserar den tidigare kända karakterseringen för ett särskilt Banach–par. Den erhållna karakteriseringen i avhandlingen möjliggör en geometrisk tolkning av den optimala uppdelningen för några viktiga fall och ger även inblick i hur dessa optimala uppdelningar kan konstrueras. En av de mest berömda modellerna i bildbehandling är den totala variationsregulariseringen som publicerades av Rudin, Osher och Fatemi. Vi föreslår en ny snabb algoritm för att konstruera den exakta minimeraren för denna modell. De ovan nämnda optimala uppdelningarna har samband med optimeringsproblem vilket också beskrivs i avhandlingen.

Acknowledgments

I thank my main advisor *Natan Kruglyak* for his guidance and advice during my Ph.D. training at Linköping University, Sweden. He introduced me to the subject of real interpolation and helped me to undersand its connections and applications to other areas of Science. I have benefited from his guidance for my mathematical development. His support all along the way and constructive discussions are invaluable.

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Linköping, May 11, 2015
Japhet Niyobuhungiro

Contents

Part I

Background and Summary

In this part, we will present the introductory background and summary of the thesis.

0

Introduction

Tʜᴇ main idea of the thesis is to develop new connections between the theory of real interpolation and applications. Near and exact minimizers for E–, K– and L– functionals of the theory of real interpolation are very important and have recently appeared in important results in image processing. The extremal problem which appears, is how to characterize and construct these minimizers. By using the duality in convex analysis, we study the properties of the exact minimizers, the possibility to construct them and investigate their usefulness for concrete applications in regularization of inverse problems, specifically image processing.

0.1   Background

Several functionals such as L–, K– and E– functionals are very important in the theory of real interpolation. A more or less detailed theory on these functionals can be found for example in the books [3, 4]. Another good reference is the book [2]. Given a couple of Banach spaces (X_0, X_1), an element $x \in X_0 + X_1$ and a positive parameter t, the K– functional is defined by the formula

$$K\left(t, x; X_0, X_1\right) = \inf_{w \in X_1} \left(\|x - w\|_{X_0} + t \|w\|_{X_1} \right).$$

The K– functional is at the center of the so–called K– method of real interpolation that is basically concerned with the construction of suitable families of real interpolation spaces between X_0 and X_1. The K– functional is a particular case

of the more general L– functional which is defined by

$$L_{p_0,p_1}(t, x; X_0, X_1) = \inf_{w \in X_1} \left(\|x - w\|_{X_0}^{p_0} + t \|w\|_{X_1}^{p_1} \right),$$ (1)

for $1 \leq p_0, p_1 < \infty$.

Definition 0.1 (Exact and near minimizers). We say that the element (which depends on x and t) $w_t \in X_1$ is a *near minimizer* for the functional (1) if there exists $C > 0$ independent of x and t such that

$$\|x - w_t\|_{X_0}^{p_0} + t \|w_t\|_{X_1}^{p_1} \leq C L_{p_0,p_1}(t, x; X_0, X_1).$$

If $C = 1$, then w_t is called *exact minimizer*. If $w_t \in X_1$ is an exact minimizer, then we will call

$$x = w_t + (x - w_t),$$ (2)

optimal decomposition for (1) corresponding to x.

The E– functional is basically seen as a distance functional and is defined by the expression

$$E(t, x; X_0, X_1) = \inf_{\|w\|_{X_1} \leq t} \|x - w\|_{X_0}.$$

Remark 0.1. It is important to note that the optimal decomposition does not always exist. See the following counter example.

Counter example. *Let f be the function defined by $f(x) = 2$ for $0 \leq x < \frac{1}{2}$ and $f(x) = -2$ for $\frac{1}{2} \leq x \leq 1$, and consider the functional*

$$E\left(1, f; L^2, C[0, 1]\right) = \inf_{\|g\|_{C[0,1]} \leq 1} \|f - g\|_{L^2}.$$ (3)

There is no $g_1 \in C[0, 1]$ such that

$$E\left(1, f; L^2, C[0, 1]\right) = \|f - g_1\|_{L^2}.$$ (4)

Let (X_0, X_1) be a regular Banach couple, i.e., X_0 and X_1 are both Banach spaces which are linearly and continuously embedded in the same Hausdorff topological vector space and moreover the intersection $X_0 \cap X_1$ is dense in both X_0 and X_1. Given an element x in $X_0 + X_1$ and some parameter $t > 0$, we consider the following L– functional

$$L_{p_0,p_1}(t, x; X_0, X_1) = \inf_{x = x_0 + x_1} \left(\frac{1}{p_0} \|x_0\|_{X_0}^{p_0} + \frac{t}{p_1} \|x_1\|_{X_1}^{p_1} \right),$$ (5)

for $1 \leq p_0, p_1 < \infty$.

Problem 1. *Suppose that the optimal decomposition $x = x_{0,opt} + x_{1,opt}$ for K– functional (respectively for L– and E– functionals) corresponding to the element x exists. Give a characterization of this decomposition. In other words, what are the mathematical properties of $x_{0,opt}$ and $x_{1,opt}$. For example, in the case of the L– functional (5), we want the mathematical properties of the decomposition $x = x_{0,opt} + x_{1,opt}$ such that*

$$L_{p_0,p_1}\left(t, x; X_0, X_1\right) = \frac{1}{p_0} \left\|x_{0,opt}\right\|_{X_0}^{p_0} + \frac{t}{p_1} \left\|x_{1,opt}\right\|_{X_1}^{p_1}.$$

The L– functional is deeply connected to the well–known Rudin–Osher–Fatemi (ROF) image denoising model. Denoising is the problem of removing noise from an image. The most commonly studied case is with additive white Gaussian noise, where the observed noisy image $f \in L^2$ is related to the underlying true image f_* by

$$f = f_* + \eta,$$

where the noise $\eta \in L^2$.

The ROF model, also known as *Total Variation* (TV) regularization technique, proposes to approximate the true image f_* by the function $f_t \in BV$ which minimizes the $L_{2,1}$– functional for the couple $\left(L^2, BV\right)$:

$$L_{2,1}\left(t, f; L^2, BV\right) = \inf_{g \in BV}\left(\|f - g\|_{L^2}^2 + t \|g\|_{BV}\right), \tag{6}$$

where L^2 and BV stand for the space of square integrable functions and the space of functions with bounded variation on a rectangular domain respectively. Since its appearance in 1992, the ROF model [14] has been successful and popular and it has since been applied to a multitude of other imaging problems (see for example the book [5]). The problem of constructing exact minimizer for the functional (6) is difficult. Let us mention that when the following estimate of the noise is known

$$\|\eta\|_{L^2} \le \varepsilon,$$

the so–called Morozov discrepancy principle (see [8]) suggests choosing $t > 0$ such that

$$\|f - f_t\|_{L^2} = \varepsilon.$$

The underlying idea of the Morozov principle can be explained from the point of view of interpolation theory. This has been done in the paper [6] by F. Cobos and N. Kruglyak who provided an algorithm that constructs an exact minimizer for the E–functional

$$E\left(t, f; L^\infty, BV\right) = \inf_{\|g\|_{L^\infty} \le t} \|f - g\|_{BV}, \tag{7}$$

where L^∞ and BV are spaces of bounded functions and functions with bounded variation on an interval $[a, b]$, respectively. They have also discussed connections

of their results with the Rudin–Osher–Fatemi denoising model.

Different approaches such as PDE and wavelet–based approaches have been proposed (see for example the books [5, 15] and the paper [7]) for approximately constructing f_t. Recently Kislyakov and Kruglyak in their book [9] considered a similar problem for the couple of Sobolev spaces $\left(L^p, \dot{W}^{q,k} \right)$, however their approach gives only near minimizer, not exact minimizers. But for applications in image processing it it crucial to have axact minimizers (see discussion in the Paper [6]).

In 2010, I. Asekritova and N. Kruglyak also presented an algorithm for the construction of a near minimizer for the couple $\left(L^2, BV \right)$ based on piecewise constant approximation and the Besicovitch covering theorem [1]. In 2002, in his book [10], Yves Meyer obtained a characterization of optimal decomposition for the ROF functional by using duality.

It is clear that the ROF model is a particular case of the L– functional (5) for $p_0 = 2$, $p_1 = 1$ and for the spaces $X_0 = L^2(D)$ and $X_1 = BV(D)$ for some rectangular domain D. Thus it is an interesting problem to study the properties of exact minimizer for L– functional in its general formulation on regular Banach couples.

0.2   Summary of the thesis

This thesis consists of two parts and the outline is as follows.

0.2.1   Summary of Part I

In Part I the background and summary are given.

0.2.2   Summary of Part II

Part II consists of five papers. We then proceed to give a short summary for each of the papers below.

Paper 1: Characterization of optimal decompositions in real interpolation

Problem statement

Let (X_0, X_1) be a Banach couple. The theory of real interpolation is based on Peetre's K– functional

$$K(t, x; X_0, X_1) = \inf_{x = x_0 + x_1} \left(\|x_0\|_{X_0} + t \|x_1\|_{X_1} \right),$$

where $t > 0$ and $x \in X_0 + X_1$. As its calculation is a difficult extremal problem, J. Peetre [13] suggested another approach to real interpolation based on a more general L_{p_0, p_1}– functional

$$L_{p_0, p_1}(t, x; X_0, X_1) = \inf_{x = x_0 + x_1} \left(\frac{1}{p_0} \|x_0\|_{X_0}^{p_0} + \frac{t}{p_1} \|x_1\|_{X_1}^{p_1} \right), \tag{8}$$

where $t > 0$ is a parameter and $1 \leq p_0, p_1 < \infty$.

It leads to the same set of interpolation spaces (see [4]) and is easier to calculate in the important cases of couples (L^p, L^q), and $(L^p, W^{k,q})$ (see [3] and [9]). Moreover, starting with the famous in image processing Rudin–Osher–Fatemi (ROF) denoising model (see [14] and [5]), the L_{p_0,p_1}– functional appeared in regularization of inverse problems, where the second term in the expression (8) is called a *regularization* or *penalty* term.

In connection with these applied problems (see, for example, the discussion in the paper [6]), the following question arises.

Problem 1. *Suppose that for a given element $x \in X_0 + X_1$ and $t > 0$ there exists an optimal decomposition for the L_{p_0,p_1}–functional, i.e. a decomposition $x = x_{0,opt} + x_{1,opt}$ such that*

$$L_{p_0,p_1}(t, x; X_0, X_1) = \inf_{x=x_0+x_1} \left(\frac{1}{p_0} \|x_0\|_{X_0}^{p_0} + \frac{t}{p_1} \|x_1\|_{X_1}^{p_1} \right) = \frac{1}{p_0} \|x_{0,opt}\|_{X_0}^{p_0} + \frac{t}{p_1} \|x_{1,opt}\|_{X_1}^{p_1}.$$

How can this optimal decomposition be characterized (constructed)?

Main contributions and outcomes

This paper consists of two parts. In the first part we use some well–known results in convex analysis to characterize the optimal decomposition for the L_{p_0,p_1}–functional. In the second part of the paper we use one result from the first part to obtain a geometrical interpretation of the optimal decomposition for the $L_{p,1}$–functional for the couple (ℓ^p, X) on $\mathbb{R}^n$, where X is any Banach space. An interesting feature of this result is the appearance of the set

$$\Omega_t = \left\{ u \in \mathbb{R}^n \ : \ \nabla \left(\frac{1}{p} \|u\|_{\ell^p}^p \right) \in t\mathcal{B}_{X^*} \right\},$$

which contains the element $x_{0,opt}$ ($\mathcal{B}_{X^*}$ is the unit ball of the dual space X^*). We demonstrate by example that for $p \neq 2$ the set Ω_t could be non–convex.

Let (X_0, X_1) be a *compatible Banach couple*. i.e., X_0 and X_1 are Banach spaces such that X_0 and X_1 are linearly and continuously embedded in some Banach space $\mathcal{X}$. Furthemore, we assume that (X_0, X_1) be a *regular couple*, i.e., $X_0 \cap X_1$ is dense in both X_0 and X_1. Let $x \in X_0 + X_1$, let $1 \leq p < +\infty$ and $t > 0$. We consider the L– functional

$$L_{p,1}(t, x; X_0, X_1) = \inf_{x=x_0+x_1} \left(\frac{1}{p} \|x_0\|_{X_0}^p + t \|x_1\|_{X_1} \right). \tag{9}$$

We need to find a characterization of *optimal decomposition* for this L– functional. i.e., $x = x_{0,opt} + x_{1,opt}$ such that

$$L_{p,1}(t, x; X_0, X_1) = \frac{1}{p} \|x_{0,opt}\|_{X_0}^p + t \|x_{1,opt}\|_{X_1}.$$

It is known in interpolation theory that (X_0^*, X_1^*) also form a Banach couple and $(X_0 \cap X_1)^* = X_0^* + X_1^*$. The norm of the dual spaces is defined by:

$$\|y\|_{X_j^*} = \sup \left\{ \langle y, x \rangle : \ x \in X_j, \ \|x\|_{X_j} \leq 1 \right\}, \ j = 0, 1.$$

The spaces $X_0 + X_1$ and $X_0 \cap X_1$ are Banach spaces with respect to the following norms

$$\|x\|_{X_0+X_1} = \inf_{x = x_0 + x_1} \left\{ \|x_0\|_{X_0} + \|x_1\|_{X_1} \right\},$$

where the infimum extends over all representations $x = x_0 + x_1$ of x with x_0 in X_0 and x_1 in X_1, and

$$\|x\|_{X_0 \cap X_1} = \max \left\{ \|x\|_{X_0}, \|x\|_{X_1} \right\}.$$

Theorem 0.1. *Let* $1 < p < +\infty$*. The decomposition* $x = x_{0,opt} + x_{1,opt}$ *is optimal for* $L_{p,1}(t, x; X_0, X_1)$ *if and only if there exists* $y_* \in X_0^* \cap X_1^*$ *such that* $\|y_*\|_{X_1^*} \leq t$ *and*

$$\begin{cases} \frac{1}{p} \|x_{0,opt}\|_{X_0}^p = \langle y_*, x_{0,opt} \rangle - \frac{1}{p'} \|y_*\|_{X_0^*}^{p'}; \\ t \|x_{1,opt}\|_{X_1} = \langle y_*, x_{1,opt} \rangle, \end{cases}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

In order to illustrate the geometry, let us consider the particular case of couple (ℓ^p, X) on $\mathbb{R}^n$, where X is any Banach couple. We have the L– functional

$$L_{p,1}(t, x; \ell_p, X) = \inf_{x = x_0 + x_1} \left(\frac{1}{p} \|x_0\|_{\ell_p}^p + t \|x_1\|_X \right),$$

where $1 < p < +\infty$. Consider the following function F_0 and its gradient:

$$F_0(u) = \frac{1}{p} \|u\|_{\ell_p}^p, \ \nabla F_0(v) = \left\{ |v|^{p-1} \operatorname{sgn}(v) \right\}.$$

Let us define the set Ω_t by

$$\Omega_t = \left\{ v \in \mathbb{R}^n : \ \nabla F_0(v) \in t\mathcal{B}_{X^*} \right\}.$$

We need to consider two cases:

(Case 1) $x \in \Omega$.

In this case, the optimal decomposition for $L_{p,1}(t, x; \ell_p, X)$ is given by

$$x_{0,opt} = x \text{ and } x_{1,opt} = 0.$$

(Case 2) $x \notin \Omega$.

In this case, the optimal decomposition for $L_{p,1}(t, x; \ell_p, X)$ is characterized by the following theorem:

Theorem 0.2. *Let x be such that $\|\nabla F_0(x)\|_{X^*} > t$. Then decomposition $x = x_{0,opt} + x_{1,opt}$ is optimal for $L_{p,1}\left(t, x; \ell_p, X\right)$ if and only if*

(a) $\left\|\nabla F_0\left(x_{0,opt}\right)\right\|_{X^*} = t$

(b) $\left\langle x_{1,opt}, \nabla F_0\left(x_{0,opt}\right)\right\rangle = t\left\|x_{1,opt}\right\|_X$.

The Figure 1 gives the geometry of optimal decomposition for couple (ℓ^p, X). The element $x_{1,opt}$ is orthogonal to the supporting hyperplane to $t\mathcal{B}_{X^*}$ at $y_* = \nabla F_0\left(x_{0,opt}\right)$.
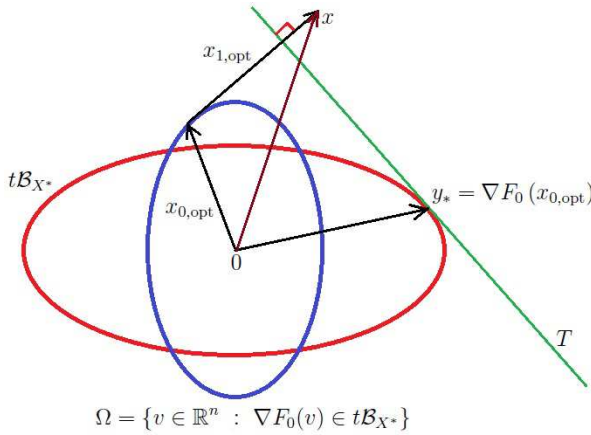

$$\Omega = \{v \in \mathbb{R}^n \ : \ \nabla F_0(v) \in t\mathcal{B}_{X^*}\}$$

Figure 1: **A geometry of the optimal decomposition.**

Remark 0.2. For the case $p = 2$, the sets Ω_t and $t\mathcal{B}_{X^*}$ coincide. This particular case was separately treated in [11] by using a different approach.

It is noted that in a general situation the set Ω_t could be non–convex and of rather complicated structure as illustrated by the following example:

──── **Example 0.1** ────

We present an example of illustration in $\mathbb{R}^2$. Consider the couple (ℓ^3, X) in the plane where space X is such that its unit ball is the rotated ball of ℓ^1 by the rotation matrix

$$R_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix},$$

for $\theta = 30°$. We have that

$$\|x\|_X = \left\|R_\theta^{-1}x\right\|_{\ell^1} = \left|\frac{\sqrt{3}}{2}x_1 - \frac{1}{2}x_2\right| + \left|\frac{1}{2}x_1 + \frac{\sqrt{3}}{2}x_2\right|,$$

and

$$\nabla F_0\left(u\right) = \left[|u_1|^2\, \text{sgn}(u_1), |u_2|^2\, \text{sgn}(u_2)\right]^T.$$

The set Ω_t can then be written as

$$\Omega_t = \left\{ v \in \mathbb{R}^2 : \left\| \left[|v_1|^2 \operatorname{sgn}(v_1), |v_2|^2 \operatorname{sgn}(v_2) \right]^T \right\|_{X^*} \leq t \right\},$$

where the norm in X^* is given by

$$\|y\|_{X^*} = \left\| R_\theta^{-1} y \right\|_{\ell^\infty} = \max \left\{ \left| \frac{\sqrt{3}}{2} y_1 - \frac{1}{2} y_2 \right|, \left| \frac{1}{2} y_1 + \frac{\sqrt{3}}{2} y_2 \right| \right\}.$$

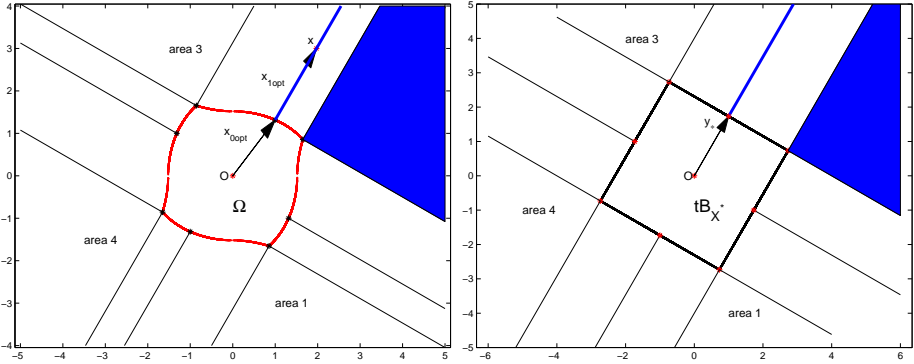The Theorem 0.2 is illustrated in Figure 2. So we see that in this situation the set Ω is not convex.

Figure 2: *Geometry of Optimal Decomposition for the Couple (ℓ^p, X) for $p = 3$, $t = 2$, $X = R_\theta \left(\ell^1 \right)$ and $\theta = 30°$. The set Ω_t is illustrated on the left and $t\mathcal{B}_{X^*}$ on the right. The unit ball of X^* is $R_\theta \left(\mathcal{B}_{\ell^\infty} \right)$, where $\mathcal{B}_{\ell^\infty}$ is the unit ball of ℓ^∞. If x belongs to the blue area on the left, then $x_{0,opt}$ is the corresponding corner point of Ω_t and y_* is the corresponding corner point of $t\mathcal{B}_{X^*}$. The same holds for areas 1, 3 and 4. In other situations, $x_{0,opt}$ belongs to the boundary of Ω_t such that the direction of $x_{1,opt}$ is the direction perpendicular to the tangent line to $t\mathcal{B}_{X^*}$ which goes through $y_* = \nabla F_0 \left(x_{0,opt} \right)$. This is illustrated by the two bold parallel lines.*

We have also obtained the results concerning optimal decomposition for K–, L– and E– functionals in general cases. For example the L– functional (9) is a particular case of the following general L– functional:

$$L_{p_0, p_1} \left(t, x; X_0, X_1 \right) = \inf_{x = x_0 + x_1} \left(\frac{1}{p_0} \|x_0\|_{X_0}^{p_0} + \frac{t}{p_1} \|x_1\|_{X_1}^{p_1} \right), \tag{10}$$

where $1 \leq p_0, p_1 < \infty$.

Theorem 0.3. *Let $x \in X_0 + X_1$, $1 < p_0, p_1 < \infty$ and let $t > 0$ be a fixed parameter. The decomposition $x = x_{0,opt} + x_{1,opt}$ is optimal for*

$$L_{p_0, p_1} \left(t, x; X_0, X_1 \right) = \inf_{x = x_0 + x_1} \left(\frac{1}{p_0} \|x_0\|_{X_0}^{p_0} + \frac{t}{p_1} \|x_1\|_{X_1}^{p_1} \right),$$

if and only if there exists $y_* \in X_0^* \cap X_1^*$ such that

$$\begin{cases} \frac{1}{p_0} \left\| x_{0,opt} \right\|_{X_0}^{p_0} = \langle y_*, x_{0,opt} \rangle - \frac{1}{p_0'} \| y_* \|_{X_0^*}^{p_0'} ; \\ \frac{t}{p_1} \left\| x_{1,opt} \right\|_{X_1}^{p_1} = \langle y_*, x_{1,opt} \rangle - \frac{t}{p_1'} \left\| \frac{y_*}{t} \right\|_{X_1^*}^{p_1'} . \end{cases}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

Remark 0.3. Results of Paper 1 for the couple (ℓ^2, X) are of special importance. We refer the reader to [11], where we have investigated the geometry of optimal decomposition for the $L_{2,1}$–functional for the couple (ℓ^2, X) on $\mathbb{R}^n$, where space ℓ^2 is defined by the standard Euclidean norm and where X is any Banach space on $\mathbb{R}^n$. Our proof is based on some geometrical considerations and Yves Meyer's duality approach which was considered for the couple (L^2, BV) in connection with the ROF model (see [10]). One of the goals here was also to investigate possibility to extend Meyer's approach to more general couples than (L^2, BV). The result therein can hence be obtained as a particular case from a result in Paper 1, but the proof uses a different and independent approach which was considered before the writing of Paper 1.

Paper 2: Subdifferentiability of Infimal Convolution on Banach Couples

Problem statement

Let (X_0, X_1) be a regular Banach couple, i.e. $X_0 \cap X_1$ is dense in both X_0 and X_1, and let $\varphi_0 : X_0 \longrightarrow \mathbb{R} \cup \{+\infty\}$ and $\varphi_1 : X_1 \longrightarrow \mathbb{R} \cup \{+\infty\}$ be convex and proper functions and let

$$\overline{\varphi}_i (u) = \begin{cases} \varphi_i (u) & \text{if } u \in X_i \\ +\infty & \text{if } u \in (X_0 + X_1) \setminus X_i \end{cases} \quad i = 0, 1 \tag{11}$$

be functions defined on the sum $X_0 + X_1$. Then the K–, L– and E– functionals (see [3, 13]) are particular cases of infimal convolution of functions $\overline{\varphi}_0$ and $\overline{\varphi}_1$ defined as follows:

$$(\overline{\varphi}_0 \oplus \overline{\varphi}_1)(x) = \inf_{x=x_0+x_1} (\varphi_0(x_0) + \varphi_1(x_1)). \tag{12}$$

The infimal convolution (12) is called *exact* at a point $x \in X_0 + X_1$ if the infimum is achieved, i.e., $(\overline{\varphi}_0 \oplus \overline{\varphi}_1)(x) = \min_{x=x_0+x_1} (\varphi_0(x_0) + \varphi_1(x_1))$. Suppose that $(\overline{\varphi}_0 \oplus \overline{\varphi}_1)(x)$ is *finite and exact*. Then the decomposition $x = x_0 + x_1$, on which the infimum is attained is called *optimal* and denoted as $x = x_{0,opt} + x_{1,opt}$. Usually, calculation of optimal decomposition is a difficult extremal problem and only near–optimal decomposition can be constructed (see [9]). However for applications, for example in image processing (see [10], [6] and [12]), exact optimal decomposition is required. In Paper 1 the following dual characterization of optimal decomposition was obtained:

Theorem 0.4. *Let $\varphi_0 : X_0 \longrightarrow \mathbb{R} \cup \{+\infty\}$ and $\varphi_1 : X_1 \longrightarrow \mathbb{R} \cup \{+\infty\}$ be convex proper functions. Suppose also that $\overline{\varphi}_0 \oplus \overline{\varphi}_1$ is subdifferentiable for a given element $x \in dom(\overline{\varphi}_0 \oplus \overline{\varphi}_1)$. Then the decomposition $x = x_{0,opt} + x_{1,opt}$ is optimal for $\overline{\varphi}_0 \oplus \overline{\varphi}_1$ if and only if there exists $y_* \in X_0^* \cap X_1^*$ such that it is dual to both $x_{0,opt}$ and $x_{1,opt}$ with respect to φ_0 and φ_1, respectively, i.e.*

$$\begin{cases} \varphi_0 \left(x_{0,opt} \right) = \langle y_*, x_{0,opt} \rangle - \varphi_0^* \left(y_* \right) \\ \varphi_1 \left(x_{1,opt} \right) = \langle y_*, x_{1,opt} \rangle - \varphi_1^*(y_*). \end{cases} \tag{13}$$

Here *dom F* is the set of points on which the functions takes finite values. Note that to use Theorem 0.4 we need to check subdifferentiability of the function $\overline{\varphi}_0 \oplus \overline{\varphi}_1$ for a given $x \in dom(\overline{\varphi}_0 \oplus \overline{\varphi}_1)$, which is often not trivial problem.

Main contributions and outcomes

In this paper we develop an approach based on Attouch–Brezis theorem that provides sufficient conditions for subdifferentiability of infimal convolution defined on a Banach couple. Important feature of this result is that it works also for boundary points of the set $dom(\overline{\varphi}_0 \oplus \overline{\varphi}_1)$. Moreover, we show how these conditions can be verified for the K–, L– and E– functionals.
For a regular Banach couple (X_0, X_1), there exist two specific convex, lower semicontinuous and proper functions $\varphi_0 : X_0 \longrightarrow \mathbb{R} \cup \{+\infty\}$ and $\varphi_1 : X_1 \longrightarrow \mathbb{R} \cup \{+\infty\}$ for each of the K–, L– and E– functionals such that they can be written as a function $F : X_0 + X_1 \longrightarrow \mathbb{R} \cup \{+\infty\}$ defined by

$$F\left(x \right) = \left(\overline{\varphi_0} \oplus \overline{\varphi_1} \right)(x) = \inf_{x = x_0 + x_1} \left(\overline{\varphi_0}(x_0) + \overline{\varphi_1}(x_1) \right), \tag{14}$$

where the infimum extends over all representations $x = x_0 + x_1$ of x with x_0 and x_1 in $X_0 + X_1$ and where $\overline{\varphi_0} : X_0 + X_1 \longrightarrow \mathbb{R} \cup \{+\infty\}$ and $\overline{\varphi_1} : X_0 + X_1 \longrightarrow \mathbb{R} \cup \{+\infty\}$ are respective extensions of φ_0 and φ_1 on $X_0 + X_1$ in the following way

$$\overline{\varphi_0}\left(u \right) = \begin{cases} \varphi_0 \left(u \right) & \text{if } u \in X_0; \\ +\infty & \text{if } u \in \left(X_0 + X_1 \right) \setminus X_0. \end{cases} \tag{15}$$

and

$$\overline{\varphi_1}\left(u \right) = \begin{cases} \varphi_1 \left(u \right) & \text{if } u \in X_1; \\ +\infty & \text{if } u \in \left(X_0 + X_1 \right) \setminus X_1. \end{cases} \tag{16}$$

For example, the L– functional can be written as the infimal convolution

$$L_{p_0,p_1} \left(t, x; X_0, X_1 \right) = \left(\overline{\varphi_0} \oplus \overline{\varphi_1} \right)(x), \tag{17}$$

where

$$\overline{\varphi_0}\left(u \right) = \begin{cases} \frac{1}{p_0} \left\| u \right\|_{X_0}^{p_0} & \text{if } u \in X_0; \\ +\infty & \text{if } u \in \left(X_0 + X_1 \right) \setminus X_0. \end{cases} \tag{18}$$

and

$$\overline{\varphi_1}\left(u \right) = \begin{cases} \frac{t}{p_1} \left\| u \right\|_{X_1}^{p_1} & \text{if } u \in X_1; \\ +\infty & \text{if } u \in \left(X_0 + X_1 \right) \setminus X_1. \end{cases} \tag{19}$$

In this case the functions $\varphi_0 : X_0 \longrightarrow \mathbb{R} \cup \{+\infty\}$ and $\varphi_1 : X_1 \longrightarrow \mathbb{R} \cup \{+\infty\}$ are defined by

$$\varphi_0 (u) = \frac{1}{p_0} \|u\|_{X_0}^{p_0} \text{ and } \varphi_1 (u) = \frac{t}{p_1} \|u\|_{X_1}^{p_1}. \tag{20}$$

However, it is important to notice that the extended functions $\overline{\varphi_0}$ and $\overline{\varphi_1}$ could stop to be lower semicontinuous even if φ_0 and φ_1 are. Since two different Banach spaces are involved, some technical difficulties appear when you would like to apply known results in convex analysis. In this regard, we reconsider the infimal convolution $F(x) = (\overline{\varphi_0} \oplus \overline{\varphi_1})(x)$ as follows:

$$F(x) = (\overline{\varphi_0} \oplus \overline{\varphi_1})(x) = \inf_{y \in X_0 \cap X_1} (S(y) + R(y)), \tag{21}$$

where S and R are functions defined on $X_0 \cap X_1$ with values in $\mathbb{R} \cup \{+\infty\}$ by

$$S(y) = \varphi_0 (a_0 - y) \text{ and } R(y) = \varphi_1 (a_1 + y), \tag{22}$$

where $a_0 \in X_0$ and $a_1 \in X_1$ are fixed elements such that $x = a_0 + a_1$. The following theorem establishes conditions for which the function $F = \overline{\varphi_0} \oplus \overline{\varphi_1}$ is subdifferentiable on its domain in $X_0 + X_1$.

Theorem 0.5 (Subdifferentiability of infimal convolution). *Let the functions S and R be defined as in* (22) *and be convex, lower semicontinuous and proper. Let φ_0^* and φ_1^* be the respective conjugate functions of φ_0 and φ_1. Suppose that*

(1) the sets dom S and dom R satisfy

$$\bigcup_{\lambda \geq 0} \lambda (\text{dom } S - \text{dom } R) = X_0 \cap X_1 \tag{23}$$

(2) The conjugate function S^* of S is given by

$$S^* (z) = \begin{cases} \varphi_0^* (-z) + \langle z, a_0 \rangle & \text{if } z \in X_0^*; \\ +\infty & \text{if } z \in (X_0^* + X_1^*) \setminus X_0^*. \end{cases} \tag{24}$$

(3) The conjugate function R^* of R is given by

$$R^* (z) = \begin{cases} \varphi_1^* (z) + \langle -z, a_1 \rangle & \text{if } z \in X_1^*; \\ +\infty & \text{if } z \in (X_0^* + X_1^*) \setminus X_1^*. \end{cases} \tag{25}$$

Then the function $(\overline{\varphi_0} \oplus \overline{\varphi_1})$ is subdifferentiable on its domain in $X_0 + X_1$.

By using this result, subdifferentiability of K–, L– and E– functionals was also proved.

Paper 3: A new reiterative algorithm for the Rudin–Osher–Fatemi denoising model on the graph

Problem statement

Let us suppose that we observed noisy image $f_{ob} \in L^2$ defined on a square domain $\Omega = [0,1]^2$ in $\mathbb{R}^2$,

$$f_{ob} = f_* + \eta,$$

where $f_* \in BV$ is the original image and $\eta \in L^2$ is the noise. *Denoising* is one of the problems which appear in image processing: "How to recover the image f_* from the noisy image f_{ob}?". Variational methods using the total variation minimization are often employed to solve this problem. The total variation regularization technique was introduced by Rudin, Osher and Fatemi in [14] and is called the ROF model. It suggests to take as an approximation to the original image f_* the function $f_{opt,t} \in BV$, which is the exact minimizer for the $L_{2,1}$–functional for the couple (L^2, BV):

$$L_{2,1}\left(t, f_{ob}; L^2, BV\right) = \inf_{g \in BV} \left(\frac{1}{2} \|f_{ob} - g\|_{L^2}^2 + t\|g\|_{BV}\right), \text{ for some } t > 0, \quad (26)$$

i.e., $f_{opt,t} \in BV$ is such that

$$L_{2,1}\left(t, f_{ob}; L^2, BV\right) = \frac{1}{2}\|f_{ob} - f_{opt,t}\|_{L^2}^2 + t\|f_{opt,t}\|_{BV}. \quad (27)$$

However the problem of actual calculation of the function $f_{opt,t}$ (see (26) and (27)) is non–trivial. Standard approach is connected with discretization of the functional (26), i.e. we divide Ω into $N \times N$ square cells and instead of the space $L^2(\Omega)$ consider its finite dimensional subspace S_N which consists of functions that are constant on each cell.

Consider the graph $G = (V, E)$, where the set of vertices V corresponds to cells and the set of edges E corresponds to set of pairs of cells which have common faces. Denote by S_V and S_E the set of real-valued functions on V and E respectively and consider the analogue of the gradient operator on the graph, i.e., $\text{grad} : S_V \longrightarrow S_E$ which maps function $f \in S_V$ to function $\text{grad}\, f \in S_E$ defined as

$$(\text{grad}\, f)(e) = f(v_j) - f(v_i) \text{ if } e = (v_i, v_j).$$

The observed image $f_{ob} \in S_N$ can be considered as an element of S_V and the ROF functional can be written as

$$L_{2,1}\left(t, f_{ob}; \ell^2(S_V), BV(S_V)\right) = \inf_{g \in S_V} \left(\frac{1}{2N^2}\|f_{ob} - g\|_{\ell^2(S_V)}^2 + \frac{t}{N}\|\text{grad}\, g\|_{\ell^1(S_E)}\right). \quad (28)$$

Notice that exact minimizer of (28) coincides with exact minimizer of

$$L_{2,1}\left(s, f_{ob}; \ell^2(S_V), BV(S_V)\right) = \inf_{g \in S_V} \left(\frac{1}{2}\|f_{ob} - g\|_{\ell^2(S_V)}^2 + s\|\text{grad}\, g\|_{\ell^1(S_E)}\right),$$

with $s = Nt$.

Problem 2. *Suppose that we know function $f_{ob} \in S_V$. For given $s > 0$, find exact minimizer of the functional*

$$L_{2,1}\left(s, f_{ob}; \ell^2(S_V), BV(S_V)\right) = \inf_{g \in BV(S_V)} \left(\frac{1}{2}\|f_{ob} - g\|^2_{\ell^2(S_V)} + s\|g\|_{BV(S_V)}\right),$$

where

$$\|f\|_{\ell^2(S_V)} = \left(\sum_{v \in V}(f(v))^2\right)^{\frac{1}{2}}; \|f\|_{BV(S_V)} = \|\mathrm{grad}\, f\|_{\ell^1(S_E)};$$

and

$$\|h\|_{\ell^1(S_E)} = \sum_{e \in E}|h(e)|,$$

and operator $\mathrm{grad} : S_V \longrightarrow S_E$ *is defined by the formula*

$$(\mathrm{grad}\, f)(e) = f(v_j) - f(v_i) \text{ if } e = (v_i, v_j)$$

Main contributions and outcomes

We consider an analogue of (26) on a general finite directed and connected graph. We consider the space BV on the graph and show that the unit ball of its dual space can be described as the image of the unit ball of the space ℓ^∞ on the graph by a divergence operator. Based on this result, we propose a new fast algorithm to find the exact minimizer for the ROF model. Convergence of the algorithm is proved and its performance illustrated on some image denoising test examples. It is known (see Paper 1) that the exact minimizer for the $L_{2,1}-$ functional for the couple $\left(\ell^2, X\right)$ on $\mathbb{R}^n$,

$$L_{2,1}\left(t, f_{ob}; \ell^2, X\right) = \inf_{g \in X}\left(\frac{1}{2}\|f_{ob} - g\|^2_{\ell^2} + t\|g\|_X\right),$$

i.e. the function $f_{opt,t}$ such that

$$L_{2,1}\left(t, f_{ob}; \ell^2, X\right) = \frac{1}{2}\|f_{ob} - f_{opt,t}\|^2_{\ell^2} + t\|f_{opt,t}\|_X,$$

is equal to the difference between f_{ob} and the nearest element to f_{ob} of the ball of radius $t > 0$ of the space X^* (see Figure 3 for illustartion). Therefore we first need to describe the ball of radius $s > 0$ of the space $BV^*(S_V)$ with norm defined by

$$\|h\|_{BV^*(S_V)} = \sup_{\|f\|_{BV(S_V)} \leq 1} \langle h, f\rangle_{S_V}, \text{ where } \langle h, f\rangle_{S_V} = \sum_{v \in V} h(v)f(v).$$

To this end, we consider divergence operator on the graph, i.e. the operator $\mathrm{div} : S_E \to S_V$ defined by

$$(\mathrm{div}\, g)(v) = \sum_{i:(v_i,v) \in E} g\left((v_i, v)\right) - \sum_{j:(v,v_j) \in E} g\left((v, v_j)\right).$$

We obtain the following result

$$\{f \in \mathbb{R}^n : \ \|f_{ob} - f\|_{\ell^2} \leq \|f_{opt,t}\|_{\ell^2}\}$$

f_{ob}

$f_{opt,t}$

0 $f_{ob} - f_{opt,t}$

T

$$t\mathcal{B}_{X^*} = \{h \in \mathbb{R}^n : \ \|h\|_{X^*} \leq t\}$$
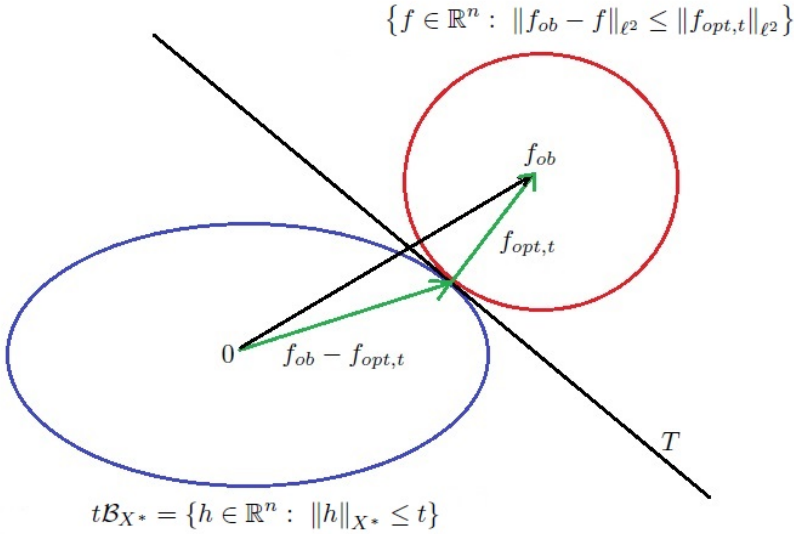
Figure 3: *Illustrating the geometry of Optimal decomposition for the couple $\left(\ell^2, X\right)$ on $\mathbb{R}^n$.*

Theorem 0.6. *The unit ball of the space $BV^*(S_V)$ is equal to the image of the unit ball of the space $\ell^\infty(S_E)$ under the operator* div, *i.e.,*

$$\mathcal{B}_{BV^*(S_V)} = \mathrm{div}\left(\mathcal{B}_{\ell^\infty(S_E)}\right).$$

Therefore the exact minimizer $f_{opt,t}$ for the $L_{2,1}-$ functional

$$L_{2,1}\left(s, f_{ob}; \ell^2(S_V), BV(S_V)\right) = \inf_{g \in BV(S_V)} \left(\frac{1}{2}\|f_{ob} - g\|^2_{\ell^2(S_V)} + s\|g\|_{BV(S_V)}\right)$$

is given by

$$f_{opt,t} = f_{ob} - \widetilde{h},$$

where $\widetilde{h}$ is such that

$$E\left(s, f_{ob}; \ell^2\left(S_V\right), BV^*\left(S_V\right)\right) = \inf_{h \in s\mathcal{B}_{BV^*(S_V)}} \|f_{ob} - h\|_{\ell^2(S_V)} = \left\|f_{ob} - \widetilde{h}\right\|_{\ell^2(S_V)},$$
(29)

where, from Theorem 0.6,

$$s\mathcal{B}_{BV^*(S_V)} = s\,\mathrm{div}\left(\mathcal{B}_{\ell^\infty(S_E)}\right) \text{ for } Nt = s > 0.$$

The proposed algorithm constructs $\widetilde{h}$ through a sequence of elements $g_n \in s\mathcal{B}_{\ell^\infty(S_E)}$ such that $\mathrm{div}(g_n) \longrightarrow \widetilde{h}$ as $n \to +\infty$ in the metric of $\ell^2\left(S_V\right)$. It consists

of several steps outlined below: Let $G = (V = \{v_1, \ldots, v_N\}, E = \{e_1, \ldots, e_M\})$, $f_{ob} \in S_V$, and t be given. Set

$$e_k = (v_i, v_j) \in E, \ k = 1, 2, \ldots, M; \text{ for some } i, j \in \{1, 2, \ldots, N\}.$$

Define the operator T as follows:

$$T = T_M T_{M-1} T_{M-2} \ldots T_2 T_1,$$

where for $k = 1, 2, \ldots, M$, $T_k : s\mathcal{B}_{\ell^\infty(S_E)} \longrightarrow s\mathcal{B}_{\ell^\infty(S_E)}$ is defined as follows:

$$(T_k g)(e) = \begin{cases} \begin{cases} Kg(e_k) & \text{,if } Kg(e_k) \in [-s, +s]; \\ -s & \text{,if } Kg(e_k) < -s; \\ +s & \text{,if } Kg(e_k) > +s. \end{cases} & \text{,if } e = e_k; \\ \\ g(e) & \text{,if } e \neq e_k. \end{cases}$$

where

$$Kg(e_k) = \frac{\left[f_{ob}(v_j) - \left(\text{div} \setminus_{e_k} g \right)(v_j) \right] - \left[f_{ob}(v_i) - \left(\text{div} \setminus_{e_k} g \right)(v_i) \right]}{2}.$$

and

$$\begin{cases} \left(\text{div} \setminus_{e_k} g \right)(v_i) = (\text{div } g)(v_i) + g(e_k); \\ \left(\text{div} \setminus_{e_k} g \right)(v_j) = (\text{div } g)(v_j) - g(e_k); \\ \text{and} \\ \left(\text{div} \setminus_{e_k} g \right)(v_\ell) = (\text{div } g)(v_\ell), \ \forall \ell \neq i, j. \end{cases}$$

Step 1. Take $g_0 = 0$, or choose any $g_0 \in s\mathcal{B}_{\ell^\infty(S_E)}$

Step 2. Calculate $g = Tg_0$. i.e., calculate $(Tg_0)(e_k)$ for $k = 1, 2, \ldots, M$. If $g = g_0$ then take $\tilde{h} = \text{div}(g_0)$, otherwise go to **Step 3.**

Step 3. Put $g_0 = g$ and go to **Step 2.**

We continue this process applying the operator T to the new element $g \in s\mathcal{B}_{\ell^\infty(S_E)}$ generating the sequence of elements $g_0, \ g_1 = Tg_0, \ g_2 = Tg_1, \ldots, g_n = Tg_{n-1}$ with $g_n \in s\mathcal{B}_{\ell^\infty(S_E)}$, $n = 0, 1, 2, \ldots$ until a maximum number of iterations is reached. It is shown in Theorem 0.7 below, that

$$\text{div}(g_n) \longrightarrow \tilde{h} \text{ as } n \to +\infty \text{ in the metric of } \ell^2(S_V).$$

The proof uses the following proposition

Proposition 0.1. *Let $\tilde{h}$ be the minimizer defined as in* (29). *The operator T is continuous and satisfies the following two conditions*

(a) For any $g \in s\mathcal{B}_{\ell^\infty(S_E)}$, $\text{div } g = \tilde{h}$ if and only if $Tg = g$;

(b) For any $g \in s\mathcal{B}_{\ell^\infty(S_E)}$, if $\operatorname{div} g \neq \widetilde{h}$ then

$$\|f_{ob} - \operatorname{div}(Tg)\|_{\ell^2(S_V)} < \|f_{ob} - \operatorname{div} g\|_{\ell^2(S_V)} . \tag{30}$$

Finally

Theorem 0.7. *Let $\widetilde{h}$ be the minimizer defined as in (29), $g \in s\mathcal{B}_{\ell^\infty(S_E)}$ and let T be the operator constructed in Algorithm. Then*

$$\operatorname{div}(T^n g) \longrightarrow \widetilde{h} \text{ as } n \to +\infty \text{ in the metric of } \ell^2(S_V) .$$

Paper 4: Exact Minimizers in Real Interpolation. Some additional results

Problem statement

In Paper 4, we consider several extensions of our previous results. In Paper 1 a characterization of optimal decomposition for real Banach couples was obtained by using duality in convex analysis. However a natural question arises as to what will happen if we have more than 2 paces. Such type of situations are important in image processing. Unfortunately for three spaces results start to be more complicated. In particular the duality formula $(X_0 + X_1)^* = X_0^* + X_1^*$ is not true even for regular triple. Another question which arises is how to characterize optimal decomposition for complex spaces since real interpolation is also used for complex spaces. In the last section of the paper we illustrate the comparison in performance of our algorithm in Paper 3 with other algorithms.

Main contributions and outcomes

Assume that the triple (X_0, X_1, X_2) is regular, i.e. $X_0 \cap X_1 \cap X_2$ is dense in each of X_j, $j = 0, 1, 2$. Let $x \in X_0 + X_1 + X_2$ and let $s, t > 0$ be fixed parameters. The L–functional for this triple is defined as follows:

$$L_{p_0, p_1, p_2}(s, t; x; X_0, X_1, X_2) = \inf_{x = x_0 + x_1 + x_2} \left(\frac{1}{p_0} \|x_0\|_{X_0}^{p_0} + \frac{s}{p_1} \|x_1\|_{X_1}^{p_1} + \frac{t}{p_2} \|x_2\|_{X_2}^{p_2} \right), \tag{31}$$

where $1 \leq p_0 < \infty$, $1 \leq p_1 < \infty$ and $1 \leq p_2 < \infty$.

We show that analogous results to Theorem 0.2 of Paper 1 are possible to obtain (see Corollary 0.1 and Theorem 0.8 below). Let us consider a regular triple (X_0, X_1, X_2) and a special case when $p_1 = p_2 = 1$. We obtain the following result

Corollary 0.1. *Let $1 < p_0 < +\infty$ and $s, t > 0$. Then the decomposition $x = x_{0,opt} + x_{1,opt} + x_{2,opt}$ is optimal for the $L_{p_0,1,1}$–functional if and only if there exists $y_* \in (X_0 + X_1 + X_2)^* \subseteq X_0^* \cap X_1^* \cap X_2^*$ such that $\|y_*\|_{X_1^*} \leq s$; $\|y_*\|_{X_2^*} \leq t$ and*

$$\begin{cases} \frac{1}{p_0} \|x_{0,opt}\|_{X_0}^{p_0} = \langle y_*, x_{0,opt} \rangle - \frac{1}{p_0'} \|y_*\|_{X_0^*}^{p_0'} \\ s \|x_{1,opt}\|_{X_1} = \langle y_*, x_{1,opt} \rangle \\ t \|x_{2,opt}\|_{X_2} = \langle y_*, x_{2,opt} \rangle \end{cases} \tag{32}$$

where $\frac{1}{p_0} + \frac{1}{p'_0} = 1$.

To understand the geometry of optimal decomposition, consider the triple (ℓ^p, X_1, X_2) on $\mathbb{R}^n$, where X_1 and X_2 are any Banach spaces. We consider the $L_{p,1,1}$–functional for the triple (ℓ^p, X_1, X_2), i.e.

$$L_{p,1,1}(s,t;x;\ell^p, X_1, X_2) = \inf_{x=x_0+x_1+x_2} \left(\frac{1}{p} \|x_0\|_{\ell^p}^p + s\|x_1\|_{X_1} + t\|x_2\|_{X_2} \right),$$

where $s, t > 0$ and $1 < p < +\infty$. Let F_0, F_1 and F_2 be functions defined on $\mathbb{R}^n$ by

$$F_0(u) = \frac{1}{p} \|u\|_{\ell^p}^p \ , \ F_1(u) = s\|u\|_{X_1} \text{ and } F_2(u) = t\|u\|_{X_2}. \tag{33}$$

It appears the consideration of two important sets Ω_{s,X_1} and Ω_{t,X_2} defined by

$$\Omega_{s,X_1} = \left\{ u \in \mathbb{R}^n \ : \ \nabla F_0(u) \in s\mathcal{B}_{X_1^*} \right\}, \ \Omega_{t,X_2} = \left\{ u \in \mathbb{R}^n \ : \ \nabla F_0(u) \in t\mathcal{B}_{X_2^*} \right\}, \tag{34}$$

where $s\mathcal{B}_{X_1^*}$ (resp. $t\mathcal{B}_{X_2^*}$) is the ball of the dual space X_1^* (resp. X_2^*) of radius s (resp. t) with its center at the origin. There will then be four cases depending on what set $x_{0,opt}$ belongs to.

Theorem 0.8. *Let* $x \in \mathbb{R}^n$ *with optimal decomposition* $x = x_{0,opt} + x_{1,opt} + x_{2,opt}$ *for* $L_{p,1,1}(s,t;x;\ell^p, X_1, X_2)$*–functional. Then*

(1) *If* $x_{0,opt} \in int\left(\Omega_{s,X_1} \cap \Omega_{t,X_2}\right)$ *then the optimal decomposition for* $L_{p,1,1}(s,t;x;\ell^p, X_1, X_2)$*–functional is given by* $x_{0,opt} = x$ *and* $x_{1,opt} = x_{2,opt} = 0$.

(2) *If* $x_{0,opt} \in int\left(\Omega_{s,X_1}\right) \cap bd\left(\Omega_{t,X_2}\right)$*, then the optimal decomposition for* $L_{p,1,1}(s,t;x;\ell^p, X_1, X_2)$*–functional is given by* $x = x_{0,opt} + 0 + x_{2,opt}$ *and is such that*

$$\langle x_{2,opt}, \nabla F_0(x_{0,opt})\rangle = \|x_{2,opt}\|_{X_2} \|\nabla F_0(x_{0,opt})\|_{X_2^*} = t\|x_{2,opt}\|_{X_2}. \tag{35}$$

(3) *If* $x_{0,opt} \in int\left(\Omega_{t,X_2}\right) \cap bd\left(\Omega_{s,X_1}\right)$ *then the optimal decomposition for* $L_{p,1,1}(s,t;x;\ell^p, X_1, X_2)$*–functional is given by* $x = x_{0,opt} + x_{1,opt} + 0$ *and is such that*

$$\langle x_{1,opt}, \nabla F_0(x_{0,opt})\rangle = \|x_{1,opt}\|_{X_1} \|\nabla F_0(x_{0,opt})\|_{X_1^*} = s\|x_{1,opt}\|_{X_1}. \tag{36}$$

(4) *If* $x_{0,opt} \in bd\left(\Omega_{s,X_1}\right) \cap bd\left(\Omega_{t,X_2}\right)$ *then the optimal decomposition for* $L_{p,1,1}(s,t;x;\ell^p, X_1, X_2)$*–functional is given by* $x = x_{0,opt} + x_{1,opt} + x_{2,opt}$ *such that*

$$\begin{cases} \langle x_{1,opt}, \nabla F_0(x_{0,opt})\rangle = \|x_{1,opt}\|_{X_1} \|\nabla F_0(x_{0,opt})\|_{X_1^*} = s\|x_{1,opt}\|_{X_1} \\ \langle x_{2,opt}, \nabla F_0(x_{0,opt})\rangle = \|x_{2,opt}\|_{X_2} \|\nabla F_0(x_{0,opt})\|_{X_2^*} = t\|x_{2,opt}\|_{X_2}. \end{cases}$$

Next we use our approach when complex spaces are considered instead of real spaces. In this case we need instead of standard conjugate functional $F^*(y_*) = \sup_{x \in E} \{\langle y_*, x \rangle - F(x)\}$, to define it as $F^*(y_*) = \sup_{x \in E} \{\Re \langle y_*, x \rangle - F(x)\}$. Let $E_{\mathbb{C}}$ be a complex Banach space and let $E_{\mathbb{R}}$ be the same space with the same norm but considered real Banach space in the sense that we restrict multiplication by scalars to real numbers only, instead of complex numbers. Let $(E_{\mathbb{C}})^*$ (resp. $(E_{\mathbb{R}})^*$) be the dual space to $E_{\mathbb{C}}$ (resp. $E_{\mathbb{R}}$) consisting of complex (resp. real) valued linear and bounded functionals $f : E_{\mathbb{C}} \to \mathbb{C}$ (resp. $g : E_{\mathbb{R}} \to \mathbb{R}$). We illustrate that the spaces $(E_{\mathbb{C}})^*$ and $(E_{\mathbb{R}})^*$ are isometric in some sense. We then show that for a regular complex Banach couple (X_0, X_1), we can use the same approach to obtain similar results as in the real situation. For example, consider the L-functional

$$L_{p_0,1}(t, x; X_0, X_1) = \inf_{x = x_0 + x_1} \left(\frac{1}{p_0} \|x_0\|_{X_0}^{p_0} + t \|x_1\|_{X_1} \right),$$

where $1 < p_0 < +\infty$.

Theorem 0.9. *Let* $1 < p_0 < +\infty$. *Then the decomposition* $x = x_{0,opt} + x_{1,opt}$ *is optimal for the* $L_{p_0,1}$*-functional if and only if there exists* $y_* \in X_0^* \cap X_1^*$ *such that* $\|y_*\|_{X_1^*} \leq t$ *and*

$$\begin{cases} \frac{1}{p_0} \|x_{0,opt}\|_{X_0}^{p_0} = \Re \langle y_*, x_{0,opt} \rangle - \frac{1}{p_0'} \|y_*\|_{X_0^*}^{p_0'} \\ t \|x_{1,opt}\|_{X_1} = \Re \langle y_*, x_{1,opt} \rangle. \end{cases} \tag{37}$$

Finally, we compare the performance of the algorithm wich was obtained in Paper 3 with the Split Bregman algorithm. The Split Bregman algorithm is like a benchmark algorithm known for the ROF model. We find out that in most cases both algorithms behave in a similar way and that in some cases our algorithm decreases the error faster with the number of iterations.

Paper 5: Optimal decomposition for infimal convolution on Banach Couples. Some Connections to Linear and Convex Programming

Problem statement

The idea of this paper was to investigate connections between our approach and two well–known optimization problems, namely (nonlinear) convex and linear programming.

Main contributions and outcomes

The main outcome of the paper is that, based on our approach, it is possible, under some additional assumptions to derive proofs for duality theorems which are central for these problems. The approach is as follows: First we reformulate the optimization problem at hand as an infimal convolution of two well–defined functions. Secondly, we check subdifferentiability of the infimal convolution by Theorem 0.5 and finally use Theorem 0.4.

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Part II

Papers

Papers

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