COLLEGE OF SCIENCE AND TECHNOLOGY

SCHOOL OF SCIENCE

DEPARTMENT OF MATHEMATICS

# PRICING INSURANCE POLICIES IN RWANDA USING MULTI-STATE MARKOV CHAIN MODELS:CASE STUDY FOR SANLAM.

By

**NDAYIRAGIJE Clementine**

Master of Science

in

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# *PRICING INSURANCE POLICIES IN RWANDA USING MULTI-STATE MARKOV CHAIN MODELS:CASE STUDY FOR SANLAM.*

By

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Student number: 218015628

Thesis Submitted in Partial Fulfillment of the Academic Degree of
Master of Science
in
Applied Mathematics
(Statistical modeling and actuarial sciences)

**Supervisors**: Prof. Juma KASOZI, Dr. Marcel NDENGO

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# Declaration

I, **NDAYIRAGIJE Clementine,** student from University of Rwanda, College of Science and Technology, with **registration number 218015628**, hereby declare and confirm that, this project entitled **Pricing insurance policies in Rwanda using multi-state Markov chain models, case study for Sanlam** is my original work and I have never been submitted or presented in any University or Institution of higher learning for academic purposes or otherwise. This thesis was conducted under the supervision of **Prof. Juma KASOZI** from Department of Mathematics at Makerere University, Uganda and **Dr. Marcel NDENGO** from University of Rwanda-College of Science and Technology, Rwanda.

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Signature:

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Supervisor: Prof. Juma KASOZI

Signature:

Date: 03/08/2022

# Dedication

I dedicate this thesis to almighty God, my supervisors and relatives.

# Abstract

This thesis seeks to apply a model for pricing insurance policies. Some insurance companies are facing increasing financial losses because of large claims, which are not equivalent to the premiums. Continuous-time Markov chain has been used to price insurance policies, at the beginning elementary theory of continuous-time Markov chain is introduced and some basic properties of Markov chain, transition probabilities, force of intensity. Next part of thesis, the reader is introduced to some application of Kolmogorov differential equation for a multi-state model as they are widely used in actuarial science because they provide a convenient way for representing changes in people's status. The multi-state models have been assumed to satisfy the Markov properties. The continuous-time Markov chain have been used in calculating the transition intensities and transition probabilities have been calculated using Kolmogorov differential equation. Transition intensities, transition probabilities are used to calculate the expected present values of benefit, the annuities of death benefit and the premiums as applied to insurance products for Sanlam. Results reveal that the premiums and the benefits of transition between disability state to death are highest compared to the others. The conclusion is increasing premiums leads to more benefits which is in line with real reality.

# Acknowledgment

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I would like to thank the government of Rwanda and Sweden for their partnership, I express my gratitude to University of Rwanda for admitting me in the master program of Applied mathematics and to the Sanlam for giving the data that I have used in this research.

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# Contents

## List of Acronyms,signs and abbreviations

**BK**: Bank of Kigali

**EAC**: East Africa Community

**GDP**: Global Disaster Protection

**MMI**: Military Medical Insurance

**MUAR**: Mauritius Union Assurance Rwanda Ltd

**RSSB**: Rwandan Social Security Board

**UAP**: Union des Assurance de Paris (UAP insurance Rwanda Ltd)

${}_tP_x^{ij}$: The probability that an individual currently age $x$ who is currently in state $i$, they will be in state $j$ at age $x+t$.

$\mu_x^{ij}$: transition intensity of moving from state $i$ to state $j$

$i$: interest rate

$v_t$: discount factor

$T_x$: future life time

$\overline{A}_x$: Expected present value of benefit

$\ddot{a}_x$: Expected present value of annuity.

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# Chapter 1

# Introduction

This chapter includes the general background, research justification, the objectives of the study, the problem statement and the structure of the thesis.

## 1.1 General background

Life insurance dates back (100 B.C) in the ancient Rome by Military leader, whose name is Caius Marius who formulated burial clubs where the members of the clubs would help cover the funeral expenses if another member get death. Romans practice towards this problem is one of the earliest form of life insurance in human history, the problem was much like that we have today, where some companies promise to the insured to pay a designated beneficiary a sum of money (benefit) in the exchange for the premium upon the death of an insured person [Dash and Khan, 2011].
England introduced the first stock companies that get into the insurance in 1720, again in 1735 was born the first insurance company in the American colonies in Charleston. Similarly, in 1759, the Presbyterian Synod of Philadelphia sponsored the first life insurance corporation in America for the benefit of ministers and their dependents. It was after 1840 that life insurance really took off in a big way [Dash and Khan, 2011]. The insurance purpose is to provide protection against future risk accidents and uncertainty [Singh, ]. It is considered as a contract of reimbursement. For example, it reimburses for losses such as fire, earthquake and accident [Mayer et al., 2012]. So,

the mostly people decide to protect their property (or their life) against these losses. The person who receives the payment is called insured and must pay some amount called the premium to insurance company.
Insurance has become a way to protect people's interests from loss and uncertainty. It may be defined as a social device to reduce or eliminate risk of loss to life and property [Adeniyi et al., 2019]. It provides safety and security against particular event.
Insurance contributes to the general economic growth of society, when it is used in mobilizing domestic savings, the people save money by paying the premiums and the insured get the benefit at the maturity of the contract [Mashayekhi and Fernandes, 2007].
Insurance can be divided into two basic categories, Life insurance and non-life insurance. Life insurance is a financial product that pays you or your dependents a sum of money either after a set period or state or upon your death as the case may be.It is in providing security for a growing family and support in items of unforeseen need [Black et al., 1994].
A life insurance policy guarantees the insurer pays a sum of money to named beneficiaries when the insured dies in exchange for the premiums paid by the policyholder during their life time It enables systematic savings due to payment of regular premium, provides a mode of investment, develops a habit of saving money by paying premiums. The insured get the lump sum amount at the maturity of the contract.
Non-life insurance covers non-life assets, such as a home, vehicle, it is the contract of indemnity, its premium has to be paid lump sum [Leiria et al., 2021]. Mainly non-life insurance includes many types of other insurance policies like auto insurance, property insurance, disaster insurance (fire, flood, earthquake, etc.), credit insurance and mortgage insurance. The time and amount of loss are uncertain and at the happening of a risk, the person will suffer the loss if she/he did not undertake insurance. Insurance guarantees the payment of loss and thus protects the insured from sufferings [Linnerooth-Bayer et al., 2011]. Insurance cannot check the happening of risk but can provide for losses at the happening of the risk.
The risk is uncertain, and therefore, the loss arising from the risk is also uncertain. When risk occurs, the loss is shared by all the persons who are exposed to the risk.

The risk-sharing in ancient times was done only at the time of damage or death; but today, on the basis of the probability of risk, (share is obtained from each and every insured in the shape of premium without which protection is not guaranteed by the insurer. The insurance joins hands with those institutions which are engaged in preventing the losses of the society because the reduction in loss causes the lesser payment to the insured and so more saving is possible which will assist in reducing the premium. Lesser premium invites more business and more business cause lesser share to the insured. So again premium is reduced and thus will stimulate more business and more protection to the masses [Mashayekhi and Fernandes, 2007].

Therefore, the insurance assists the health organization, fire brigade, educational institutions and other organizations against losses arising from death or damage, it eliminates worries and miseries of losses at death and destruction of property. The carefree person can devote his body and soul together for better achievement, it improves not only his efficiency but the efficiencies of the masses are also advanced. Insurance market in Rwanda is currently small relative to overall financial sector in EAC, the percentage represented by GDP in Rwanda is 1.6%. In 2017 Rwanda had in total of 12 insurance companies, 2 public insurers and 10 private. In public there are RSSB and MMI, in private these 10 insurers are SORAS(Sanlam), SONARWA, RADIANT, UAP, BK insurance Company, Prime insurance, Saham, MUAR, Britam and Mayfair insurance company. Private companies take 0.9% and public are 0.7%, the corresponding premiums in USD is 143 million [Group, 2019].

## 1.2 Research Justification

Many people will benefit from the findings of this work. The results from this study will provide a helpful literature for individuals, students and the management of Insurance companies in Rwanda and beyond. First of all, this study will be helpful to the researcher to understand how insurance policies are priced between different states. About individuals, this research will help people who want to get knowledge of impact of pricing of insurances policies in different states in Rwanda and it will open the way to other researchers to research more on to price insurance policies. Finally, the students who are doing their thesis in the same or related areas will benefit from the findings of this research.

## 1.3 Problem statement

Some insurance companies are always facing increasing financial losses because of large claims. The theory of continuous time Markov chain will be applied to analyze pricing insurance policies in Rwanda using multi-state markov chain models, means that it will be used to analyze life insurance model with payment depending on the state of health of insured including alive, disabilities and dead.

## 1.4 The General and specific objectives

### 1.4.1 Main objective

The main objective of this thesis is to price insurance policies in Rwanda using multi-state Markov chain models:case study of Sanlam.

### 1.4.2 Specific objectives

- To price insurance policies by using a Markov chain model for alive, disabilities and dead.
- To evaluate the premiums and benefits in the multi-states model.
- To model alive, disabilities and dead using Kolmogorov differential equations.

## 1.5 Key definitions

**Premiums** is the amount charged by an insurance company for an insurance policy, it help insurance company to provide the funds needed to meet the benefit paying

liabilities and it cover the expenses of running the business [Gupta and Varga, 2013]
**Death benefit** is the money lump sum that gets paid to your beneficiaries if you die and it helps maintain your family's financial stability [Browne and Kim, 1993].
**Life insured** is the person whose life is covered by the company. In case of unfortunate death of the Life Insured the death benefits of the policy are received by the nominee or the Policyholder [Fordney, 2015].
**Net single premium** is the present value of the future death benefit [Gupta and Varga, 2013].
Interest rate: is the reward paid by insurer to a insured for the use of money for a period and they are expressed in percentage terms per year [Faure, 2014].

## 1.6 Structure of the thesis

This study is structured as follows: Chapter one describes the introduction and background of the life insurance, statement of the problem, the research objectives, research justification. Chapter two describes Markov chain and its property, multi-state models and literature reviewed based on pricing insurance. Chapter three describes the methodology with permanent disability, application of Chapman Kolmogorov equations, Kolmogorov forward differential equation, calculation of transition intensities and transition probabilities, the calculation of premiums using equivalence principal. Chapter four outlines the application of the pricing insurance policies in three states alive, disability and death within kolmogorov differential equation, the findings, conclusion and recommendations based on the study findings.

# Chapter 2

# Literature review

In this chapter the relevant literature regarding the Markov Chain and pricing insurance policies are introduced. Different authors proposed the use of Markov chains in life contingencies, for both the time-continuous case and the time-discrete case [Alegre et al., 2002] There has been an important research which makes use of Multi-state models to provide a powerful tool for application in many areas of actuarial science, particularly in the actuarial assessment of alive insurance, disability and death income benefits.

## 2.1 Theoretical review

### 2.1.1 Probability

**Definition 2.1.** *Let* $(\Omega, \mathcal{A})$ *be a measurable space; let* $P$ *be a real valued set function defined on the* $\sigma$ *-algebra. The set function* $P : \mathcal{A} \to [0, 1]$ *is called a probability measure if it has the following properties*

- For all $B \in \mathcal{A}; P(B) \geq 0$
- $P(\Omega) = 1$
- If $B_n \cap B_m = \phi$ for $n \neq m$ then $P(\cup_{n=1}^{\infty} B_n) = \sum_{n=1}^{\infty} P(B_n)$ where all $B_n \in \mathcal{A}$. The probability measure $P$ along with the measurable space $(\Omega, \mathcal{A})$ define probability space, $(\Omega, \mathcal{A}, P)$ [Grimmett and Welsh, 2014].

### 2.1.2 Stochastic process

A stochastic process is a family of a random variables, $X(t) : t \in T$, on a probability space $(\Omega, \mathcal{A}, P)$ taking values in state space of the process [Lawler, 2018].

### 2.1.3 Markov Chain

Markov chain, first introduced by Russian Andrei Andreyevich Markov, it is the mathematical model that uses the concept probability to describe how a system changes from one state to another [Behrends, 2000].

In Brief, Markov chain is a discrete sequence of state where each variable drawn from discrete state space (finite or infinite) follows the Markov property, the above statements can be written mathematically as follows:

**Definition 2.2.** *Let $X(t), t \in T$ be a stochastic process on $(\Omega, \mathcal{A}, P)$ with state $S$, $t_0, \cdots, t_{n+1} \in T$ and $t_0 < t_1 < \cdots < t_{n+1}$ for any $i, j \in S$,*

- $P\left[X(t_0)\right] = i_0$
- $P\left(X(t_{n+1}) = j | X(t_0) = i_0, X(t_1) = i_1, \cdots, X(t_n) = i\right)$
  $= P\left(X(t_{n+1}) = j | X(t_n) = i\right) = p^{ij}$.

*Where $p^{ij}$ is the probability that the Markov chain jumps from state $i$ to state $j$. Hence the process $X(t_n)$, $n \geq 0$ is called Markov chain with state space $\mathcal{S}$ [Serfozo, 2009].*

#### 2.1.3.1 Markov property

**Definition 2.3.** *Suppose we have state space $\mathcal{S} = \{0, 1, \cdots, n\}$ with $j_k \in \mathcal{S}$ , assume $X(t)$ is a stochastic process on some probability space $(\Omega, \mathcal{A}, P)$ with values in $\mathcal{S}$ . Further let $t_k \in T \subset R$ for all $k$, where $T$ is some ordered index set. The Markov property is defined as:*

$$P\left(X(t_k) = j_k | X(t_{k-1}) = j_{k-1}\right) = P\left(X(t_k) = j_k | X(t_h) = j_h, h = 0, 1, \cdots, k-1\right), \tag{2.1}$$

*where $X(t_0) = j_0$ and $0 = t_0 < t_1 < \cdots < t_k$.*

In Markov chain, there is a finite state space which we may call the discrete space that contains the set of all possible values of $\mathcal{S}$. We can classify the Markov chain depending on time parameter index $T$ discrete or continuous. Markov chain with discrete time $T = \{0, 1, \cdots\}$ is called discrete time Markov chain while when $T$= $(0, \infty)$ it is called a continuous Markov chain [Pasaribu et al., 2019]. The continuous-time Markov chain (CTMC'S) is the natural sequel to discrete time Markov chain, most properties of CTMC'S follows directly the result about DTMC'S, the Poisson process and the exponential distribution [Whitt, 2006]. The continuous-time Markov chain (CTMC) is a continuous stochastic process for each state, the process will change state according to an exponential random variable and then move to a different state as specified by the probabilities of a stochastic matrix.

#### 2.1.3.2 The Chapman-Kolmogorov Equations

For a fixed $t \in [0, n]$ the events $\{X(t) = j\}$,$j \in S$, are disjoint. It follows that

$$\begin{aligned} P\left[X(u) = k|X(s) = i\right] &= \sum_{j \in S} P\left[X(t) = j, X(u) = k|X(s) = i\right] \\ &= \sum_{j \in S} P\left[X(t) = j|\ X(s) = i\right] P[X(u) = k|X(s) = i, X(t) = j]. \end{aligned}$$

If $X$ is Markov, and $0 \leq s \leq t \leq u$ this reduces to

$$p^{ik}(s, u) = \sum_{j \in S} p^{ij}(s, t) p^{jk}(t, u), \tag{2.2}$$

which is known as the Chapman-Kolmogorov equations. Where $p^{ik}(s, u)$ represents the transition probability from state $i$ at time $s$ to state $k$ at time $u$, $p^{ij}(s, t)$ represents transition probability from state $i$ at time $s$ to state $j$ at time $t$ and $p^{jk}(t, u)$ represents transition probability from state $j$ at time $t$ to state $k$ at time $u$ [Norberg, 2000].

### 2.1.4 Multi-State Models

In this section, the reader is introduced to the importance of those models and their applications in the real life problems. Different authors had used multiple state models for analyzing the actuarial problems. It has importance of reformulating the life table model as Markov model, i.e it generalizes the more complicated problems that have the form of insurance contracts.

Assume that there is a finite set of $n + 1$ states labeled $0, 1, 2, \ldots, n$. All states represent different conditions for an individual or groups of individuals. For each $t \geq 0$, the random variable $X(t)$ takes one of the values $0, 1, \ldots, n$, and the event $X(t) = S$ is interpreted to mean that the individual is in state $S$ at age $x + t$ where $x + t$ represents the age of an individual after tim $t$.

#### 2.1.4.1 Survival model

Consider two states ***alive*** denoted by 0 or ***dead*** denoted by 1, an individual is at any one of these two state so the transition which can be possible it is from ***alive*** to ***dead*** but the transition in the opposite direction is impossible, and the following model can be formulated as an individual aged $x \geq 0$ at time $t = 0$, for each $t \geq 0$, a random variable $X(t)$ takes one of the two values 0 or 1. $X(t) = 0$, it means that our individual is alive ate age $x + t$, and $X(t) = 1$ if the individual died before age $x + t$.

Let show how the transition matrix P and generator matrix Q are computed for the survival model represented in Figure 2.1,

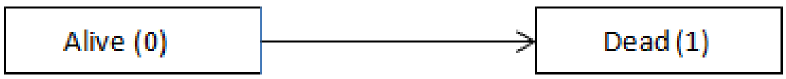



Figure 2.1: Survival model.

Applying equation (2.2) on the survival model,

$$
\begin{aligned}
{}_tP_x^{00}(s, t+h) &= {}_tP_x^{00}(s,t)\, {}_tP_x^{00}(t, t+h) + {}_tP_x^{01}(s,t)\, {}_tP_x^{10}(t, t+h) \\
&= {}_tP_x^{00}(s,t) \left(1 - \mu_{x+t}^{01} h + 0(h)\right)
\end{aligned}
$$

Rearranging the above formula, the answer will be:

$$
{}_tP_x^{00}(s, t+h) - {}_tP_x^{00}(s,t) = \left(-\mu_{x+t}^{01} h + 0(h)\right) {}_tP_x^{00}(s,t) \tag{2.3}
$$

Considering the definition of derivative, i.e by dividing the equation (2.3) by $h$ and letting $h \to 0^+$,

$$\frac{d}{dt}\,{}_tP_x^{00}(s,t) = \lim_{h\to 0^+} \frac{{}_tP_x^{00}(s,t+h) - {}_tP_x^{00}(s,t)}{h}. \tag{2.4}$$

Finally,

$$\frac{d}{dt}\,{}_tP_x^{00}(s,t) = -\mu_{x+t}^{01}\,{}_tP_x^{00}(s,t). \tag{2.5}$$

Similar case using the Chapman-Kolmogorov Equation,

$$\begin{aligned} {}_tP_x^{01}(s,t+h) &= {}_tP_x^{00}(s,t)\,{}_tP_x^{01}(t,t+h) + {}_tP_x^{01}(s,t){}_tP_x^{11}(t,t+h) \\ &= {}_tP_x^{00}(s,t)\left(\mu_{x+t}^{01}h + 0(h)\right) + {}_tP_x^{01}(s,t)\left(1 - \mu_{x+t}^{10}h + 0(h)\right), \end{aligned}$$

$${}_tP_x^{01}(s,t+h) - {}_tP_x^{01}(s,t) = {}_tP_x^{00}(s,t)\left(\mu_{x+t}^{01}h + 0(h)\right) + {}_tP_x^{01}(s,t)\left(-\mu_{x+t}^{10}h + 0(h)\right), \tag{2.6}$$

such that:

$$\frac{d}{dt}\,{}_tP_x^{01}(s,t) = \lim_{h\to 0^+} \frac{{}_tP_x^{01}(s,t+h) - {}_tP_x^{01}(s,t)}{h} \tag{2.7}$$

replace

$$\frac{{}_tP_x^{01}(s,t+h) - {}_tP_x^{01}(s,t)}{h}$$

by equation 2.6 , the answer will be

$$\frac{d}{dt}\,{}_tP_x^{01}(s,t) = \mu_{x+t}^{01}\,{}_tP_x^{00}(s,t) - \mu_{x+t}^{10}\,{}_tP_x^{01}(s,t) \tag{2.8}$$

$$\frac{d}{dt}\,{}_tP_x^{01}(s,t) = \mu_{x+t}^{01}\,{}_tP_x^{00}(s,t). \tag{2.9}$$

Derive the system of Kolmogorov Differential Equations for survival model

$$\begin{cases} \frac{d}{dt}\,{}_tP_x^{00}(s,t) = -\mu_{x+t}^{01}\,{}_tP_x^{00}(s,t) \\ \frac{d}{dt}\,{}_tP_x^{01}(s,t) = \mu_{x+t}^{01}\,{}_tP_x^{00}(s,t). \end{cases} \tag{2.10}$$

This system Kolmogorov Differential Equations can be represented as matrix as follows:

$$\begin{bmatrix} \frac{d}{dt}\,{}_tP_x^{00}(s,t) \\ \frac{d}{dt}\,{}_tP_x^{01}(s,t) \end{bmatrix}^T = \left({}_tP_x^{00}(s,t),\, {}_tP_x^{01}(s,t)\right) \begin{bmatrix} -\mu_{x+t}^{01} & \mu_{x+t}^{01} \\ 0 & 0 \end{bmatrix} \tag{2.11}$$

which is Kolmogorov forward differential equation

$$\frac{d}{dt}\,{}_tP_x = \,{}_tP_x\; Q(t), \tag{2.12}$$

where $Q(t)$ represent transition matrix and is given by

$$Q(t) = \begin{bmatrix} -\mu_{x+t}^{01} & \mu_{x+t}^{01} \\ 0 & 0 \end{bmatrix}. \tag{2.13}$$

#### 2.1.4.2 Survival model and die with different causes

An individual starts by being alive and after some future time he/she dies with the different causes for example deaths due to accident, disease,...

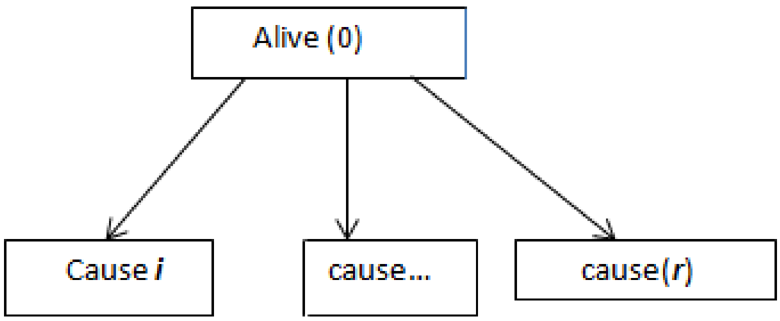



Figure 2.2: Different causes of death model

According equation (2.2), let derive the transition probabilities and their corresponding transition intensities using the Figure 2.2:

$$ {}_tP_x^{0j}(s,t+h) = \sum_{k}^{r} {}_tP_x^{0k}(s,t)\,{}_tP_x^{kj}(t,t+h)$$

$$ {}_tP_x^{ij}(s,t+h) = \sum_{k}^{r} {}_tP_x^{ik}(s,t)\,{}_tP_x^{kj}(t,t+h).$$

$$
\begin{aligned}
{}_tP_x^{00}(s,t{+}h) &= {}_tP_x^{00}(s,t)\,{}_tP_x^{00}(t,t{+}h){+}h\left({}_tP_x^{01}(s,t)\,{}_tP_x^{10}(t,t+h)+\cdots+{}_tP_x^{0r}(s,t)\,{}_tP_x^{r0}(t,t+h)\right)\\
&= {}_tP_x^{00}(s,t)\left(1-h\sum_{k=1}^{r}\mu_{x+t}^{0k}\right)
\end{aligned}
$$

which is simply

$$
\frac{{}_tP_x^{00}(s,t+h)-{}_tP_x^{00}(s,t)}{h} = -\,{}_tP_x^{00}(s,t)\sum_{k=1}^{r}\mu_{x+t}^{0k} \tag{2.14}
$$

represent the Kolmogorov forward differential equation

$$
\frac{d}{dt}\,{}_tP_x^{00}(s,t) = {}_tP_x^{00}(s,t)\left(-\sum_{k=1}^{r}\mu_{x+t}^{0k}\right). \tag{2.15}
$$

#### 2.1.4.3 Permanent disability model

Ballentine's Law Dictionary defines a permanent disability is one that ”will remain with a person throughout” his or her lifetime, or he or she will not recover, or ”that in all possibility, will continue indefinitely. Total and Permanent Disablement Insurance is designed to provide a lump sum benefit to the life insured in the event of a medically diagnosed event that renders the claimant unable to work again. If Insurance is generally used to cover debts and the ongoing living expenses of an individual to reduce the ongoing financial burden of loss of income. Data on this model are available from Sanlam, it will be used for determining the transition probabilities, force of intensities, premiums and benefits for states of alive, disabilities and death.

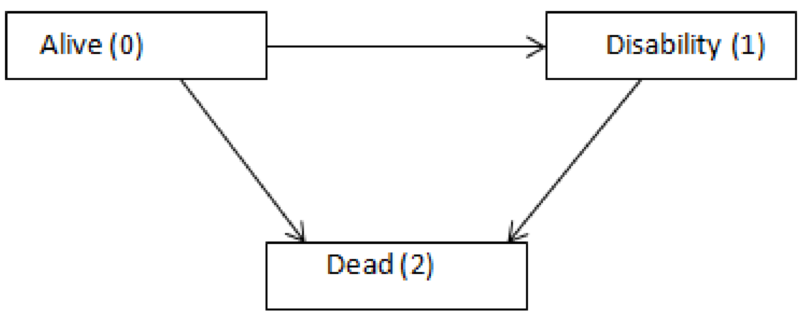



Figure 2.3: Permanent disability model

According the equation (2.2) for the Figure 2.3,

$$\begin{aligned} {}_tP_x^{00}(s,t+h) &= {}_tP_x^{00}(s,t)\,{}_tP_x^{00}(t,t+h) + {}_tP_x^{01}(s,t)\,{}_tP_x^{10}(t,t+h) + {}_tP_x^{02}(s,t)\,{}_tP_x^{20}(t,t+h) \\ &= {}_tP_x^{00}(s,t)\left(1-(\mu_{x+t}^{01}+\mu_{x+t}^{02})h+0(h)\right) \end{aligned}$$

and applying definition of derivative as,

$$\frac{d}{dt}\,{}_tP_x^{00}(s,t) = \lim_{h\to 0}\frac{{}_tP_x^{00}(s,t+h) - {}_tP_x^{00}(s,t)}{h} \tag{2.16}$$

rearranging the above expression,

$$\frac{d}{dt}\,{}_tP_x^{00}(s,t) = -\left(\mu_{x+t}^{01}+\mu_{x+t}^{02}\right)\,{}_tP_x^{00}(s,t) \tag{2.17}$$

Again the Champoman Kolomogrov equation,

$$\begin{aligned} {}_tP_x^{01}(s,t+h) &= {}_tP_x^{00}(s,t)\,{}_tP_x^{01}(t,t+h) + {}_tP_x^{01}(s,t)\,{}_tP_x^{11}(t,t+h) \\ &= {}_tP_x^{00}(s,t)\left(\mu_{x+t}^{01}h+0(h)\right) + {}_tP_x^{01}(s,t)(1-\mu_{x+t}^{12}(h)+0(h), \end{aligned}$$

this yields that:

$$\frac{d}{dt}\,{}_tP_x^{01}(s,t) = \mu_{x+t}^{01}\,{}_tP_x^{00}(s,t) - \mu_{x+t}^{12}\,{}_tP_x^{01}(s,t). \tag{2.18}$$

and,

$${}_tP_x^{02}(s,t+h) = {}_tP_x^{00}(s,t)\,{}_tP_x^{02}(t,t+h) + {}_tP_x^{01}(s,t)\,{}_tP_x^{12}(t,t+h) + {}_tP_x^{02}(s,t)\,{}_tP_x^{22}(t,t+h) \tag{2.19}$$

rearranging the expression and using definition of derivatives,

$$\frac{d}{dt}\,{}_tP_x^{02}(s,t) = \mu_{x+t}^{02}\,{}_tP_x^{00}(s,t) + \mu_{x+t}^{12}\,{}_tP_x^{01}(s,t). \tag{2.20}$$

Therefore, the forward Kolmogorov Differential Equations can be written as,

$$\begin{bmatrix} \frac{d}{dt}\,{}_tP_x^{00}(s,t) \\ \frac{d}{dt}\,{}_tP_x^{01}(s,t) \\ \frac{d}{dt}\,{}_tP_x^{02}(s,t) \end{bmatrix}^T = \left({}_tP_x^{00}(s,t),{}_t P_x^{01}(s,t),{}_t P_x^{02}(s,t)\right)\ Q(t), \tag{2.21}$$

where $Q(t)$ is transition matrix

$$\mathbf{Q} = \begin{bmatrix} -(\mu_{x+t}^{01}+\mu_{x+t}^{02}) & \mu_{x+t}^{01} & \mu_{x+t}^{02} \\ 0 & -\mu_{x+t}^{12} & \mu_{x+t}^{12} \\ 0 & 0 & 0 \end{bmatrix}. \tag{2.22}$$

### 2.1.4.4 Permanent /no permanent model

This look like the the model described in Figure 2.3 but this model it is possible to move from state 2 to state 0 because an individual can be in sick but then recover. Also it can be possible to be translated from state 2 to state 1 (when the individual die) but it is impossible to return to 0 from 1.

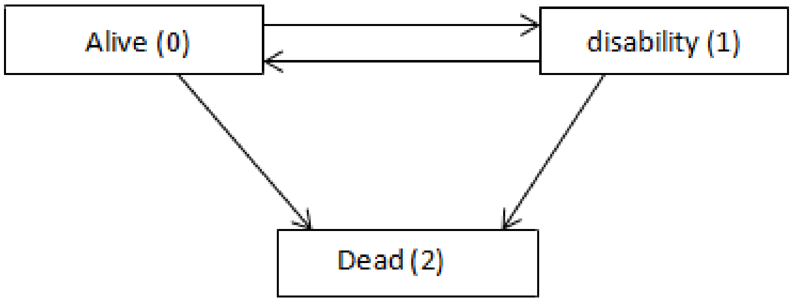



Figure 2.4: Permanent and no permanent disability model

According to Chapman Kolmogorov differential equation,

$$\begin{aligned} {}_tP_x^{00}(s,t+h) &= {}_tP_x^{00}(s,t)\,{}_tP_x^{00}(t,t+h) + {}_tP_x^{01}(s,t)\,{}_tP_x^{10}(t,t+h) + {}_tP_x^{02}(s,t)\,{}_tP_x^{20}(t,t+h) \\ &= {}_tP_x^{00}(s,t)\left(1-(\mu_{x+t}^{01}+\mu_{x+t}^{02})h+0(h)\right) + {}_tP_x^{01}(s,t)\left(\mu_{x+t}^{10}h+0(h)\right) \end{aligned}$$

After doing rearrangements, and using the definition of derivatives,

$$\frac{d}{dt}\,{}_tP_x^{00}(s,t) = -(\mu_{x+t}^{01}+\mu_{x+t}^{02})\,{}_tP_x^{00}(s,t) + \mu_{x+t}^{10}\,{}_tP_x^{01}(s,t). \tag{2.23}$$

Using (2.2),

$$\begin{aligned} {}_tP_x^{01}(s,t+h) &= {}_tP_x^{00}(s,t)\,{}_tP_x^{01}(t,t+h) + {}_tP_x^{01}(s,t)\,{}_tP_x^{11}(t,t+h) + {}_tP_x^{02}(s,t)\,{}_tP_x^{21}(t,t+h) \\ &= {}_tP_x^{00}(s,t)\left(\mu_{x+t}^{01}h+0(h)\right) + {}_tP_x^{01}(s,t)\left(1-(\mu_{x+t}^{10}+\mu_{x+t}^{12})h+0(h)\right) \end{aligned}$$

by arrangements and using definition of derivative,

$$\frac{d}{dt}\,{}_tP_x^{01}(s,t) = \mu_{x+t}^{01}\,{}_tP_x^{00}(s,t) - (\mu_{x+t}^{10}+\mu_{x+t}^{12})\,{}_tP_x^{01}(s,t) \tag{2.24}$$

and

$${}_tP_x^{02}(s,t+h) = {}_tP_x^{00}(s,t)\,{}_tP_x^{02}(t,t+h) + {}_tP_x^{01}(s,t){}_tP_x^{12}(t,t+h) + {}_tP_x^{02}(s,t)\,{}_tP_x^{22}(t,t+h), \tag{2.25}$$

it will become,

$$\frac{d}{dt}\,{}_tP_x^{02}(s,t) = \mu_{x+t}^{02}\,{}_tP_x^{00}(s,t) + \mu_{x+t}^{12}\,{}_tP_x^{01}(s,t). \tag{2.26}$$

Finally,

$$\begin{bmatrix} \frac{d}{dt}\,{}_tP_x^{00}(s,t) \\ \frac{d}{dt}\,{}_tP_x^{01}(s,t) \\ \frac{d}{dt}\,{}_tP_x^{02}(s,t) \end{bmatrix}^T = \left({}_tP_x^{00}(s,t)\;{}_tP_x^{01}(s,t)\;{}_tP_x^{02}(s,t)\right)\, Q(t) \tag{2.27}$$

with transition matrix

$$Q(t) = \begin{bmatrix} -(\mu_{x+t}^{01} + \mu_{x+t}^{02}) & \mu_{x+t}^{01} & \mu_{x+t}^{02} \\ \mu_{x+t}^{10} & -(\mu_{x+t}^{10} + \mu_{x+t}^{12}) & \mu_{x+t}^{12} \\ 0 & 0 & 0 \end{bmatrix}. \tag{2.28}$$

### 2.1.5 Discount rate

Discount rate in finance as well as in insurance policies has been defined by many authors. According to Makarov (2018), the following definition help us to understand deeply:

**Definition 2.4.** *Let $i_t$ be the interest rate in year $t$. Then*

$$v_t = \frac{1}{(1+i_t)}$$

*is discount rate in year $t$. The discount rate can be used to calculate the net present value (the present value of the future benefits). Let consider the interest intensity at time $t$ which is denoted by $\delta(t)$. A yearly interest rate $i$ yields*

$$e^{\delta} = 1 + i \text{ and thus } \delta = ln(1+i)$$

*In continuous time the discount rate from $t$ to $0$ is*

$$v^t = e^{-\delta t}$$

*where $\delta$ is known as the continuously compounded rate of interest or is the force of interest per year [Campolieti and Makarov, 2018].*

## 2.2 Empirical review

The price of insurance is the monetary value for which two parties agree to exchange risk and certainty. There are two commonly encountered situations in which the price of insurance is subject of consideration, when an individual agent (for example, a household), bearing an insurable risk, buys insurance from an insurer at an agreed periodic premium, and when insurance portfolios (that is, a collection of insurance contracts) are traded in the financial industry (e.g being transferred from an insurer to another insurer or from an insurer to the financial market) [Melnick and Everitt, 2008]. Pricing in the former situation is usually referred to as premium calculation while pricing in the latter situation is usually referred to as insurance pricing, although such a distinction is not unambiguous. The reader may verify that some of the methods discussed are more applicable to the former situation and other methods are more applicable to the latter situation [Melnick and Everitt, 2008]. .

Premium calculation principle introduced by Roger J.A in 2007 and by Jerusha in 2018, both talk about the factors that influence pricing of life insurance policies, where they consider that the influence of demographics on premium pricing, the age can affect how insurance products are priced. Jerusha examine the influences of socia-economic factors on premium pricing also determine the influence of regulatory framework on insurance premium pricing. Spellmam, Witt, and Rentz (1975) developed an insurance pricing method based on microeconomics theory in which investment income and the effect of the elasticity of demand are considered, and to determine the price [Abachi, 2018].

In 1992, Shalom Feldblum introduce pricing insurance policies, by using the internal rate of return model (IRR). Kliger and Levikson (1998) talk about pricing of short-term insurance contracts based on economic and probabilistic arguments. Their objective function in the maximization problem is an expectation of net profit, the loss resulting from insolvency. Price and the number of insurance policies are determined by optimizing an objective function.

Min Ming (2006) illustrated the applications of three commonly used pricing models like the risk-free model, CAPM and the R-L model to pricing insurance policies. Pricing insurance policies can be calculated by using the capital asset pricing model (CAPM) which require the information about the expected payoff and its co-movements with the market returns. Also the Rubinstein-Leland model can be used to price insurance contract [Wen, 2006].

In life insurance the are some basic types of policies for example Term insurance that can protect for a limited time (it contains limitation means that the money will be paid if and only if the death occurs during the term of policy, it means that if the death occurs between 0 and $n^{th}$ year).

Whole life insurance is simplest contract, in this insurance the premiums are paid throughout the client's life and the death benefit will be paid on or after death of the policyholder and the net single premium is denoted by $A_x = E\left[z\right] = E\left[V^{k+1}\right] = \sum_{i=0}^{\infty} \nu_k^{k+1} p_x q_{x+k}$ . Endowment insurance is paid at the end of year of death in which the insured died in. If this occurs within the first $n$ years, otherwise at the end of the $n^{th}$year; there exist also pure endowments of duration $n$ provides for payment of sum insured only if the insured is alive at the end of $n$ years where the net single premium is given by $A_{x:\overline{n}} = \nu_n^n\, p_x$ or consider the interest rate $r$ and that the amount of money paid out 1 unit [Molen, 2017]. The pricing in insurance is in the form of premium rates, and there are three main factors that determine the premium rates under a life insurance plan like mortality, expense and interest [Basaula, 2017]. The premium rates can be revised if there are any significant changes in any of these factors that we have mentioned. Pricing is a very important key element in the life insurance marketing strategy which hugely affects the final sale of the product. Price is the valuation placed upon the product by the life insurer. The management must take decisions regarding pricing (premium), investment return, premium level, premium mode, commission rate, insured sum amount, life covered, pricing strategy, under writing and price related contingencies. Mukherjee (2005) has mentioned about pricing product in Indian life insurance sector. Pricing of life insurance products plays a major role in marketing them. In his study, he explained extensively the pricing methodology of insurance companies. Low pricing attracts the customers whereas high pricing drives them away. Therefore, competitive pricing which takes care of the interests of both the insurer and the insured should be implemented. His study helps in understanding the seller's view point of the various elements associated with policy pricing which will give a chance to the insurers to make a better pricing policy. In the work [Dash, 2012], the impact of life insurance policy pricing on the customers buying behavior was found to be highly positive. The pricing strategies of Indian life insurers positively influence the customers decision to buy a life insurance policy.

# Chapter 3

# Methodology

This chapter introduce to the reader a clear view how the thesis was carried out. It starts by defining the thesis design, data sources and data analysis technique.

## 3.1 Pricing insurance policies using Markov chain model

In Rwandan, disabilities policies are gaining importance. The regular premiums are paid while the policy is in force. Describing the actuarial of disability in Rwanda and relatively to the actual problem such as pricing, to consider the multi-states as alive, disability and death is needed. The contribution in this study is to use Markov process for analysis of personal insurance in Rwanda using the Sanlam data and analysis disabilities relating to the insurance benefit

The research first introduced by Ragnar Norberg where the Morkov chain in life contingency have been proposed it knows that Markov process assumes the probabilities at any time depend only on the current state not on the past. In this study, Markov process will consider state space $\mathcal{Z} = \{\text{alive, disability, death}\}$ means that $\mathcal{Z} = \{0, 1, 2\}$ where "0" means alive, "1" means disability and "2" means death [Wen, 2006].

## 3.2 Premiums and Insurance Benefits

### 3.2.1 Insurance Benefits

Let $T_x$ be the future life time then the formula of insurance benefits can be derived where the policies are whole life insurance, term insurance and endowment insurance.

### 3.2.2 Whole Life Insurance

In Chapter 2 we defined discount rate so benefits for whole life insurance in continuous case

$$\begin{aligned}\overline{A}_x &= E[e^{-\delta T_x}] \\ &= \int_0^\infty {}_tP_x\,\mu_{x+t}\,e^{-\delta t}dt \\ &= \int_0^\infty {}_tP_x\,\mu_{x+t}v^t dt\end{aligned}$$

The second moment of the present value of the death benefits is

$$\begin{aligned}E[z^2] &= E[(e^{-\delta T_x})^2] \\ &= E[e^{-2\delta T_x}] \\ &= \int_0^\infty {}_tP_x\mu_{x+t}\,e^{-2\delta T_x}dt \\ &= {}^2\overline{A}_x\end{aligned}$$

where $Z$ represent the present value of benefit of 1 dollar.
The variance of the present value of a unit benefit payable immediately on death is

$$\begin{aligned}Var[Z] &= Var[e^{-\delta T_x}] \\ &= E[Z^2] - E[Z]^2 \\ &= {}^2\overline{A}_x - (\overline{A}_x)^2\end{aligned}$$

and whole life continuous annuity is represented by $\ddot{a}_x = \int_0^n {}_tP_x e^{-\delta t}dt$ if $\delta = 0$ then $\ddot{a}_x = e^0$.

### 3.2.3 Term Insurance

In a term insurance policy, benefit must be paid upon death of the policy holder before a fixed term of years. The present value of benefit of 1 dollar is

$$Z = \begin{cases} e^{-\delta T_x} \text{ if } T_x \leq n \\ 0 \text{ otherwise.} \end{cases} \tag{3.1}$$

The expected present value is given by

$$\overline{A}^{1}_{x:\overline{n|}} = \int_0^n {}_tP_x\, \mu_{x+t} e^{-\delta t} dt, \tag{3.2}$$

the expected value of the square of the present value is

$${}^2\overline{A}^{1}_{x:\overline{n|}} = \int_0^n {}_tP_x\, \mu_{x+t}\, e^{-2\delta t} dt \tag{3.3}$$

and the variance is

$$V[Z] =^2 \overline{A}^{1}_{x:\overline{n|}} - (\overline{A}^{1}_{x:\overline{n|}})^2. \tag{3.4}$$

The term continuous annuity is given by $\ddot{a}_{x:\overline{n|}} = \int_0^n {}_tP_x e^{-\delta t} dt$.

### 3.2.4 Endowment Insurance

This is combination of a term insurance pure endowment. The benefit at death will be paid at the end of the year in which the insured dies. The present value of the benefit is

$$Z = \begin{cases} e^{-\delta T_x} \text{ if } T_x < n \\ e^{-\delta n} \text{ if } T_x \geq n \end{cases}, \tag{3.5}$$

where $n$ is the present the years a person survives. The expected present value of the benefit is

$$\begin{aligned} E[Z] &= \int_0^n {}_tP_x\, \mu_{x+t} e^{-\delta t} dt + \int_n^\infty {}_tP_x\, \mu_{x+t} e^{-\delta n} dt \\ &= \int_0^n {}_tP_x\, \mu_{x+t} e^{-\delta t} dt + {}_nP_x e^{-\delta n} \\ &= \overline{A}^{1}_{x:\overline{n|}} + A^{1}_{x:\overline{n|}} \end{aligned}$$

The Actuarial notation of expected present value of benefit payment $(E(Z) = E(e^{-\delta T_x}))$ is $\overline{A}_{x:\overline{n}|}$, then

$$\overline{A}_{x:\overline{n}|} = \overline{A}^{1}_{x:\overline{n}|} + A_{x:\overline{n}|}^{\ \ 1} \tag{3.6}$$

## 3.3 Permanent Disability Model with Transition Probabilities

The $x$ in stochastic process represents $X(t)$, but let we consider it in this section as the age $x$ of a person at time 0 in state $i$ and after time $t$ he will be in state $j$; the transition probability is written as ${}_tP_x^{ij} = Pr[X_x(t) = j|X_x(0) = i]$. The transition probability matrix is given by

$$\begin{bmatrix} p^{00} & p^{01} & p^{02} & \cdots & p^{0k} \\ p^{10} & p^{11} & p^{12} & \cdots & p^{1k} \\ p^{20} & p^{21} & p^{22} & \cdots & p^{2k} \\ p^{k0} & p^{k1} & p^{k2} & \cdots & p^{kk} \end{bmatrix},$$

for a model with $k$ states, each row of this matrix must sum to 1 for $i = 0, 1, 2, \cdots, k$ and when $i = k$, since state $k$ is an absorbing state for all $t \geq 0$ ${}_tp_x^{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 \text{ otherwise} \end{cases}$

When $t = 0$, we have ${}_0p_x^{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$. When $t = 1$, it is the one-year transition probability defined as

$$p_x^{ij} = \begin{cases} q_x^i \text{ for j=i} \\ (1 - q_x^i)p_x^{ij}. \end{cases}$$

When $t > 0$, the t-years transition probability ${}_tp_x^{ij}$, can be calculated by using chapman Kolmogorov equation.

Now the concepts of transition intensity or force of transition for the case of continuous time process are introduce. The transition intensity describes the instantaneous rate at which the Continuous time Markov Chain transits between states, it may be define as:

**Definition 3.1.** *The force of mortality of transition $i \to j$ at time $x$ or transition intensity is*

$$\mu_{x+t}^{ij} = \frac{d \; {}_tP_x^{ij}}{dt}|_{t=0} = \lim_{t \to 0^+} \frac{{}_tP_x^{ij}}{t} \quad \text{for } i \neq j, \tag{3.7}$$

*and if it is not possible to move from state $i$ to state $j$ at any time, it means that the transition intensity is zero ($\mu_{x+t}^{ij} = 0$).*

Using this definition in the permanent disability model in Figure 2.3, it will be impossible to transit from disability to alive. Consider the probability of $x$ year old dying at age $x + 1$ $q_x$. In continuous case, calculating the probability of person who has attained the age $x$ dying between $x$ and $x + t$ using conditional probability is as follows:

$$p_x(t) = p(x < X < x + t | X > x) = \frac{F_X(x + t) - F_X(x)}{1 - F_X(x)}, \tag{3.8}$$

where $F_X(x)$ represent the cumulative distribution function of age $x$ at death for random variable $X$. Considering that $t \to 0$ and taking into account that probability is in the continuous case. force of mortality may approximated by taking this probability divided by $t$. Thus the force of mortality:

$$\mu_{x+t} = \lim_{t \to 0^+} \frac{F_X(x + t) - F_X(x)}{t(1 - F_X(x + t))} = \frac{F_X'(x)}{1 - F_X(x)} = \frac{\frac{d}{dt} F_X(t)}{S_x(t)}, \tag{3.9}$$

where $S_x(t)$ is the survival function and using the actuarial notation for survival and mortality probabilities is

$${}_tp_x = p_r \left[ X_x > t \right] = S_x(t), \tag{3.10}$$

$${}_tq_x = p_r \left[ X_x \leq t \right] = 1 - S_x(t) = F_X(t), \tag{3.11}$$

force of mortality becomes

$$\mu_{x+t} = \frac{\frac{d}{dt} \, {}_tq_x}{{}_tp_x}. \tag{3.12}$$

Using the assumption of linearity of ${}_tq_x$, assuming that ${}_tq_x$ is linear function of $t$ and interpolation between 0 and 1 yields ${}_tq_x = tq_x$, and it is known that ${}_tq_x +_t p_x = 1$.

Finally, force of mortality expression for multiple state model can be calculated as

$$\mu_{x+t}^{ij} = \frac{q_x^{ij}}{1 - tq_x^{ij}}. \tag{3.13}$$

As the main objective of this study, the continuous-time Markov chain is needed to price insurance policies. Since the transition intensities can be used to write down how the transitions probabilities will behave over time. Using the the equation (2.2) and insert in the definition transition probability. For transition probability ${}_{(t,u)}P_x^{ij}$, let divide the interval $(t, u)$ into two: $(t, t + h)$ and $[t + h, u)$. Such that in the first interval, there are two possibilities: For the probability

$$_{(t,t+h)}P_x^{\overline{ii}} = 1 - \mu_{x+t}^{ii} h + 0(h) \tag{3.14}$$

the process $X$ remains in state $i$ after it goes to state $j$ by probability ${}_{(t+h,u)}P_x^{ij}$. Secondly, $X$ can switch to any state $k$ with probability $\mu_{x+t}^{ik} h + 0(h)$, after the probability of ending up to state $l$ given by ${}_{(t+h,u)}P_x^{kl}$. Hence by combining the above statements,

$$_{(t,u)}P_x^{ik} = (1 - \mu_{x+t}^{ii})\ h\ {}_{(t+h,u)}P_x^{ik} + \sum_{l \neq i} \mu_{x+t}^{il}\ h\ {}_{(t+h,u)}P_x^{lk} + 0(h). \tag{3.15}$$

By definition of derivative, it is known that

$$\frac{d}{dt}\ {}_{(t,u)}P_x^{ik} = \lim_{h \to 0} \frac{{}_{(t+h,u)}P_x^{ik} - {}_{(t,u)}P_x^{ik}}{h}, \tag{3.16}$$

and the equation (3.15), it may be written

$$\lim_{h \to 0} \frac{{}_{(t+h,u)}P_x^{ik} - {}_{(t,u)}\ P_x^{ik}}{h} = \mu_{x+t}^{ii}\ {}_{(t+h,u)}P_x^{ik} - \sum_{l \neq i} \mu_{x+t}^{il}\ {}_{(t+h,u)}P_x^{lk} + \lim_{h \to 0} \frac{0(h)}{h}, \tag{3.17}$$

this yields,

$$\frac{d}{dt}\ {}_{(t,u)}P_x^{ik} = \mu_{x+t}^{ii}\ {}_{(t,u)}P_x^{ik} - \sum_{l \neq i} \mu_{x+t}^{il}\ {}_{(t,u)}P_x^{lk}. \tag{3.18}$$

called the *Kolmogorov backward differential equations* already given in Basic life insurance Mathematics [Norberg, 2000] .Let make a distinction between ${}_{(t,u)}P_x^{ii}$ and staying in state $i$ the entire time. When a person currently age $x$ and is currently in state $i$, the probability that a person continuously remains in the same state for the period $t$ is called *an occupancy probability*, and this probability can be computed by using

$$ {}_tP_x^{\overline{ii}} = \exp\left( -\sum_{j=0,j\neq i}^{n} \int_0^t \mu_{x+s}^{ij} ds \right). \tag{3.19} $$

*Proof.* Now

$$ {}_hP_x^{\overline{ii}} = 1 - \sum_{j\neq i} {}_hP_x^{ij}, \tag{3.20} $$

and by definition ${}_hP_x^{ij} = h\mu_x^{ij} + 0(h)$. Comparing those two similar expressions,

$$ {}_hP_x^{ii} = 1 - \sum_{j\neq i} \left(h\mu_x^{ij} + 0(h)\right) = {}_hP_x^{\overline{ii}} = 1 - \sum_{j\neq i} h\mu_x^{ij} + \sum_{j\neq i} 0(h). \tag{3.21} $$

and

$$ {}_hP_x^{\overline{ii}} = 1 - h\sum_{j\neq i} \mu_x^{ij}. \tag{3.22} $$

By using the derivative formula,

$$ \begin{aligned} \frac{d}{dt}\ {}_tP_x^{\overline{ii}} &= \lim_{h\to 0} \frac{{}_{t+h}P_x^{\overline{ii}} - {}_tP_x^{\overline{ii}}}{h} \\ &= \lim_{h\to 0} \frac{{}_tP_x^{\overline{ii}}({}_hP_{x+t}^{\overline{ii}} - 1)}{h} \\ &= {}_tP_x^{\overline{ii}} \lim_{h\to 0} \frac{-h\int_{j\neq i} \mu_{x+t}^{ij} + 0(h)}{h} \\ &= -\ {}_tP_x^{\overline{ii}} \lim_{h\to 0} \frac{h\int_{j\neq i} \mu_{x+t}^{ij} + 0(h)}{h} \\ &= -\ {}_tP_x^{\overline{ii}} \int_{j\neq i} \mu_{x+t}^{ij} dt. \end{aligned} $$

Simply, the above equation can be written in this way,

$$ \frac{\frac{d}{dt}\ {}_tP_x^{\overline{ii}}}{{}_tP_x^{\overline{ii}}} = -\sum_{j\neq i} \mu_{x+t}^{ij}, \tag{3.23} $$

integrate both side over interval $[0, t]$

$$ \int_0^t \frac{\frac{d}{dt}\ {}_tP_x^{\overline{ii}}}{{}_tP_x^{\overline{ii}}} = -\int_0^t \sum_{j\neq i} \mu_{x+s}^{ij} ds. \tag{3.24} $$

Hence

$$ {}_tP_x^{\overline{ii}} = \exp\left(-\int_0^t \sum_{j\neq i} \mu_{x+s}^{ij} ds\right) \tag{3.25} $$

which is called *differential equations for transition probabilities.* □

Let consider the states $i$ and state $j$, using the Kolmogorov's Forward Equations, now the transition probability is given by the following equation [Esquível et al., 2021] :

$$ \frac{d}{dt}\,{}_tp_x^{ij} = \sum_{k\neq j}\, {}_tp_x^{ik}\mu_{x+t}^{kj} - {}_tp_x^{ij}\mu_{x+t}^{jk}, \tag{3.26} $$

with this condition $k \neq j$. For the case that $k = j$, $\sum_{k=j} \mu_{x+t}^{jk} = 0$ gives zero. This can be shown from equation (3.26) and by taking into account, the properties transition matrix and generator matrix,

$$ \begin{aligned} \sum_k \mu_{x+t}^{jk} &= \frac{d}{dt}\sum_k\, {}_tp_x^{jk} \\ &= \frac{d}{dt}1 \\ &= 0. \end{aligned} $$

The differential equations for transition probabilities according to the model in Figure 2.3 based on the Chapman Kolmogolov Equations are:

$$ \begin{aligned} \frac{d}{dt}\,{}_tp_x^{00} &= -\,{}_tp_x^{00}\mu_{x+t}^{00} \\ \frac{d}{dt}\,{}_tp_x^{01} &= {}_tp_x^{00}\mu_{x+t}^{01} - {}_tp_x^{01}\mu_{x+t}^{11} \\ \frac{d}{dt}\,{}_tp_x^{10} &= 0 \\ \frac{d}{dt}\,{}_tp_x^{02} &= {}_tp_x^{00}\mu_{x+t}^{02} - {}_tp_x^{02}\mu_{x+t}^{22} \\ \frac{d}{dt}\,{}_tp_x^{11} &= -\,{}_tp_x^{11}\mu_{x+t}^{11} \\ \frac{d}{dt}\,{}_tp_x^{12} &= {}_tp_x^{11}\mu_{x+t}^{12} - {}_tp_x^{12}\mu_{x+t}^{22} \\ \frac{d}{dt}\,{}_tp_x^{21} &= 0 \\ \frac{d}{dt}\,{}_tp_x^{22} &= -\,{}_tp_x^{22}\mu_{x+t}^{22} \end{aligned} $$

Also the Kolmogorov's Forward differential equations can be written in the form of matrix as

$$\frac{d}{dt}\,{}_tP_x = \begin{bmatrix} {}_tp_x^{00} & {}_tp_x^{01} & {}_tp_x^{02} \\ 0 & {}_tp_x^{11} & {}_tp_x^{12} \\ 0 & 0 & {}_tp_x^{22} \end{bmatrix} \begin{bmatrix} -\mu_{x+t}^{00} & \mu_{x+t}^{01} & \mu_{x+t}^{02} \\ 0 & -\mu_{x+t}^{11} & \mu_{x+t}^{12} \\ 0 & 0 & -\mu_{x+t}^{22} \end{bmatrix}$$

$$\frac{d}{dt}\,{}_tP_x = {}_tP_xQ_x \tag{3.27}$$

where $Q_x$ represent the generator of Markov process and so the Kolmogorov's Backward equations is

$$\frac{d}{dt}\,{}_tP_x = -Q_x\,{}_tP_x. \tag{3.28}$$

The following transition probability notations are defined as fallows: ${}_tp_x^{00}$ : the probability of individual age $x$ currently in alive state (0), at age $x+t$ still alive state in (0).

${}_tp_x^{01}$ : the probability of individual age $x$ currently in alive state (0), at age $x+t$ move to disability state (1).

${}_tp_x^{02}$ : the probability of individual age $x$ currently in alive state (0), at age $x+t$ move to death state (2).

${}_tp_x^{11}$ : the probability of individual age $x$ currently in disability state (1), at age $x+t$ still in the same state (1).

${}_tp_x^{12}$ : the probability of individual age $x$ currently in disability state (1), at age $x+t$ move to death state (2) still in the same state (2).

## 3.4 Pricing Permanent Disabilities Insurance using Multi-State Model

Suppose a person aged $x$ currently in state $i$ . Let $\ddot{a}_x^{ij}$ denote the expected present value of the annuity that pays continuously 1 dollar per year while the life is in some state $j$ (which may be equal to $i$). Then

$$\ddot{a}_x^{ij} = \int_0^\infty {}_tP_x^{ij} e^{-\delta t} dt \tag{3.29}$$

For insurance benefits, the payment is usually conditional on making a transition. So, suppose a unit benefit is payable immediately on each future transfer into state $j$, given that the life is currently in state $i$ (which may be equal to $j$. Then the expected present value of the benefit is

$$\overline{A}_x^{ij} = \int_0^{\infty} (\sum_{k \neq j} {}_tP_x^{ik} \mu_{x+t}^{kj}) e^{-\delta t} dt. \tag{3.30}$$

The premium is given by

$$P = B \frac{\overline{A}_x}{\ddot{a}_x} \tag{3.31}$$

where $B$ is the benefit insured payable on death; so the premium payment is calculated based on the present value. In this permanent disability, the net single premium will be used, so that the premium payment is made once and the additional costs for policy making are not included in the calculation. In general, the calculation of premium is based on three things, death probability, interest rate and cost of policy [de, 2014].

# Chapter 4

# Data Analysis

In this chapter, the data collected from Sanlam will be analysed. To price insurance policies that are contingent using continuous Markov chain model for finite state $\mathcal{S}$ such that the states are mutually exclusive, assume that state policy at time $t$ in period of insurance coverage and the process $X(t)$ be a Markov process, with transition probabilities satisfy the Chapman-Kolmogrov equation (2.2) and (3.26).

## 4.1 Data description

The data have been used in this research are secondary data obtained from Sanlam. They are the insurance claims from January 2010 up to December 2018, thus constituting 123 observations with the age of clients, the period(time) of insurance.

## 4.2 Computation of probabilities for the three state under investigation

To express the probabilities in terms of transition intensity is most important thing. It determines that the transition intensities $\{\mu_x^{ij}, x \geq 0 \text{ for } i, j = 0, \cdots, n, \text{with } i \neq j\}$ are fundamental quantities which tell every thing that is needed to know about the multistate model.

In the multi-state model, the following notation is used for transition probabilities: ${}_tP_x^{ij} = P\{X(x+t) = j/X(x) = i\}$ which is the probability to transit from state $i$ at time $x$ to state $j$. Let derive that ${}_tP_x^{\overline{ii}} = 1$ is always 1. Writing the formula for ${}_tP_x^{\overline{ii}}$ in terms of transition intensities out of state $i, \mu_x^{ij}$ in multi-state model of any state $i$ ,

$$ {}_tP_x^{\overline{ii}} = \exp\left(-\int_0^t \sum_{j=0,\ j\neq i}^{n} \mu_{x+s} ds\right), \tag{4.1} $$

by considering any $h > 0$, calculation of (4.1) at ${}_{t+h}P_x^{\overline{ii}}$, is needed where the probability of an individual stay in state $i$ through the time period $[0, t+h]$; given that the probability was in state $i$ ate age $x$, use the condition of probabilities; the probabilities of ${}_tP_x^{\overline{ii}}$ and ${}_hP_{x+t}^{\overline{ii}}$ that can help in computing

$$ {}_{t+h}P_x^{\overline{ii}} = {}_tP_x^{\overline{ii}}\ {}_hP_{x+t}^{\overline{ii}} \tag{4.2} $$

This equation (4.2) can be expressed as

$$ {}_{t+h}P_x^{\overline{ii}} = {}_tP_x^{\overline{ii}}\left(1 - h\sum_{j=0,\ j\neq i}^{n} \mu_{x+t}^{ij} + 0/h\right). $$

From which

$$ \frac{{}_{t+h}P_x^{\overline{ii}} -_t P_x^{\overline{ii}}}{h} = -{}_tP_x^{\overline{ii}} \sum_{j=0,\ j\neq i}^{n} \mu_{x+t}^{ij} + 0/h $$

taking $h \to 0$,

$$ \frac{d}{dt}\,{}_tP_x^{\overline{ii}} = -{}_tP_x^{\overline{ii}} \sum_{j=0,\ j\neq i}^{n} \mu_{x+t}^{ij} $$

Solving this ODE at interval $[0, t]$

$$ log({}_tP_x^{\overline{ii}}) - log({}_0P_x^{ii}) = -\int_0^t \sum_{j=0,\ j\neq i}^{n} \mu_{x+s}^{ij} ds. \tag{4.3} $$

Finally,

$$ {}_tP_x^{\overline{ii}} = {}_0P_x^{ii}\ \exp\left(-\int_0^t \sum_{j=0,\ j\neq i}^{n} \mu_{x+s}^{ij} ds\right). \tag{4.4} $$

Therefore, it claims that ${}_0P_x^{\overline{ii}} = 1$. Using (4.1), the proposed model of permanent disability is applied, then

$$\begin{aligned} {}_tP_x^{\overline{00}} &= \exp\left(-\int_0^t \mu_{x+s}^{01} + \mu_{x+s}^{02} ds\right) \\ &= \exp\left(-[\mu_{x+s}^{01} + \mu_{x+s}^{02}][s]_0^t\right) \\ &= \exp\left(-(\mu_{x+s}^{01} + \mu_{x+s}^{02})t\right), \end{aligned}$$

the properties refer to generator matrix said that: $\mu_{x+s}^{00} + \mu_{x+s}^{01} + \mu_{x+s}^{02} = 0$ and

$$ {}_tP_x^{\overline{00}} = \exp(\mu_{x+t}^{00} t). \tag{4.5}$$

So ${}_tP_x^{\overline{ii}}$ probability that individual does not leave state $i$ between age $x$ and $x+t$; and ${}_tP_x^{ii}$ is the probability that an individual is in state $i$ at age $x+t$, in both cases that the individual was in state $i$. Hence it is important to note that ${}_tP_x^{ii}$ include the possibility for a person leaving state $i$ between $x$ and $x+t$ provided that it is back in state $i$ at age $x+t$.

This shows that ${}_tP_x^{\overline{00}} = {}_tP_x^{00}$ and ${}_tP_x^{\overline{11}} = {}_tP_x^{11}$. Similar case using (4.1),

$$\begin{aligned} {}_tP_x^{11} &= \exp(-\int_0^t \mu_{x+s}^{12} ds) \\ {}_tP_x^{11} &= \exp(\mu_{x+s}^{11} t). \end{aligned} \tag{4.6}$$

Using the expressions of ${}_tP_x^{00}$ and ${}_tP_x^{11}$, derive ${}_tP_x^{01}$. To derive the probability ${}_tP_x^{01}$, equation (3.26) have been used.

$$\frac{d}{dt}({}_tP_x^{ij}) = \sum_{k=0,k\neq j}^{n} ({}_tP_x^{ik}\mu_{x+t}^{kj} - {}_tP_x^{ij}\mu_{x+t}^{jk}),$$

obtain the following ordinary differential equations

$$\begin{aligned} \frac{d}{dt}({}_tP_x^{01}) &= {}_tP_x^{00}\mu_{x+t}^{01} - {}_tP_x^{01}\mu_{x+t}^{12} \\ \frac{d}{dt}({}_tP_x^{01}) &= {}_tP_x^{00}\mu_{x+t}^{01} + {}_tP_x^{01}\mu_{x+t}^{11} \end{aligned}$$

Which is rewriten as

$$\frac{d}{dt}({}_tP_x^{01}) - {}_tP_x^{01}\mu_{x+t}^{11} = {}_tP_x^{00}\mu_{x+t}^{01} \tag{4.7}$$

The solution to the homogeneous part

$$\frac{d}{dt}({}_tP_x^{01}) = {}_tP_x^{01}\mu_{x+t}^{11}$$

can declared, it has the form

$${}_tP_x^{01} = A(t)\ \exp\left\{\mu_{x+t}^{11}t\right\}$$

and by doing the substitution in (4.7),

$$\begin{aligned} A'(t)\ \exp\left\{(\mu_{x+t}^{11})t\right\} &= \exp\left\{(\mu_{x+t}^{00})t\right\}\mu_{x+t}^{01} \\ A'(t) &= \exp\left\{(\mu_{x+t}^{00} - \mu_{x+t}^{11})t\right\}\mu_{x+t}^{01}. \end{aligned}$$

Therefore,

$$A(t) = \frac{\mu_{x+t}^{01}}{\mu_{x+t}^{00} - \mu_{x+t}^{11}}\ \exp\left\{(\mu_{x+t}^{00} - \mu_{x+t}^{11})t\right\} + K. \tag{4.8}$$

Using initial condition, $t = 0,\ {}_0P_{x+t}^{01} = 0,\ A(0) = 0$, So

$$K = \frac{-\mu_{x+t}^{01}}{\mu_{x+t}^{00} - \mu_{x+t}^{11}}$$

Therefore the solution of (4.7) is

$$\begin{aligned} {}_tP_x^{01} &= A(t)\ \exp\left\{(\mu_{x+t}^{11})t\right\} \\ &= \left(\frac{\mu_{x+t}^{01}}{\mu_{x+t}^{00} - \mu_{x+t}^{11}}\ \exp\left\{(\mu_{x+t}^{00} - \mu_{x+t}^{11})t\right\} - \frac{\mu_{x+t}^{01}}{\mu_{x+t}^{00} - \mu_{x+t}^{11}}\right)\ \exp\left\{(\mu_{x+t}^{11})t\right\} \end{aligned}$$

$${}_tP_x^{01} = \frac{\mu_{x+t}^{01}}{\mu_{x+t}^{00} - \mu_{x+t}^{11}}\left(\ \exp\left\{(\mu_{x+t}^{00} - \mu_{x+t}^{11})t\right\} - 1\right)\ \exp\left\{(\mu_{x+t}^{11})t\right\} \tag{4.9}$$

and by using the properties of transition matrix, now

$${}_tP_x^{02} = 1 - {}_tP_x^{00} - {}_tP_x^{01}. \tag{4.10}$$

Using The model Figure 2.3, the transition probabilities becomes

$${}_tP_x^{ij} = \begin{bmatrix} {}_tp_x^{00} & {}_tp_x^{01} & {}_tp_x^{02} \\ {}_tp_x^{10} & {}_tp_x^{11} & {}_tp_x^{12} \\ {}_tp_x^{20} & {}_tp_x^{21} & {}_tp_x^{22} \end{bmatrix}$$

and by considering state 2 as absorbing state, then the transition probabilities yield

$$ {}_tP_x^{ij} = \begin{bmatrix} {}_tp_x^{00} & {}_tp_x^{01} & {}_tp_x^{02} \\ 0 & {}_tp_x^{11} & {}_tp_x^{12} \\ 0 & 0 & 1 \end{bmatrix}. $$

The generator matrix for the model of Figure 2.3 [Jones, 1994],

$$ Q_x = \begin{bmatrix} \mu_{x+t}^{00} & \mu_{x+t}^{01} & \mu_{x+t}^{02} \\ 0 & \mu_{x+t}^{11} & \mu_{x+t}^{12} \\ 0 & 0 & 0 \end{bmatrix}. $$

The report of transition intensities and transition probabilities from the data in the table 4.1 and by considering the results, the transition probabilities for both remaining in alive and disabled state appear higher than to move from one state to another.
If the age is increased, transition probability of disability to death also will increase and its transition intensity decrease. The transition probability of remaining in disability state decrease in the case the age increase, and its transition intensity increase. The transition probability of moving from alive to death increase in the case of increasing the age and its transition intensity increase. An x years old person buys a permanent disability specifying that premium is paid continuously at fixed rate per time unit in active state ,the policy terminates years after issue, assume that the interest rate is constant. The relevant results are given in Table 4.1 shows the values of transition intensities and probabilities, where transition intensity $\mu_{x+t}^{02} = \frac{q_x}{1-t\, q_x}$ as is refer to equation (3.13) and to find the transition probability, apply chapman-kolmogorov equation ${}_tp_x^{00} = \exp(\mu_{x+t}^{00}\, t)$, ${}_tp_x^{11} = \exp(\mu_{x+t}^{11}\, t)$ but ${}_tp_x^{01}$ is given by the equation (4.9) the others probabilities are given by the properties of transition matrix which says that $\displaystyle\sum_{j=0}^{k} {}_tp_x^{ij} = 1$ and the remains transition intensities are given by the property of generating matrix says that $\displaystyle\sum_{j=0}^{k} \mu_{x+t}^{ij} = 0$. The model of permanent disability shown in Figure 2.3, the transition from state 1 to state 0 is impossible that why ${}_tp_x^{10}, {}_tp_x^{20}, {}_tp_x^{21}$ are equal to zero , this is defined in section2.4.3.

Table 4.1: ${}_tP_x^{ij}$ the probabilities and $\mu_{x+t}^{ij}$transitions intensities

| $\mu_{x+t}^{00}$ | $\mu_{x+t}^{01}$ | $\mu_{x+t}^{02}$ | $\mu_{x+t}^{11}$ | $\mu_{x+t}^{12}$ | ${}_tp_x^{00}$ | ${}_tp_x^{01}$ | ${}_tp_x^{02}$ | ${}_tp_x^{11}$ | ${}_tp_x^{12}$ |
|---|---|---|---|---|---|---|---|---|---|
| -0.0062 | 0.0026 | 0.0036 | -0.0037 | 0.0037 | 0.8815 | 0.0474 | 0.0711 | 0.9281 | 0.0719 |
| -0.0072 | 0.0040 | 0.0031 | -0.0011 | 0.0011 | 0.8528 | 0.0818 | 0.0654 | 0.9755 | 0.0245 |
| -0.0057 | 0.0034 | 0.0023 | -0.0014 | 0.0014 | 0.8614 | 0.0807 | 0.0579 | 0.9647 | 0.0353 |
| -0.0023 | 0.0010 | 0.0013 | -0.0022 | 0.0022 | 0.9237 | 0.0318 | 0.0445 | 0.9258 | 0.0742 |
| -0.0041 | 0.0012 | 0.0029 | -0.0015 | 0.0015 | 0.9101 | 0.0254 | 0.0645 | 0.9667 | 0.0333 |
| -0.0030 | 0.0012 | 0.0018 | -0.0021 | 0.0021 | 0.9133 | 0.0346 | 0.0520 | 0.9395 | 0.0605 |
| -0.0087 | 0.0065 | 0.0022 | -0.0024 | 0.0024 | 0.7922 | 0.1509 | 0.0569 | 0.9385 | 0.0615 |
| -0.0059 | 0.0032 | 0.0027 | -0.0011 | 0.0011 | 0.8686 | 0.0700 | 0.0614 | 0.9747 | 0.0253 |
| -0.0115 | 0.0066 | 0.0048 | -0.0049 | 0.0049 | 0.8350 | 0.0917 | 0.0733 | 0.9263 | 0.0737 |
| -0.0035 | 0.0025 | 0.0010 | -0.0036 | 0.0036 | 0.8667 | 0.0891 | 0.0441 | 0.8623 | 0.1377 |
| -0.0047 | 0.0027 | 0.0020 | -0.0025 | 0.0025 | 0.8754 | 0.0692 | 0.0554 | 0.9314 | 0.0686 |
| -0.0025 | 0.0012 | 0.0013 | -0.0024 | 0.0024 | 0.9171 | 0.0379 | 0.0449 | 0.9201 | 0.0799 |
| -0.0036 | 0.0024 | 0.0013 | -0.0039 | 0.0039 | 0.8807 | 0.0722 | 0.0471 | 0.8727 | 0.1273 |
| -0.0044 | 0.0024 | 0.0020 | -0.0011 | 0.0011 | 0.8833 | 0.0619 | 0.0548 | 0.9689 | 0.0311 |
| -0.0023 | 0.0011 | 0.0012 | -0.0022 | 0.0022 | 0.9201 | 0.0376 | 0.0423 | 0.9227 | 0.0773 |
| -0.0106 | 0.0027 | 0.0078 | -0.0008 | 0.0008 | 0.9127 | 0.0225 | 0.0648 | 0.9927 | 0.0073 |
| -0.0108 | 0.0019 | 0.0088 | -0.0029 | 0.0029 | 0.9310 | 0.0123 | 0.0568 | 0.9808 | 0.0192 |
| -0.0085 | 0.0016 | 0.0069 | -0.0052 | 0.0052 | 0.9140 | 0.0155 | 0.0705 | 0.9462 | 0.0538 |
| -0.0038 | 0.0024 | 0.0014 | -0.0024 | 0.0024 | 0.8808 | 0.0711 | 0.0481 | 0.9245 | 0.0755 |
| -0.0045 | 0.0009 | 0.0036 | -0.0019 | 0.0019 | 0.9140 | 0.0170 | 0.0690 | 0.9629 | 0.0371 |
| -0.0053 | 0.0036 | 0.0016 | -0.0015 | 0.0015 | 0.8508 | 0.1005 | 0.0487 | 0.9561 | 0.0439 |
| -0.0107 | 0.0082 | 0.0025 | -0.0016 | 0.0016 | 0.7688 | 0.1734 | 0.0577 | 0.9621 | 0.0379 |
| -0.0030 | 0.0015 | 0.0015 | -0.0016 | 0.0016 | 0.9078 | 0.0440 | 0.0482 | 0.9507 | 0.0493 |
| -0.0026 | 0.0013 | 0.0013 | -0.0018 | 0.0018 | 0.9133 | 0.0423 | 0.0445 | 0.9386 | 0.0614 |
| -0.0056 | 0.0014 | 0.0042 | -0.0019 | 0.0019 | 0.9051 | 0.0231 | 0.0718 | 0.9661 | 0.0339 |
| -0.0020 | 0.0009 | 0.0011 | -0.0019 | 0.0019 | 0.9268 | 0.0313 | 0.0419 | 0.9289 | 0.0711 |
| -0.0057 | 0.0012 | 0.0045 | -0.0024 | 0.0024 | 0.9090 | 0.0192 | 0.0718 | 0.9613 | 0.0387 |
| -0.0022 | 0.0012 | 0.0009 | -0.0024 | 0.0024 | 0.9116 | 0.0480 | 0.0403 | 0.9024 | 0.0976 |
| -0.0050 | 0.0017 | 0.0034 | -0.0024 | 0.0024 | 0.9014 | 0.0319 | 0.0668 | 0.9518 | 0.0482 |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| -0.0088 | 0.0010 | 0.0078 | -0.0018 | 0.0018 | 0.9266 | 0.0081 | 0.0653 | 0.9846 | 0.0154 |
| -0.0029 | 0.0019 | 0.0009 | -0.0018 | 0.0018 | 0.8876 | 0.0724 | 0.0400 | 0.9277 | 0.0723 |
| -0.0062 | 0.0010 | 0.0052 | -0.0053 | 0.0053 | 0.9135 | 0.0132 | 0.0733 | 0.9254 | 0.0746 |
| -0.0047 | 0.0034 | 0.0014 | -0.0015 | 0.0015 | 0.8526 | 0.1025 | 0.0449 | 0.9516 | 0.0484 |
| -0.0031 | 0.0010 | 0.0022 | -0.0019 | 0.0019 | 0.9196 | 0.0244 | 0.0559 | 0.9500 | 0.0500 |
| -0.0074 | 0.0010 | 0.0064 | -0.0019 | 0.0019 | 0.9173 | 0.0108 | 0.0719 | 0.9778 | 0.0222 |
| -0.0123 | 0.0113 | 0.0009 | -0.0019 | 0.0019 | 0.5993 | 0.3541 | 0.0466 | 0.9229 | 0.0771 |
| -0.0023 | 0.0012 | 0.0011 | -0.0019 | 0.0019 | 0.9136 | 0.0447 | 0.0417 | 0.9268 | 0.0732 |
| -0.0034 | 0.0009 | 0.0025 | -0.0009 | 0.0009 | 0.9193 | 0.0214 | 0.0593 | 0.9777 | 0.0223 |
| -0.0023 | 0.0009 | 0.0014 | -0.0009 | 0.0009 | 0.9259 | 0.0293 | 0.0448 | 0.9695 | 0.0305 |
| -0.0032 | 0.0021 | 0.0012 | -0.0009 | 0.0009 | 0.8889 | 0.0698 | 0.0413 | 0.9670 | 0.0330 |
| -0.0053 | 0.0042 | 0.0010 | -0.0009 | 0.0009 | 0.8120 | 0.1488 | 0.0392 | 0.9643 | 0.0357 |
| -0.0032 | 0.0019 | 0.0013 | -0.0012 | 0.0012 | 0.8952 | 0.0614 | 0.0434 | 0.9583 | 0.0417 |
| -0.0030 | 0.0019 | 0.0011 | -0.0012 | 0.0012 | 0.8916 | 0.0680 | 0.0403 | 0.9536 | 0.0464 |
| -0.0047 | 0.0011 | 0.0036 | -0.0012 | 0.0012 | 0.9121 | 0.0198 | 0.0682 | 0.9772 | 0.0228 |
| -0.0024 | 0.0009 | 0.0015 | -0.0017 | 0.0017 | 0.9247 | 0.0268 | 0.0485 | 0.9475 | 0.0525 |
| -0.0052 | 0.0037 | 0.0015 | -0.0017 | 0.0017 | 0.8452 | 0.1061 | 0.0487 | 0.9475 | 0.0525 |
| -0.0068 | 0.0052 | 0.0015 | -0.0017 | 0.0017 | 0.8030 | 0.1481 | 0.0488 | 0.9472 | 0.0528 |
| -0.0025 | 0.0015 | 0.0011 | -0.0017 | 0.0017 | 0.9053 | 0.0532 | 0.0415 | 0.9360 | 0.0640 |
| -0.0038 | 0.0027 | 0.0011 | -0.0019 | 0.0019 | 0.8616 | 0.0959 | 0.0425 | 0.9271 | 0.0729 |
| -0.0048 | 0.0027 | 0.0020 | -0.0019 | 0.0019 | 0.8747 | 0.0702 | 0.0551 | 0.9471 | 0.0529 |
| -0.0047 | 0.0027 | 0.0020 | -0.0027 | 0.0027 | 0.8749 | 0.0692 | 0.0559 | 0.9258 | 0.0742 |
| -0.0039 | 0.0019 | 0.0020 | -0.0027 | 0.0027 | 0.8953 | 0.0490 | 0.0557 | 0.9258 | 0.0742 |
| -0.0023 | 0.0009 | 0.0014 | -0.0028 | 0.0028 | 0.9277 | 0.0259 | 0.0465 | 0.9141 | 0.0859 |
| -0.0047 | 0.0032 | 0.0015 | -0.0028 | 0.0028 | 0.8610 | 0.0898 | 0.0493 | 0.9162 | 0.0838 |
| -0.0030 | 0.0015 | 0.0015 | -0.0027 | 0.0027 | 0.9092 | 0.0425 | 0.0483 | 0.9169 | 0.0831 |
| -0.0037 | 0.0022 | 0.0015 | -0.0027 | 0.0027 | 0.8883 | 0.0630 | 0.0487 | 0.9169 | 0.0831 |
| -0.0085 | 0.0070 | 0.0015 | -0.0039 | 0.0039 | 0.7639 | 0.1814 | 0.0547 | 0.8828 | 0.1172 |
| -0.0026 | 0.0011 | 0.0015 | -0.0040 | 0.0040 | 0.9209 | 0.0304 | 0.0487 | 0.8816 | 0.1184 |
| -0.0031 | 0.0016 | 0.0015 | -0.0040 | 0.0040 | 0.9064 | 0.0444 | 0.0493 | 0.8812 | 0.1188 |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| -0.0030 | 0.0010 | 0.0020 | -0.0027 | 0.0027 | 0.9201 | 0.0251 | 0.0547 | 0.9268 | 0.0732 |
| -0.0041 | 0.0021 | 0.0020 | -0.0028 | 0.0028 | 0.8928 | 0.0522 | 0.0550 | 0.9263 | 0.0737 |
| -0.0031 | 0.0009 | 0.0022 | -0.0028 | 0.0028 | 0.9188 | 0.0232 | 0.0580 | 0.9268 | 0.0732 |
| -0.0033 | 0.0012 | 0.0022 | -0.0020 | 0.0020 | 0.9124 | 0.0298 | 0.0578 | 0.9453 | 0.0547 |
| -0.0036 | 0.0021 | 0.0015 | -0.0021 | 0.0021 | 0.8928 | 0.0594 | 0.0478 | 0.9372 | 0.0628 |
| -0.0036 | 0.0020 | 0.0015 | -0.0020 | 0.0020 | 0.8931 | 0.0592 | 0.0478 | 0.9375 | 0.0625 |
| -0.0050 | 0.0021 | 0.0029 | -0.0021 | 0.0021 | 0.8945 | 0.0427 | 0.0627 | 0.9548 | 0.0452 |
| -0.0047 | 0.0029 | 0.0018 | -0.0014 | 0.0014 | 0.8707 | 0.0793 | 0.0500 | 0.9600 | 0.0400 |
| -0.0027 | 0.0012 | 0.0015 | -0.0014 | 0.0014 | 0.9181 | 0.0346 | 0.0473 | 0.9573 | 0.0427 |
| -0.0027 | 0.0012 | 0.0015 | -0.0014 | 0.0014 | 0.9178 | 0.0349 | 0.0473 | 0.9573 | 0.0427 |
| -0.0036 | 0.0021 | 0.0015 | -0.0014 | 0.0014 | 0.8926 | 0.0603 | 0.0472 | 0.9571 | 0.0429 |
| -0.0055 | 0.0032 | 0.0023 | -0.0019 | 0.0019 | 0.8647 | 0.0758 | 0.0594 | 0.9503 | 0.0497 |
| -0.0036 | 0.0012 | 0.0023 | -0.0019 | 0.0019 | 0.9100 | 0.0303 | 0.0597 | 0.9503 | 0.0497 |
| -0.0036 | 0.0012 | 0.0023 | -0.0019 | 0.0019 | 0.9098 | 0.0305 | 0.0597 | 0.9503 | 0.0497 |
| -0.0069 | 0.0053 | 0.0016 | -0.0019 | 0.0019 | 0.8050 | 0.1440 | 0.0510 | 0.9412 | 0.0588 |
| -0.0069 | 0.0053 | 0.0016 | -0.0014 | 0.0014 | 0.8050 | 0.1453 | 0.0498 | 0.9574 | 0.0426 |
| -0.0070 | 0.0054 | 0.0016 | -0.0014 | 0.0014 | 0.8023 | 0.1480 | 0.0497 | 0.9573 | 0.0427 |
| -0.0071 | 0.0054 | 0.0016 | -0.0014 | 0.0014 | 0.8008 | 0.1494 | 0.0497 | 0.9573 | 0.0427 |
| -0.0066 | 0.0054 | 0.0012 | -0.0014 | 0.0014 | 0.7859 | 0.1720 | 0.0422 | 0.9507 | 0.0493 |
| -0.0065 | 0.0054 | 0.0012 | -0.0014 | 0.0014 | 0.7884 | 0.1695 | 0.0421 | 0.9507 | 0.0493 |
| -0.0036 | 0.0024 | 0.0012 | -0.0014 | 0.0014 | 0.8781 | 0.0801 | 0.0418 | 0.9506 | 0.0494 |
| -0.0060 | 0.0024 | 0.0036 | -0.0021 | 0.0021 | 0.8842 | 0.0452 | 0.0707 | 0.9590 | 0.0410 |
| -0.0060 | 0.0024 | 0.0036 | -0.0021 | 0.0021 | 0.8860 | 0.0445 | 0.0695 | 0.9596 | 0.0404 |
| -0.0068 | 0.0032 | 0.0036 | -0.0021 | 0.0021 | 0.8720 | 0.0587 | 0.0693 | 0.9595 | 0.0405 |
| -0.0064 | 0.0027 | 0.0036 | -0.0021 | 0.0021 | 0.8799 | 0.0507 | 0.0694 | 0.9594 | 0.0406 |
| -0.0064 | 0.0028 | 0.0036 | -0.0021 | 0.0021 | 0.8796 | 0.0510 | 0.0694 | 0.9594 | 0.0406 |
| -0.0049 | 0.0027 | 0.0022 | -0.0017 | 0.0017 | 0.8754 | 0.0680 | 0.0565 | 0.9555 | 0.0445 |
| -0.0044 | 0.0028 | 0.0016 | -0.0017 | 0.0017 | 0.8709 | 0.0788 | 0.0502 | 0.9485 | 0.0515 |
| -0.0044 | 0.0028 | 0.0016 | -0.0012 | 0.0012 | 0.8705 | 0.0798 | 0.0497 | 0.9619 | 0.0381 |
| -0.0039 | 0.0022 | 0.0016 | -0.0012 | 0.0012 | 0.8860 | 0.0643 | 0.0498 | 0.9619 | 0.0381 |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| -0.0039 | 0.0022 | 0.0016 | -0.0012 | 0.0012 | 0.8857 | 0.0646 | 0.0497 | 0.9638 | 0.0362 |
| -0.0039 | 0.0022 | 0.0016 | -0.0024 | 0.0024 | 0.8855 | 0.0635 | 0.0509 | 0.9280 | 0.0720 |
| -0.0062 | 0.0045 | 0.0016 | -0.0024 | 0.0024 | 0.8246 | 0.1240 | 0.0514 | 0.9284 | 0.0716 |
| -0.0062 | 0.0045 | 0.0016 | -0.0016 | 0.0016 | 0.8246 | 0.1256 | 0.0498 | 0.9522 | 0.0478 |
| -0.0027 | 0.0016 | 0.0012 | -0.0016 | 0.0016 | 0.9035 | 0.0538 | 0.0427 | 0.9432 | 0.0568 |
| -0.0027 | 0.0016 | 0.0012 | -0.0016 | 0.0016 | 0.9035 | 0.0538 | 0.0427 | 0.9433 | 0.0567 |
| -0.0027 | 0.0016 | 0.0012 | -0.0016 | 0.0016 | 0.9033 | 0.0540 | 0.0427 | 0.9432 | 0.0568 |
| -0.0029 | 0.0016 | 0.0014 | -0.0014 | 0.0014 | 0.9044 | 0.0499 | 0.0457 | 0.9536 | 0.0464 |
| -0.0029 | 0.0016 | 0.0014 | -0.0011 | 0.0011 | 0.9044 | 0.0502 | 0.0454 | 0.9641 | 0.0359 |
| -0.0050 | 0.0037 | 0.0014 | -0.0017 | 0.0017 | 0.8418 | 0.1120 | 0.0462 | 0.9444 | 0.0556 |
| -0.0038 | 0.0015 | 0.0023 | -0.0020 | 0.0020 | 0.9065 | 0.0352 | 0.0583 | 0.9483 | 0.0517 |
| -0.0038 | 0.0015 | 0.0023 | -0.0025 | 0.0025 | 0.9065 | 0.0349 | 0.0585 | 0.9364 | 0.0636 |
| -0.0047 | 0.0022 | 0.0025 | -0.0056 | 0.0056 | 0.8886 | 0.0485 | 0.0629 | 0.8688 | 0.1312 |
| -0.0026 | 0.0010 | 0.0015 | -0.0019 | 0.0019 | 0.9208 | 0.0306 | 0.0485 | 0.9403 | 0.0597 |
| -0.0045 | 0.0012 | 0.0034 | -0.0019 | 0.0019 | 0.9102 | 0.0227 | 0.0671 | 0.9611 | 0.0389 |
| -0.0052 | 0.0020 | 0.0031 | -0.0017 | 0.0017 | 0.8934 | 0.0415 | 0.0652 | 0.9643 | 0.0357 |
| -0.0040 | 0.0015 | 0.0025 | -0.0024 | 0.0024 | 0.9062 | 0.0337 | 0.0601 | 0.9432 | 0.0568 |
| -0.0106 | 0.0079 | 0.0027 | -0.0029 | 0.0029 | 0.7719 | 0.1638 | 0.0643 | 0.9310 | 0.0690 |
| -0.0030 | 0.0019 | 0.0011 | -0.0018 | 0.0018 | 0.8927 | 0.0657 | 0.0416 | 0.9351 | 0.0649 |
| -0.0104 | 0.0015 | 0.0089 | -0.0014 | 0.0014 | 0.9287 | 0.0100 | 0.0613 | 0.9902 | 0.0098 |
| -0.0042 | 0.0015 | 0.0027 | -0.0020 | 0.0020 | 0.9041 | 0.0329 | 0.0630 | 0.9516 | 0.0484 |
| -0.0060 | 0.0015 | 0.0045 | -0.0020 | 0.0020 | 0.9025 | 0.0235 | 0.0741 | 0.9654 | 0.0346 |
| -0.0060 | 0.0015 | 0.0045 | -0.0011 | 0.0011 | 0.9025 | 0.0237 | 0.0739 | 0.9811 | 0.0189 |
| -0.0033 | 0.0015 | 0.0019 | -0.0012 | 0.0012 | 0.9070 | 0.0400 | 0.0530 | 0.9665 | 0.0335 |
| -0.0038 | 0.0015 | 0.0023 | -0.0104 | 0.0104 | 0.9067 | 0.0315 | 0.0617 | 0.7646 | 0.2354 |
| -0.0036 | 0.0015 | 0.0022 | -0.0012 | 0.0012 | 0.9062 | 0.0372 | 0.0565 | 0.9688 | 0.0312 |
| -0.0087 | 0.0039 | 0.0048 | -0.0029 | 0.0029 | 0.8706 | 0.0564 | 0.0730 | 0.9547 | 0.0453 |
| -0.0038 | 0.0018 | 0.0020 | -0.0019 | 0.0019 | 0.8987 | 0.0463 | 0.0550 | 0.9478 | 0.0522 |
| -0.0045 | 0.0018 | 0.0027 | -0.0029 | 0.0029 | 0.8975 | 0.0394 | 0.0631 | 0.9318 | 0.0682 |
| -0.0045 | 0.0016 | 0.0029 | -0.0042 | 0.0042 | 0.9010 | 0.0330 | 0.0660 | 0.9067 | 0.0933 |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| -0.0068 | 0.0039 | 0.0029 | -0.0012 | 0.0012 | 0.8557 | 0.0817 | 0.0627 | 0.9723 | 0.0277 |
| -0.0057 | 0.0036 | 0.0020 | -0.0020 | 0.0020 | 0.8552 | 0.0907 | 0.0541 | 0.9450 | 0.0550 |
| -0.0069 | 0.0027 | 0.0042 | -0.0029 | 0.0029 | 0.8851 | 0.0442 | 0.0707 | 0.9497 | 0.0503 |
| -0.0052 | 0.0016 | 0.0036 | -0.0016 | 0.0016 | 0.9034 | 0.0287 | 0.0680 | 0.9698 | 0.0302 |

The Table 4.1 is summarized in following example where a person aged 45 may be in one of these three states: alive (A), disability (Di) or dead(De). Now is initially alive and he buy insurance contract of 20 years, the transition intensities are as follows

| | $A$ | $Di$ | $De$ |
|---|---|---|---|
| $A$ | -0.0052 | 0.0016 | 0.0036 |
| $Di$ | 0 | -0.0016 | 0.0016 |
| $De$ | 0 | 0 | 0 |

The transition probabilities given by the following equations (4.5), (4.6) and (4.10), their results are shown by using this table

| | $A$ | $Di$ | $De$ |
|---|---|---|---|
| $A$ | 0.9034 | 0.0287 | 0.0680 |
| $Di$ | 0 | 0.9698 | 0.0302 |
| $De$ | 0 | 0 | 1 |

and they satisfy these properties $\sum_{j=0}^{k} {}_tp_x^{ij} = 1$ and $\sum_{j=0}^{k} \mu_{x+t}^{ij} = 0$.

Figure 4.1: Transition intensity of alive state to dead state against time.

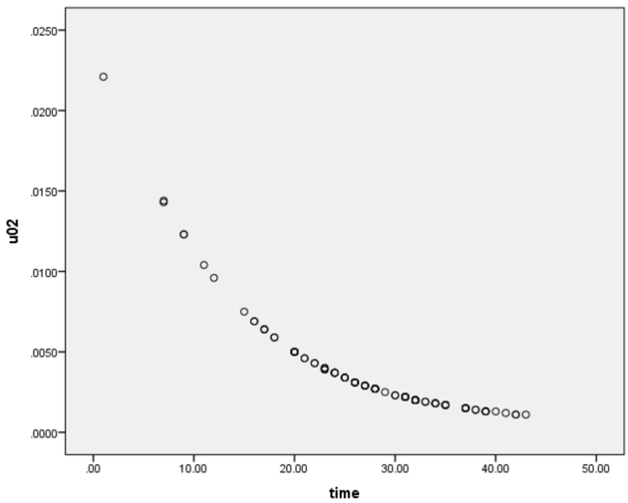



Figure 4.2: Transition intensity of staying to disability state against time.

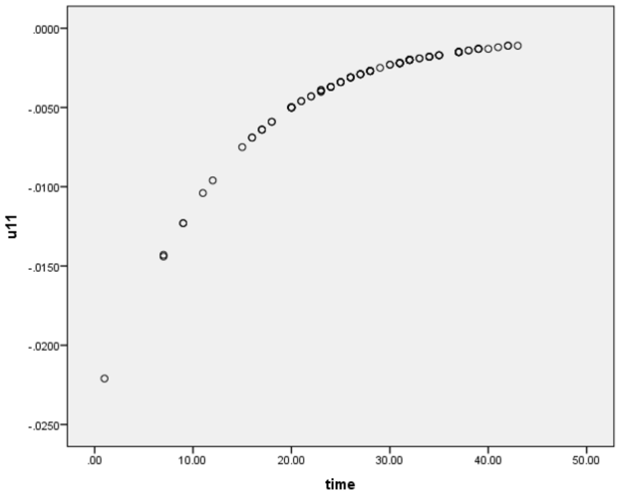
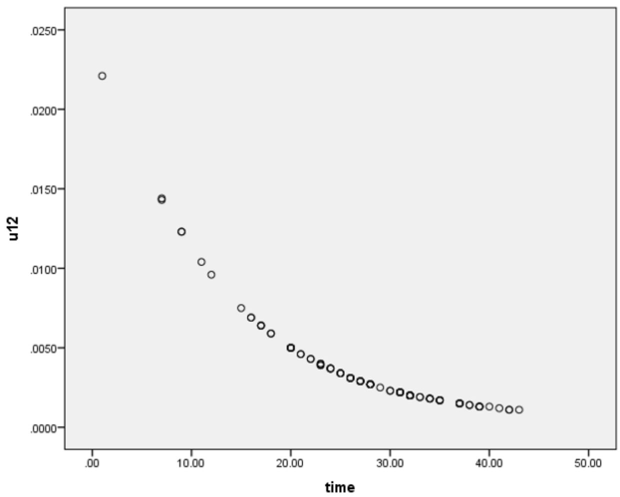



Figure 4.3: Transition intensity of disability state to dead state against time.



Observing the graph in Figure 4.1,Figure 4.2,Figure 4.3, the transition intensities $\mu^{02}_{x+t}$,$\mu^{11}_{x+t}$ and $\mu^{12}_{x+t}$ , they are exponentially with respect to ages, comparing to the other researcher's, like David C. M. Dickson in [Dickson et al., 2019] where there is Gompert-Makeham (GM) formula

$$\mu_{x+t} = \beta_0 + \exp(\beta_1 x + t),$$

where $\beta_0, \beta_1$ are constants to be estimated, in the analysis of the data, the force of mortality or force of intensities $\mu^{02}_{x+t}$,$\mu^{11}_{x+t}$ and $\mu^{12}_{x+t}$, they follow the GM formula. The estimated coefficient of $\beta_0$ and $\beta_1$ together with the coefficient of determination from the data is shown as follows:

Table 4.2: Estimated coefficients.

| Transition intensities | $\beta_0$ | $\beta_1$ | $R^2$ |
|---|---|---|---|
| $\mu^{02}_{x+t}$ | 0.014 | -0.067 | 0.994 |
| $\mu^{11}_{x+t}$ | -0.012 | -0.065 | 0.990 |
| $\mu^{12}_{x+t}$ | 0.012 | -0.065 | 0.990 |

## 4.3 Calculation of Expectation of Present and Annuities of Death Benefit

As it has mentioned previously in formulas (3.29) and (3.30), it has used first to understand the expectation of present value in life insurances and annuities as the application in multistate models. After it has reflected to the mathematical model of permanent disability model, so that the transition intensities and the transition probabilities may be used to calculate the expected present value and the annuities of death benefit for permanent disability model.

Now apply the formulas (3.29) and (3.30) to the data obtained from Sanlam, Table 4.3 summarize the results of the expected present value and the annuities of our mathematical model where $A^{01}, A^{02}, A^{12}$ are expected present value for alive to disability, expected present value for alive to death, expected present value disability to death respectively and $a^{01}, a^{02}, a^{12}$ stands the annuity in alive state to disability, annuity in alive state to death and annuity in disability to death respectively.

Table 4.3: Expected Present Value of Benefit and Annuity of Death Benefit

| $\overline{A}_x^{01}$ | $\overline{A}_x^{02}$ | $\overline{A}_x^{12}$ | $\ddot{a}_x^{01}$ | $\ddot{a}_x^{02}$ | $\ddot{a}_x^{12}$ |
|---|---|---|---|---|---|
| 0.0011 | 0.0493 | 0.0017 | 0.0236 | 1.4175 | 0.0482 |
| 0.0016 | 0.0429 | 0.0005 | 0.0380 | 1.4122 | 0.0422 |
| 0.0012 | 0.0365 | 0.0005 | 0.0329 | 1.3561 | 0.0320 |
| 0.0003 | 0.0256 | 0.0006 | 0.0095 | 1.1700 | 0.0172 |
| 0.0005 | 0.0427 | 0.0006 | 0.0115 | 1.3920 | 0.0394 |
| 0.0004 | 0.0315 | 0.0007 | 0.0122 | 1.2972 | 0.0242 |
| 0.0020 | 0.0363 | 0.0009 | 0.0599 | 1.3190 | 0.0299 |
| 0.0012 | 0.0395 | 0.0005 | 0.0306 | 1.3789 | 0.0368 |
| 0.0032 | 0.0545 | 0.0026 | 0.0534 | 1.2539 | 0.0601 |
| 0.0005 | 0.0257 | 0.0008 | 0.0220 | 1.0457 | 0.0118 |
| 0.0009 | 0.0349 | 0.0009 | 0.0264 | 1.3129 | 0.0279 |
| 0.0003 | 0.0259 | 0.0006 | 0.0113 | 1.1774 | 0.0172 |
| 0.0006 | 0.0287 | 0.0010 | 0.0218 | 1.1579 | 0.0172 |
| 0.0008 | 0.0335 | 0.0004 | 0.0233 | 1.3414 | 0.0279 |
| 0.0003 | 0.0239 | 0.0006 | 0.0106 | 1.1111 | 0.0151 |
| 0.0019 | 0.0536 | 0.0006 | 0.0167 | 0.7503 | 0.0746 |
| 0.0014 | 0.0491 | 0.0023 | 0.0098 | 0.5381 | 0.0722 |

| | | | | | |
|---|---|---|---|---|---|
| 0.0010 | 0.0568 | 0.0034 | 0.0108 | 0.9355 | 0.0729 |
| 0.0007 | 0.0285 | 0.0007 | 0.0226 | 1.2284 | 0.0197 |
| 0.0004 | 0.0480 | 0.0009 | 0.0086 | 1.3572 | 0.0478 |
| 0.0011 | 0.0292 | 0.0005 | 0.0350 | 1.2285 | 0.0226 |
| 0.0027 | 0.0365 | 0.0006 | 0.0743 | 1.3283 | 0.0342 |
| 0.0004 | 0.0285 | 0.0005 | 0.0145 | 1.2401 | 0.0211 |
| 0.0004 | 0.0255 | 0.0005 | 0.0126 | 1.1720 | 0.0172 |
| 0.0007 | 0.0513 | 0.0010 | 0.0125 | 1.3360 | 0.0540 |
| 0.0002 | 0.0231 | 0.0005 | 0.0084 | 1.1152 | 0.0141 |
| 0.0006 | 0.0524 | 0.0013 | 0.0108 | 1.2767 | 0.0568 |
| 0.0003 | 0.0214 | 0.0005 | 0.0109 | 1.0300 | 0.0104 |
| 0.0007 | 0.0459 | 0.0011 | 0.0157 | 1.3440 | 0.0447 |
| 0.0007 | 0.0544 | 0.0013 | 0.0060 | 0.7503 | 0.0746 |
| 0.0004 | 0.0215 | 0.0004 | 0.0173 | 1.0306 | 0.0111 |
| 0.0005 | 0.0554 | 0.0030 | 0.0080 | 1.1988 | 0.0630 |
| 0.0009 | 0.0261 | 0.0004 | 0.0322 | 1.1687 | 0.0184 |
| 0.0004 | 0.0355 | 0.0007 | 0.0098 | 1.3014 | 0.0298 |
| 0.0006 | 0.0568 | 0.0013 | 0.0072 | 1.0156 | 0.0710 |

| | | | | | |
|---|---|---|---|---|---|
| 0.0016 | 0.0277 | 0.0004 | 0.0845 | 1.0306 | 0.0111 |
| 0.0003 | 0.0228 | 0.0005 | 0.0115 | 1.1074 | 0.0132 |
| 0.0004 | 0.0385 | 0.0004 | 0.0091 | 1.3283 | 0.0342 |
| 0.0003 | 0.0257 | 0.0003 | 0.0091 | 1.1820 | 0.0184 |
| 0.0005 | 0.0228 | 0.0003 | 0.0198 | 1.1111 | 0.0151 |
| 0.0009 | 0.0208 | 0.0002 | 0.0380 | 1.0600 | 0.0125 |
| 0.0005 | 0.0248 | 0.0004 | 0.0186 | 1.1489 | 0.0172 |
| 0.0005 | 0.0220 | 0.0003 | 0.0180 | 1.0752 | 0.0133 |
| 0.0005 | 0.0475 | 0.0006 | 0.0101 | 1.3366 | 0.0476 |
| 0.0003 | 0.0286 | 0.0005 | 0.0088 | 1.2496 | 0.0211 |
| 0.0010 | 0.0288 | 0.0005 | 0.0349 | 1.2496 | 0.0211 |
| 0.0014 | 0.0289 | 0.0005 | 0.0487 | 1.2496 | 0.0211 |
| 0.0003 | 0.0226 | 0.0004 | 0.0138 | 1.1039 | 0.0132 |
| 0.0006 | 0.0237 | 0.0005 | 0.0248 | 1.1039 | 0.0132 |
| 0.0009 | 0.0343 | 0.0007 | 0.0266 | 1.3248 | 0.0279 |
| 0.0009 | 0.0352 | 0.0010 | 0.0262 | 1.3248 | 0.0279 |
| 0.0006 | 0.0350 | 0.0010 | 0.0186 | 1.3248 | 0.0279 |
| 0.0003 | 0.0276 | 0.0008 | 0.0084 | 1.1848 | 0.0197 |

| | | | | | |
|---|---|---|---|---|---|
| 0.0009 | 0.0303 | 0.0009 | 0.0302 | 1.2114 | 0.0211 |
| 0.0004 | 0.0292 | 0.0008 | 0.0143 | 1.2114 | 0.0211 |
| 0.0007 | 0.0296 | 0.0008 | 0.0212 | 1.2114 | 0.0211 |
| 0.0018 | 0.0364 | 0.0012 | 0.0610 | 1.2114 | 0.0211 |
| 0.0003 | 0.0296 | 0.0012 | 0.0102 | 1.2114 | 0.0211 |
| 0.0005 | 0.0303 | 0.0012 | 0.0149 | 1.2114 | 0.0211 |
| 0.0003 | 0.0344 | 0.0010 | 0.0097 | 1.2966 | 0.0278 |
| 0.0007 | 0.0348 | 0.0010 | 0.0201 | 1.2966 | 0.0278 |
| 0.0003 | 0.0365 | 0.0010 | 0.0090 | 1.3700 | 0.0299 |
| 0.0004 | 0.0362 | 0.0008 | 0.0116 | 1.3700 | 0.0299 |
| 0.0006 | 0.0287 | 0.0007 | 0.0201 | 1.2031 | 0.0211 |
| 0.0006 | 0.0287 | 0.0006 | 0.0200 | 1.2031 | 0.0211 |
| 0.0008 | 0.0420 | 0.0009 | 0.0197 | 1.3309 | 0.0391 |
| 0.0009 | 0.0303 | 0.0005 | 0.0287 | 1.2424 | 0.0242 |
| 0.0004 | 0.0281 | 0.0004 | 0.0117 | 1.2054 | 0.0211 |
| 0.0004 | 0.0281 | 0.0004 | 0.0118 | 1.2054 | 0.0211 |
| 0.0006 | 0.0280 | 0.0004 | 0.0203 | 1.2054 | 0.0211 |
| 0.0011 | 0.0376 | 0.0007 | 0.0305 | 1.3889 | 0.0321 |

| | | | | | |
|---|---|---|---|---|---|
| 0.0005 | 0.0379 | 0.0007 | 0.0122 | 1.3889 | 0.0321 |
| 0.0005 | 0.0379 | 0.0007 | 0.0123 | 1.3889 | 0.0321 |
| 0.0014 | 0.0307 | 0.0006 | 0.0488 | 1.2823 | 0.0226 |
| 0.0014 | 0.0293 | 0.0004 | 0.0492 | 1.2823 | 0.0226 |
| 0.0015 | 0.0293 | 0.0004 | 0.0501 | 1.2823 | 0.0226 |
| 0.0015 | 0.0293 | 0.0004 | 0.0506 | 1.2823 | 0.0226 |
| 0.0012 | 0.0239 | 0.0004 | 0.0490 | 1.1040 | 0.0151 |
| 0.0012 | 0.0239 | 0.0004 | 0.0483 | 1.1040 | 0.0151 |
| 0.0006 | 0.0235 | 0.0004 | 0.0228 | 1.1040 | 0.0151 |
| 0.0011 | 0.0484 | 0.0010 | 0.0224 | 1.4288 | 0.0483 |
| 0.0011 | 0.0479 | 0.0010 | 0.0223 | 1.3884 | 0.0480 |
| 0.0014 | 0.0476 | 0.0010 | 0.0294 | 1.3884 | 0.0480 |
| 0.0012 | 0.0478 | 0.0010 | 0.0254 | 1.3884 | 0.0480 |
| 0.0012 | 0.0478 | 0.0010 | 0.0256 | 1.3884 | 0.0480 |
| 0.0009 | 0.0354 | 0.0006 | 0.0268 | 1.3363 | 0.0299 |
| 0.0008 | 0.0300 | 0.0005 | 0.0268 | 1.2759 | 0.0226 |
| 0.0008 | 0.0293 | 0.0004 | 0.0271 | 1.2759 | 0.0226 |
| 0.0007 | 0.0294 | 0.0004 | 0.0218 | 1.2759 | 0.0226 |

| | | | | | |
|---|---|---|---|---|---|
| 0.0007 | 0.0293 | 0.0004 | 0.0220 | 1.2759 | 0.0226 |
| 0.0007 | 0.0308 | 0.0008 | 0.0216 | 1.2759 | 0.0226 |
| 0.0013 | 0.0315 | 0.0008 | 0.0423 | 1.2660 | 0.0226 |
| 0.0013 | 0.0297 | 0.0005 | 0.0429 | 1.2660 | 0.0226 |
| 0.0004 | 0.0238 | 0.0004 | 0.0150 | 1.1382 | 0.0151 |
| 0.0004 | 0.0238 | 0.0004 | 0.0150 | 1.1382 | 0.0151 |
| 0.0004 | 0.0238 | 0.0004 | 0.0150 | 1.1382 | 0.0151 |
| 0.0004 | 0.0262 | 0.0004 | 0.0153 | 1.2060 | 0.0184 |
| 0.0004 | 0.0259 | 0.0003 | 0.0154 | 1.2060 | 0.0184 |
| 0.0009 | 0.0268 | 0.0005 | 0.0344 | 1.2060 | 0.0184 |
| 0.0005 | 0.0374 | 0.0008 | 0.0144 | 1.3373 | 0.0320 |
| 0.0005 | 0.0377 | 0.0010 | 0.0143 | 1.3373 | 0.0320 |
| 0.0008 | 0.0420 | 0.0020 | 0.0204 | 1.3735 | 0.0344 |
| 0.0003 | 0.0287 | 0.0006 | 0.0101 | 1.2458 | 0.0211 |
| 0.0005 | 0.0461 | 0.0009 | 0.0111 | 1.3560 | 0.0448 |
| 0.0009 | 0.0438 | 0.0008 | 0.0196 | 1.3638 | 0.0420 |
| 0.0006 | 0.0391 | 0.0009 | 0.0144 | 1.3429 | 0.0343 |
| 0.0026 | 0.0423 | 0.0012 | 0.0706 | 1.4150 | 0.0369 |

| | | | | | |
|---|---|---|---|---|---|
| 0.0005 | 0.0233 | 0.0005 | 0.0180 | 1.0910 | 0.0142 |
| 0.0011 | 0.0524 | 0.0011 | 0.0079 | 0.6212 | 0.0767 |
| 0.0006 | 0.0412 | 0.0008 | 0.0143 | 1.3953 | 0.0369 |
| 0.0007 | 0.0534 | 0.0011 | 0.0130 | 1.3497 | 0.0577 |
| 0.0007 | 0.0531 | 0.0006 | 0.0131 | 1.3497 | 0.0577 |
| 0.0005 | 0.0323 | 0.0004 | 0.0147 | 1.3057 | 0.0260 |
| 0.0005 | 0.0417 | 0.0033 | 0.0130 | 1.3316 | 0.0320 |
| 0.0005 | 0.0354 | 0.0004 | 0.0146 | 1.3394 | 0.0299 |
| 0.0020 | 0.0535 | 0.0016 | 0.0327 | 1.2709 | 0.0603 |
| 0.0006 | 0.0342 | 0.0007 | 0.0176 | 1.3179 | 0.0279 |
| 0.0007 | 0.0416 | 0.0012 | 0.0172 | 1.3852 | 0.0368 |
| 0.0006 | 0.0442 | 0.0017 | 0.0148 | 1.4139 | 0.0395 |
| 0.0015 | 0.0409 | 0.0005 | 0.0372 | 1.3694 | 0.0393 |
| 0.0012 | 0.0340 | 0.0007 | 0.0351 | 1.2830 | 0.0278 |
| 0.0013 | 0.0507 | 0.0015 | 0.0241 | 1.3024 | 0.0537 |
| 0.0007 | 0.0473 | 0.0008 | 0.0146 | 1.3337 | 0.0476 |

## 4.4 Premiums

In insurance the contract between insurer and insured consists of some arrangement of the insured paying a sum of money to the insurer and receiving benefits to be paid out by the insurer at any certain events. That payment or a series of payments paid by the insured is called premium, so that an insurance policy contract continues to protect the insured. Premium payments are calculated based on the expected present value and annuities. In this study net single premium will be used so that premium payments are made once and the additional costs for policy-making are not included in the calculation. However, because what is used is the net single premium so the amount of additional costs for policy-making is not included. So after obtaining transition probability and transition intensity , the amount of premiums was calculated and it is given by equation (3.31) with a 3.5% of interest. Table 4.4 gives the results of the data in calculation of premiums, $\text{Premium}^{01}, \text{Premium}^{02}, \text{Premium}^{12}, \text{Benefit}^{01}, \text{Benefit}^{02}, \text{Benefit}^{12}$ present premium of alive to disability, premium of alive to death, premium of disability to death, benefit of alive to disability, benefit of alive to death and benefit of disability to death respectively.

Table 4.4: Premiums and Benefits: $\text{Premium}^{01}$, $\text{Premium}^{02}$, $\text{Premium}^{12}$, $\text{Benefit}^{01}$, $\text{Benefit}^{02}$, $\text{Benefit}^{12}$

| Age | $\text{Premium}^{02}$ | $\text{Premium}^{01}$ | $\text{Premium}^{12}$ | $\text{Benefit}^{02}$ | $\text{Benefit}^{01}$ | $\text{Benefit}^{12}$ |
|---|---|---|---|---|---|---|
| 45 | 68,041.70 | 60,414.73 | 68,261.66 | 1,418,695.00 | 1,273,490.86 | 1,440,000.00 |
| 43 | 271,700.19 | 28,492.58 | 2,263.09 | 6,472,745.00 | 675,542.55 | 51,000.00 |
| 39 | 21,988.30 | 340.17 | 11,169.44 | 608,008.00 | 9,317.91 | 297,000.00 |
| 30 | 53,151.52 | 53,239.73 | 91,849.34 | 1,865,723.00 | 1,888,033.33 | 3,348,000.00 |
| 42 | 2,308.26 | 1,996.46 | 23,921.72 | 55,003.00 | 48,051.28 | 562,500.00 |
| 35 | 39,250.45 | 834.39 | 114,339.71 | 1,208,943.00 | 25,947.41 | 3,582,747.00 |
| 38 | 543.41 | 25,560.85 | 33,800.65 | 15,909.00 | 702,311.88 | 937,500.00 |
| 41 | 1,945.64 | 3,217.73 | 104,573.09 | 49,645.00 | 81,847.90 | 2,550,000.00 |
| 49 | 161,470.60 | 22,596.34 | 160,641.20 | 2,675,333.00 | 369,054.91 | 2,625,000.00 |
| 24 | 62,177.74 | 11,605.59 | 82,095.26 | 2,522,971.28 | 435,804.87 | 3,600,000.00 |
| 37 | 277,992.35 | 58,467.55 | 141,770.00 | 8,034,031.00 | 1,670,045.00 | 4,116,000.00 |
| 30 | 3,036.28 | 21,112.79 | 31,869.80 | 107,069.00 | 748,509.14 | 1,170,000.00 |
| 30 | 19,852.10 | 2,594.12 | 45,228.65 | 688,231.53 | 86,524.89 | 1,687,500.00 |
| 64 | 40,545.31 | 3,114.43 | 19,504.83 | 1,205,879.71 | 92,314.64 | 562,500.00 |
| 37 | 10,073.69 | 13,169.65 | 33,011.10 | 369,772.00 | 485,081.00 | 1,260,000.00 |
| 28 | 110,705.69 | 32,294.19 | 134,468.93 | 997,351.40 | 291,621.00 | 1,165,473.00 |
| 56 | 259,886.29 | 117,725.95 | 538,329.21 | 1,769,495.37 | 807,671.00 | 3,605,550.00 |
| 58 | 11,425.55 | 33,593.81 | 123,457.09 | 123,651.33 | 371,440.00 | 1,350,000.00 |
| 54 | 11,449.44 | 37,854.64 | 63,690.17 | 391,133.10 | 1,265,153.00 | 2,209,200.00 |
| 32 | 21,783.35 | 161,924.56 | 408,453.10 | 443,108.32 | 3,340,751.00 | 8,249,158.00 |
| 45 | 11,211.98 | 678.27 | 41,794.89 | 364,495.28 | 21,435.00 | 1,310,178.86 |
| 34 | 5,088.19 | 9,474.45 | 17,188.29 | 140,581.21 | 249,203.00 | 431,916.35 |
| 40 | 15,744.41 | 106,724.88 | 819.18 | 518,337.58 | 3,517,543.00 | 27,027.43 |
| 33 | 17,583.04 | 34,219.93 | 2,981.93 | 625,722.27 | 1,218,198.00 | 108,055.00 |
| 30 | 56,298.79 | 128,887.20 | 3,523.31 | 1,040,614.53 | 2,409,889.00 | 64,124.00 |
| 47 | 9,475.96 | 17,495.44 | 1,619.35 | 361,976.56 | 674,025.00 | 64,124.00 |
| 27 | 5,592.60 | 129,485.34 | 2,556.58 | 95,721.42 | 2,248,213.00 | 43,394.27 |
| 48 | 29,700.74 | 10,566.00 | 957.90 | 1,271,035.00 | 447,115.00 | 43,394.00 |
| 22 | 57,435.42 | 26,883.52 | 2,050.51 | 1,218,535.78 | 577,451.00 | 43,394.00 |

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|---|---|---|---|---|---|---|
| 44 | 54,656.13 | 48,296.29 | 26,581.55 | 486,611.50 | 432,934.00 | 231,339.00 |
| 56 | 15,840.09 | 32,538.47 | 5,347.69 | 674,667.20 | 1,340,427.00 | 231,339.00 |
| 23 | 27,159.13 | 33,615.43 | 2,711.96 | 400,132.61 | 511,216.00 | 41,279.00 |
| 50 | 10,166.34 | 38,974.92 | 5,665.68 | 361,620.20 | 1,336,232.00 | 195,472.00 |
| 31 | 14,768.93 | 295,204.30 | 711.25 | 400,000.00 | 8,108,510.00 | 19,447.00 |
| 38 | 177,856.77 | 44,234.41 | 1,652.25 | 2,137,489.00 | 536,717.00 | 19,447.00 |
| 53 | 257.72 | 10,926.51 | 2,246.24 | 13,426.53 | 406,940.00 | 97,429.00 |
| 23 | 17,114.90 | 13,479.76 | 2,379.83 | 679,211.80 | 532,773.00 | 97,429.00 |
| 26 | 17,438.82 | 11,630.06 | 2,767.87 | 443,330.00 | 297,758.00 | 68,992.00 |
| 40 | 11,229.70 | 22,103.13 | 2,003.93 | 389,570.52 | 770,597.00 | 68,992.94 |
| 31 | 17,773.40 | 30,746.58 | 1,850.79 | 679,681.32 | 1,159,926.00 | 68,992.00 |
| 28 | 1,505.88 | 45,766.34 | 1,708.26 | 65,161.22 | 1,865,723.00 | 68,992.00 |
| 25 | 17,761.59 | 19,427.86 | 4,725.70 | 636,922.88 | 689,309.00 | 167,293.03 |
| 30 | 9,192.56 | 64,368.51 | 4,228.01 | 367,363.03 | 2,524,792.00 | 167,293.03 |
| 26 | 37,369.98 | 3,796.23 | 3,008.55 | 759,993.00 | 77,832.00 | 59,787.00 |
| 45 | 37,889.09 | 4,581.55 | 688.64 | 1,239,726.69 | 151,653.00 | 22,870.36 |
| 33 | 6,236.38 | 4,597.01 | 688.64 | 213,547.00 | 151,653.00 | 22,870.36 |
| 33 | 11,627.25 | 4,607.29 | 688.51 | 408,626.42 | 151,653.00 | 22,870.00 |
| 33 | 17,988.47 | 3,814.72 | 562.60 | 719,231.11 | 150,840.00 | 22,870.00 |
| 26 | 62,976.72 | 3,897.27 | 3,166.58 | 2,569,253.79 | 150,840.00 | 129,347.00 |
| 26 | 47,530.22 | 1,762.93 | 4,468.04 | 1,393,576.24 | 51,177.00 | 129,347.00 |
| 37 | 22,305.77 | 1,787.45 | 3,198.34 | 646,433.28 | 51,177.00 | 93,663.00 |
| 37 | 130,783.02 | 1,781.21 | 3,198.34 | 3,746,441.86 | 51,177.00 | 93,663.00 |
| 37 | 19,062.33 | 2,738.33 | 2,747.43 | 616,567.71 | 90,291.00 | 93,663.00 |
| 32 | 19,171.95 | 9,720.99 | 2,827.80 | 626,894.52 | 305,163.93 | 93,663.00 |
| 33 | 21,734.28 | 9,551.42 | 1,281.79 | 691,722.83 | 305,164.00 | 42,439.00 |
| 33 | 12,441.50 | 9,623.37 | 1,281.79 | 400,610.98 | 305,164.00 | 42,439.00 |
| 33 | 1,619.64 | 9,604.34 | 794.72 | 55,230.72 | 278,715.65 | 26,824.00 |
| 33 | 5,768.24 | 8,785.12 | 1,588.34 | 178,887.27 | 278,716.00 | 53,648.00 |

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|---|---|---|---|---|---|---|
| 33 | 46,563.51 | 8,875.96 | 793.97 | 1,455,169.07 | 278,716.00 | 26,824.00 |
| 33 | 15,066.76 | 3,156.19 | 2,757.46 | 420,379.14 | 89,837.00 | 79,644.30 |
| 37 | 118,461.97 | 3,172.20 | 2,756.55 | 3,354,606.97 | 89,837.00 | 79,644.00 |
| 37 | 16,151.56 | 11,075.40 | 2,788.23 | 446,007.12 | 312,738.00 | 79,644.00 |
| 38 | 87,137.55 | 11,030.14 | 3,812.20 | 2,438,617.56 | 312,738.00 | 107,813.80 |
| 38 | 23,345.94 | 6,276.21 | 3,307.05 | 754,901.93 | 200,911.00 | 107,813.80 |
| 33 | 1,664.97 | 6,275.05 | 4,635.47 | 53,836.00 | 200,911.00 | 151,103.80 |
| 33 | 2,314.00 | 5,008.60 | 6,557.25 | 53,836.00 | 117,281.00 | 151,103.80 |
| 42 | 3,652.12 | 1,202.48 | 1,827.43 | 113,412.00 | 36,800.00 | 55,139.00 |
| 35 | 3,090.54 | 1,643.38 | 1,707.37 | 99,711.73 | 53,294.95 | 55,139.00 |
| 33 | 3,089.94 | 1,692.62 | 1,707.43 | 99,712.00 | 54,894.00 | 55,139.00 |
| 33 | 2,022.98 | 1,690.63 | 1,707.27 | 66,201.02 | 54,894.00 | 55,139.00 |
| 33 | 1,922.81 | 1,783.22 | 4,717.92 | 53,427.00 | 49,067.00 | 128,248.00 |
| 39 | 854.39 | 1,790.47 | 2,358.93 | 23,134.99 | 49,067.00 | 64,124.00 |
| 39 | 854.30 | 1,790.43 | 4,717.92 | 23,134.99 | 49,067.00 | 128,248.00 |
| 39 | 2,004.64 | 6,318.19 | 1,975.72 | 68,311.11 | 201,668.00 | 64,124.00 |
| 34 | 1,987.17 | 6,170.12 | 16,973.03 | 68,311.11 | 201,668.00 | 546,164.00 |
| 34 | 1,983.81 | 6,169.12 | 4,243.19 | 68,311.11 | 201,668.00 | 136,541.00 |
| 34 | 1,982.02 | 6,169.00 | 4,243.06 | 68,311.00 | 201,668.00 | 136,541.00 |
| 34 | 1,698.91 | 6,978.08 | 14,587.96 | 68,311.00 | 256,029.00 | 546,164.00 |
| 28 | 2,074.85 | 6,977.04 | 3,646.92 | 83,288.71 | 256,029.00 | 136,541.00 |
| 28 | 6,353.42 | 6,920.93 | 3,646.80 | 241,356.00 | 256,029.00 | 136,541.00 |
| 28 | 12,393.65 | 1,157.83 | 2,806.75 | 263,526.00 | 24,784.00 | 58,504.82 |
| 45 | 12,623.10 | 9,117.26 | 2,854.47 | 263,820.00 | 191,803.00 | 58,504.82 |
| 45 | 884.51 | 9,087.99 | 2,854.27 | 18,635.05 | 191,803.00 | 58,504.82 |
| 45 | 6,997.55 | 8,244.20 | 2,854.16 | 146,744.88 | 173,669.00 | 58,505.00 |
| 45 | 6,996.65 | 8,243.90 | 2,854.04 | 146,744.88 | 173,669.00 | 58,505.00 |
| 45 | 6,581.00 | 18,828.08 | 1,328.24 | 186,234.90 | 529,220.00 | 36,800.00 |
| 38 | 5,684.57 | 4,662.08 | 1,141.85 | 186,234.90 | 149,984.00 | 36,800.00 |

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|---|---|---|---|---|---|---|
| 34 | 5,642.67 | 4,610.19 | 5,227.86 | 186,235.00 | 149,984.00 | 167,293.00 |
| 34 | 1,775.38 | 4,767.30 | 5,227.73 | 58,073.13 | 154,785.28 | 167,293.00 |
| 34 | 1,773.29 | 4,760.83 | 1,870.17 | 58,073.00 | 154,785.00 | 59,787.00 |
| 34 | 1,807.39 | 4,874.44 | 6,719.40 | 58,073.00 | 154,785.00 | 218,953.00 |
| 34 | 1,515.95 | 4,750.21 | 4,197.30 | 50,262.64 | 148,213.00 | 136,076.00 |
| 34 | 1,496.44 | 12,492.16 | 6,826.94 | 50,262.64 | 401,419.00 | 218,512.00 |
| 34 | 855.83 | 5,952.72 | 5,697.63 | 32,567.92 | 224,410.00 | 218,512.00 |
| 28 | 855.80 | 2,976.20 | 6,225.49 | 32,567.92 | 112,205.00 | 238,746.00 |
| 28 | 855.76 | 2,976.44 | 6,225.22 | 32,567.92 | 112,205.00 | 238,746.00 |
| 28 | 766.41 | 7,652.05 | 4,270.25 | 27,027.43 | 268,883.00 | 150,181.00 |
| 31 | 762.16 | 7,605.84 | 28,592.24 | 27,027.43 | 268,883.00 | 1,000,000.00 |
| 31 | 27,478.07 | 7,743.46 | 28,295.75 | 1,000,000.00 | 268,883.00 | 1,000,000.00 |
| 31 | 18,875.91 | 3,208.03 | 37,600.94 | 500,000.00 | 85,812.00 | 1,000,000.00 |
| 39 | 18,996.24 | 1,610.06 | 37,361.95 | 500,000.00 | 42,906.00 | 1,000,000.00 |
| 39 | 20,109.39 | 8,682.87 | 37,037.05 | 500,000.00 | 218,953.00 | 1,000,000.00 |
| 40 | 30,675.14 | 30,349.55 | 30,053.22 | 1,000,000.00 | 1,000,000.00 | 1,000,000.00 |
| 33 | 23,446.11 | 46,291.05 | 47,241.91 | 500,000.00 | 1,000,000.00 | 1,000,000.00 |
| 44 | 44,125.03 | 43,864.95 | 45,025.68 | 1,000,000.00 | 1,000,000.00 | 1,000,000.00 |
| 43 | 39,508.18 | 39,032.99 | 39,141.42 | 1,000,000.00 | 1,000,000.00 | 1,000,000.00 |
| 40 | 37,193.67 | 39,844.72 | 39,459.62 | 1,000,000.00 | 1,000,000.00 | 1,000,000.00 |
| 41 | 25,978.99 | 26,555.76 | 25,710.09 | 1,000,000.00 | 1,000,000.00 | 1,000,000.00 |
| 27 | 135,514.43 | 134,956.54 | 139,265.81 | 1,000,000.00 | 1,000,000.00 | 1,000,000.00 |
| 58 | 40,199.96 | 39,751.16 | 40,232.45 | 1,000,000.00 | 1,000,000.00 | 1,000,000.00 |
| 41 | 56,308.10 | 55,646.87 | 57,232.69 | 1,000,000.00 | 1,000,000.00 | 1,000,000.00 |
| 48 | 55,851.75 | 55,496.67 | 57,696.59 | 1,000,000.00 | 1,000,000.00 | 1,000,000.00 |
| 48 | 33,188.77 | 33,073.94 | 33,685.56 | 1,000,000.00 | 1,000,000.00 | 1,000,000.00 |
| 36 | 42,109.12 | 39,473.88 | 33,750.56 | 1,000,000.00 | 1,000,000.00 | 1,000,000.00 |
| 39 | 35,658.20 | 35,485.59 | 36,295.51 | 1,000,000.00 | 1,000,000.00 | 1,000,000.00 |
| 38 | 60,258.92 | 60,027.20 | 61,676.54 | 1,000,000.00 | 1,000,000.00 | 1,000,000.00 |

| | | | | | | |
|---|---|---|---|---|---|---|
| 49 | 34,675.25 | 34,571.84 | 34,667.11 | 1,000,000.00 | 1,000,000.00 | 1,000,000.00 |
| 37 | 40,653.38 | 40,161.67 | 39,980.11 | 1,000,000.00 | 1,000,000.00 | 1,000,000.00 |
| 41 | 42,724.67 | 41,734.22 | 40,796.06 | 1,000,000.00 | 1,000,000.00 | 1,000,000.00 |
| 42 | 40,946.12 | 41,199.36 | 43,064.36 | 1,000,000.00 | 1,000,000.00 | 1,000,000.00 |
| 42 | 34,425.31 | 35,232.26 | 35,189.26 | 1,000,000.00 | 1,000,000.00 | 1,000,000.00 |
| 37 | 54,739.62 | 54,266.57 | 55,261.37 | 1,000,000.00 | 1,000,000.00 | 1,000,000.00 |
| 47 | 49,181.95 | 48,762.76 | 50,194.11 | 1,000,000.00 | 1,000,000.00 | 1,000,000.00 |

From the data collect from Sanlam, there are claims and we require to calculate the amount of premiums of the permanent disability model for each state. From the Table 4.4 and Figure 4.4, the premiums of moving from disability state to death state and the benefit of moving disability state to death state are highest comparing other states, it shows that to invest more money means to gain more benefit.

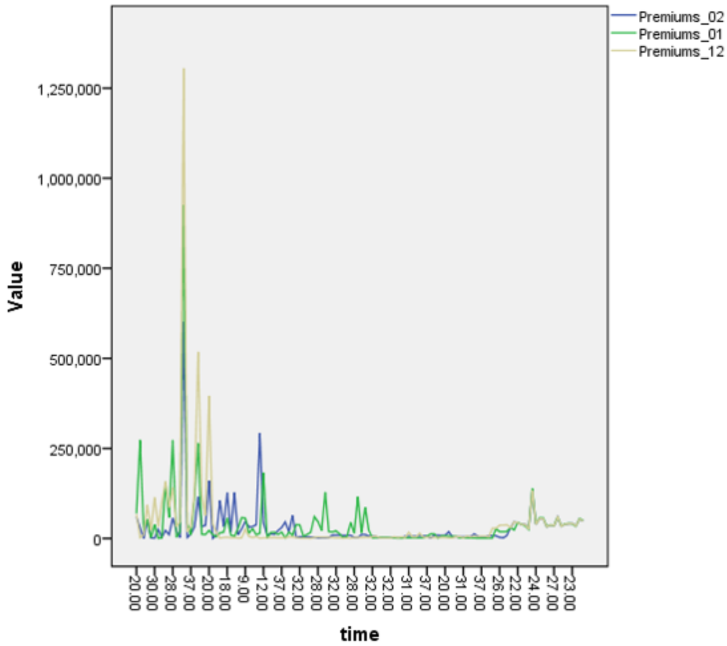


Figure 4.4: Premiums against time.

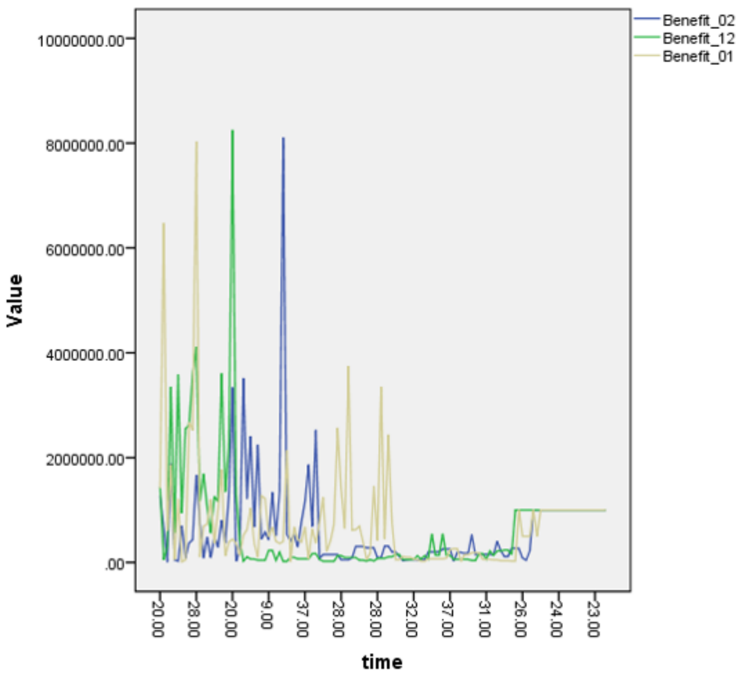


Figure 4.5: Benefit against time.

The following graph represent the amount of premiums paid and amount of money gained for state space against time, this is also represent how to price the insurance policies by using the Markov Chain model for alive, disabilities and dead it can be applicable. The amount of premiums paid and amount of money gained for state space against time. it is observed that the premiums of disability state to death state are higher than the amount of alive state to dead and alive to disability, the benefits gained of disability state to death state are higher than the benefits gained in alive state to death and the benefits gained in alive state to disability, this means that to invest more amounts of premiums implies to gain more benefits.

# Chapter 5

# Results discussion, Conclusion and Recommendations

This chapter summarizes the study findings referring to the research objectives (called results discussion). Conclusions of research thesis. Finally, some recommendations are mentioned based on the study findings.

## 5.1 Results discussion

In this research the applications of multi-state models of continuous-time Markov chain to the actuarial problems in life insurance have been presented by considering the 3 states which are alive, disability and dead. Due to the availability of Sanlam data and to achieve the goal of this research of pricing insurance policies, to use permanent disability model is preferred. The research objectives are achieved as it seems in chapter 4. In modeling disabilities using Kolmogorov differential equations, transition intensities, transition probabilities, expected present values, annuities of death benefit and premiums have been calculated. From the results force of intensities, $\mu^{02}_{x+t}$,$\mu^{11}_{x+t}$ and $\mu^{12}_{x+t}$ are exponentially with respect to ages, comparing to the other researcher's, like David C. M. Dickson in [Dickson et al., 2019]. To evaluate the premiums and benefits in the multi-state model ,by comparing the premiums and the benefits, so it shows that the amount of premiums obtained are related to the number of states space. From the data observed, the premiums paid from disability state to death state and the benefit of moving disability state to death state are highest comparing other states, which shows that to invest more money to gain more benefit which match to reality of life insurance. The investigation of this actuarial problem, the researcher formulated the new results, there is no research done in Rwanda for pricing insurance policies using Multi-states model.

## 5.2 Conclusion

The aim of this research was to price insurance policies, a multi-state model of continuous-time Markov chain have been proposed to determine the premium of insurance policy. It was done via the application of Kolmogorov differential equation for a multi-state model. The amount of premium obtained is related to the number of state in multi-states, the transition intensities and also the transition probabilities. The Kolmogorov differential equation appeared as the equation that decompose the expected present value and the annuity of death benefit, these two give the premium. The insured whose a contract depend on death will get the more future benefit, according to the transition probabilities that are in Table 4.1.

## 5.3 Recommendations

Due to the findings, Its possible to develop a model for pricing insurance using the currently available data in Rwanda. Insurance companies may use this model because it allows them to calculate all necessary numbers for life insurance contracts.

1. To do a research to the the application of Thiele's differential equations for a multi-state model.

2. To do further improvement based on the following:

   (a) To allow for all possible models of recovery in the multi-state.

(b) Based on the project findings, the insurer handle the extreme claim of clients.

(c) The further study on other type of multi-state model.

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## Appendex

Table 5.1: Rwanda Mortality Table

| $x$ | $l_x$ | $d_x$ | $q_x$ | $x$ | $l_x$ | $d_x$ | $q_x$ | $x$ | $l_x$ | $d_x$ | $q_x$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1,000,000 | 3,359 | 0.00335900 | 38 | 967,959 | 1,980 | 0.00204554 | 76 | 609,042 | 29,294 | 0.04809849 |
| 1 | 996,641 | 455 | 0.00045653 | 39 | 965,979 | 2,119 | 0.00219363 | 77 | 579,748 | 31,032 | 0.05352670 |
| 2 | 996,186 | 348 | 0.00034933 | 40 | 963,860 | 2,270 | 0.00235511 | 78 | 548,716 | 32,692 | 0.05957909 |
| 3 | 995,838 | 297 | 0.00029824 | 41 | 961,590 | 2,433 | 0.00253018 | 79 | 516,024 | 34,221 | 0.06631668 |
| 4 | 995,541 | 253 | 0.00025413 | 42 | 959,157 | 2,608 | 0.00271905 | 80 | 481,803 | 35,545 | 0.07377497 |
| 5 | 995,288 | 232 | 0.00023310 | 43 | 956,549 | 2,796 | 0.00292301 | 81 | 446,258 | 36,589 | 0.08199069 |
| 6 | 995,056 | 208 | 0.00020903 | 44 | 953,753 | 3,000 | 0.00314547 | 82 | 409,669 | 37,278 | 0.09099541 |
| 7 | 994,848 | 196 | 0.00019702 | 45 | 950,753 | 3,213 | 0.00337943 | 83 | 372,391 | 37,559 | 0.10085904 |
| 8 | 994,652 | 197 | 0.00019806 | 46 | 947,540 | 3,441 | 0.00363151 | 84 | 334,832 | 37,383 | 0.11164703 |
| 9 | 994,455 | 185 | 0.00018603 | 47 | 944,099 | 3,682 | 0.00390001 | 85 | 297,449 | 36,707 | 0.12340603 |
| 10 | 994,270 | 196 | 0.00019713 | 48 | 940,417 | 3,939 | 0.00418857 | 86 | 260,742 | 35,489 | 0.13610772 |
| 11 | 994,074 | 197 | 0.00019817 | 49 | 936,478 | 4,212 | 0.00449770 | 87 | 225,253 | 33,722 | 0.14970722 |
| 12 | 993,877 | 231 | 0.00023242 | 50 | 932,266 | 4,504 | 0.00483124 | 88 | 191,531 | 31,437 | 0.16413531 |
| 13 | 993,646 | 275 | 0.00027676 | 51 | 927,762 | 4,816 | 0.00519099 | 89 | 160,094 | 28,714 | 0.17935713 |
| 14 | 993,371 | 338 | 0.00034026 | 52 | 922,946 | 5,147 | 0.00557671 | 90 | 131,380 | 25,690 | 0.19553966 |
| 15 | 993,033 | 445 | 0.00044812 | 53 | 917,799 | 5,495 | 0.00598715 | 91 | 105,690 | 22,486 | 0.21275428 |
| 16 | 992,588 | 564 | 0.00056821 | 54 | 912,304 | 5,855 | 0.00641782 | 92 | 83,204 | 19,220 | 0.23099851 |
| 17 | 992,024 | 700 | 0.00070563 | 55 | 906,449 | 6,229 | 0.00687187 | 93 | 63,984 | 16,002 | 0.25009377 |
| 18 | 991,324 | 829 | 0.00083626 | 56 | 900,220 | 6,614 | 0.00734709 | 94 | 47,982 | 12,941 | 0.26970531 |
| 19 | 990,495 | 923 | 0.00093186 | 57 | 893,606 | 7,009 | 0.00784350 | 95 | 35,041 | 10,151 | 0.28968922 |
| 20 | 989,572 | 836 | 0.00084481 | 58 | 886,597 | 7,410 | 0.00835780 | 96 | 24,890 | 7,712 | 0.30984331 |

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 21 | 988,736 | 852 | 0.00086171 | 59 | 879,187 | 7,814 | 0.00888776 | 97 | 17,178 | 5,670 | 0.33007335 |
| 22 | 987,884 | 874 | 0.00088472 | 60 | 871,373 | 8,226 | 0.00944027 | 98 | 11,508 | 4,031 | 0.35027807 |
| 23 | 987,010 | 900 | 0.00091184 | 61 | 863,147 | 8,890 | 0.01029952 | 99 | 7,477 | 2,769 | 0.37033570 |
| 24 | 986,110 | 930 | 0.00094310 | 62 | 854,257 | 9,636 | 0.01127998 | 100 | 4,708 | 1,836 | 0.38997451 |
| 25 | 985,180 | 963 | 0.00097749 | 63 | 844,621 | 10,473 | 0.01239964 | 101 | 2,872 | 1,176 | 0.40947075 |
| 26 | 984,217 | 1,002 | 0.00101807 | 64 | 834,148 | 11,408 | 0.01367623 | 102 | 1,696 | 727 | 0.42865566 |
| 27 | 983,215 | 1,045 | 0.00106284 | 65 | 822,740 | 12,442 | 0.01512264 | 103 | 969 | 433 | 0.44685243 |
| 28 | 982,170 | 1,092 | 0.00111182 | 66 | 810,298 | 13,572 | 0.01674939 | 104 | 536 | 249 | 0.46455224 |
| 29 | 981,078 | 1,145 | 0.00116708 | 67 | 796,726 | 14,798 | 0.01857351 | 105 | 287 | 138 | 0.48083624 |
| 30 | 979,933 | 1,204 | 0.00122866 | 68 | 781,928 | 16,124 | 0.02062082 | 106 | 149 | 74 | 0.49664430 |
| 31 | 978,729 | 1,272 | 0.00129964 | 69 | 765,804 | 17,551 | 0.02291840 | 107 | 75 | 39 | 0.52000000 |
| 32 | 977,457 | 1,346 | 0.00137704 | 70 | 748,253 | 19,084 | 0.02550474 | 108 | 36 | 19 | 0.52777778 |
| 33 | 976,111 | 1,429 | 0.00146397 | 71 | 729,169 | 20,638 | 0.02830345 | 109 | 17 | 9 | 0.52941176 |
| 34 | 974,682 | 1,521 | 0.00156051 | 72 | 708,531 | 22,271 | 0.03143264 | 110 | 8 | 4 | 0.50000000 |
| 35 | 973,161 | 1,621 | 0.00166571 | 73 | 686,260 | 23,974 | 0.03493428 | 111 | 4 | 2 | 0.50000000 |
| 36 | 971,540 | 1,731 | 0.00178171 | 74 | 662,286 | 25,730 | 0.03885029 | 112 | 2 | 1 | 0.50000000 |
| 37 | 969,809 | 1,850 | 0.00190759 | 75 | 636,556 | 27,514 | 0.04322322 | 113 | 1 | 1 | 1.00000000 |