GENERALIZATIONS OF INJECTIVE MODULES: RED-INJECTIVE AND STRONGLY RED-INJECTIVE MODULES

By

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Declaration

I hereby declare that this dissertation is my own work and that it has never been submitted to any other University for assessment or completion of any postgraduate qualification or any other qualification.

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Approval

This dissertation has been under our supervision and has been submitted with our approval.

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Dedication

I dedicate this dissertation to my loving wife Ms. Niyomahoro Delphine for being there for me at all times.

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Firstly, I thank the Almighty God who has seen me through all situations till the accomplishment of this work. I will always give Him glory and honour for who He is.

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Contents

Abstract

As generalizations of injective modules, Red-injective and strongly Red-injective modules are introduced. The whole study is based on extensive use of definitions, propositions and theorems. Properties of semi-Artinian, quasi-Frobenius and right V-rings have provided a basis for other properties so derived. Many properties of Red-injective and strongly Red-injective modules are derived. Among them, there are: (1) The class of Red-injective modules is closed under direct products and summands. (2) A semi-simple module is Soc-injective if and only if it is Red-injective. (3) Over a Principal Ideal Domain (P.I.D), every projective module is Red-injective if and only if every free module is Red-injective. (4) For a Noetherian module M_R, any direct sum of Red-M-injective modules is Red-injective. (5) Quasi-Frobenius and right V-rings are characterised in terms of strongly Red-injective modules. It is shown that an injective module is strongly Red-injective, a strongly Red-injective module is strongly Soc-injective, and a strongly Soc-injective module is strongly min-injective. Furthermore, it is shown that Red-injectivity is not a Morita invariant property.

List of symbols

Category	Symbol	Description
Rings		
	$M_n(R)$	the ring of $n \times n$ matrices over the ring R
	P.I.D.	Principal Ideal Domain
Modules		
	$<$	proper submodule
	$\leq$	submodule
	M_R	right R-module M
	$\subseteq^e$	essential submodule
	$\mathrm{Soc}(M)$	largest semi-simple submodule of M_R
	Red(M)	sum of all reduced submodules of M_R
	$Mor_{\mathcal{C}}(A, B)$	the set of morphisms from A to B
	$\mathrm{Hom}_R(M, N)$	the group of homomorphisms from M to N
	$End_R(M)$	the group of endomorphisms of M
	$F : \mathfrak{M}_R \to \mathfrak{M}_S$	the functor from $\mathfrak{M}_R$ to $\mathfrak{M}_S$

Chapter 1

Introduction

In Chapter 1, I give a background, a statement of the problem, objectives, significance of study and the methodology used to achieve the stated objectives of the study.

1.1 Background

In module theory, an injective module is a module that shares certain desirable properties with the $\mathbb{Z}$-module $\mathbb{Q}$ of all rational numbers. Specifically, if an injective module I is a submodule of some other module, then it is already a direct summand of that module; also, given a submodule of a module Y, then any module homomorphism from this submodule to I can be extended to a homomorphism from all of Y to I. By Baer's test, a right R-module I is injective if and only if for any right ideal $\mathfrak{U}$ of a ring R, any homomorphism from $\mathfrak{U}$ to I can be extended to a homomorphism from R to I. The theory of injective modules there has an injective producing lemma (Lam, 1999, Lemma 3.5) which is used to produce an injective module given another injective module, and the Modified Injective Test (Lam, 1999, p.63) which is used to check if a given module is injective. Injective modules have been generalized differently by many authors like Johnson and Wong, (1961), Jain and Singh, (1975), Azumaya, *et al.*, (1975), Nicholson and Yousif, (2003). In this work, my contribution is to make a new generalization of injective modules.

The following key words are defined: modules, simples and semi-simple modules,Semi-reduced modules, maximal injective modules, simple-injective modules, mini-injective modules and Soc-injective modules.

Definitions 1 and 2 are used to define a module in Definition 4.

Definition 1. A group is a pair $(G; *)$ with a non-empty set G and a binary operation $*$, i.e., a map $G \times G \to G, (a, b) \mapsto a * b$, called the "group law", satisfying the following conditions

G1 : Group multiplication is "associative", i.e., for all $a, b, c \in G$ we have $(a*b)*c = a*(b*c)$.

G2 : Existence of a "identity element": There is an element $e \in G$ such that $e*a = a = a*e$ for all elements $a \in G$.

G3 : Existence of "inverse elements": For all $a \in G$ there is an element $a^{-1} \in G$, such that $a * a^{-1} = a^{-1} * a = e$.

A group G is commutative or Abelian if $a * b = b * a$ for all $a, b \in G$.

Example 1.1. $\mathbb{R}$ with ordinary addition is a commutative group.

Definition 2. A nonempty set R is called a ring if it has two binary operations called addition denoted by $a + b$ and multiplication denoted by ab for $a,\ b \in R$ satisfying the following axioms:

1. $(R, +)$ is an Abelian group.
2. Multiplication is associative, i.e., $a(bc) = (ab)c$ for all $a,\ b,\ c \in R$.
3. Distributive laws hold: $a(b + c) = ab + ac$ and $(b + c)a = ba + ca \ \forall\ a,\ b,\ c \in R$.

If multiplication in R is commutative, then R is called a commutative ring. If there is an identity for multiplication in R, then R is said to have unity (or a ring with unity $1 \neq 0$).

Example 1.2. The sets $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are commutative rings with unity.

A ring R with at least two elements is called a *division* ring if R has a unity and every non-zero element of R has a multiplicative inverse in R. A division ring in which multiplication is commutative is called a *field*. Like $\mathbb{R}$ is a field, but $\mathbb{Z}$ is not a field since 3 doesn't have a multiplicative inverse in $\mathbb{Z}$.

Definition 3. Let A be a nonempty set of a ring R with the property that, with respect to the property of addition, A is a subgroup of the additive group R . Then :

1. A is a right ideal in R if $ar \in R$ for each $a \in A$ and $r \in R$.
2. A is a left ideal in R if $ra \in R$ for each $a \in A$ and $r \in R$.

3. A is an ideal in R if it is both a right ideal in R and a left ideal in R.

Example 1.3. Let a be a fixed element of the ring $\mathbb{Z}$, and let $A = \{ai|\ i \in \mathbb{Z}\}$. Then A is an ideal in $\mathbb{Z}$.

Definition 4. Let R be a ring. A right R-module consists of an Abelian group $(M, +)$ together with a function : $M \times R \to M$ such that

1. $m(r + s) = mr + ms$;
2. $(m + n)r = mr + nr$;
3. $m(rs) = (mr)s$; $\forall\ r, s \in R,\ m, n \in M$.

A left R-module is defined in a similar way as the right R-module but the elements of R are applied on the left. In general, a left R-module need not be a right R-module. If R has a unity $1 \neq 0$ such that $m \cdot 1 = m = m$ for all $m \in M$, then an R-module M is unital. Let K be a field, a K-module M is called a *vector space*.

Example 1.4. An Abelian group is a $\mathbb{Z}$-module.

Definition 5. Let R be a ring. An R-module homomorphism $f : M \to N$ is a function satisfying

1. $f(a + b) = f(a) + f(b)$ for all $a, b \in M$;
2. $f(ar) = f(a)r$ for all $r \in R, a \in M$.

Definition 6. A subset X of an R-module M is called a submodule of M if it is an R-module such that the inclusion map from X to M is an R-module homomorphism. The R-module M is simple if its only submodules are M and $\{0\}$.

Example 1.5. If M is an Abelian group then, a submodule of M is a$\mathbb{Z}$-submodule.

In this work, R denotes a commutative ring with identity $1 \neq 0$ and all modules are unital right R-modules.

Definition 7. A non-empy subset X of an R-module M is called a basis of M if:

1. every $m \in M$ is a linear combination of elements of X, i.e., $m = \sum_{i=1}^{n} x_i r_i,\ \forall\ r_i \in R;\ x_i \in X,\ i = 1, 2, \cdots, n$;
2. $\sum_{i=1}^{n} x_i r_i = 0$ if and only if $r_i = 0,\ \forall\ i = 1, 2, \cdots, n$.

Definition 8. An R-module M is free if it has a basis.

Remark 1. Not every module possesses a basis and a basis need not be finite.

Example 1.6. $M = \mathbb{Z}/3\mathbb{Z} \bigoplus \mathbb{Z}/3\mathbb{Z}$ is not a free $\mathbb{Z}$-module on the set $\{(\overline{1}, \overline{0}); (\overline{0}, \overline{1})\}$. This is because the expression $(\overline{1}, \overline{2}) = 1(\overline{1}, \overline{0}) + 2(\overline{0}, \overline{1}) = 1(\overline{1}, \overline{0}) + 5(\overline{0}, \overline{1})$ is not written uniquely. Hence M is not free.

Definition 9. Let M_1 and M_2 be R-modules. An R-module I is called injective if for any homomorphism $f : M_1 \to I$ and every monomorphism $h : M_1 \to M_2$ there exists a homomorphism $g : M_2 \to I$ such that $gh = f$.

Example 1.7. $\mathbb{Q}$ is an injective $\mathbb{Z}$-module.

Remark 2. From Example 1.7, a submodule of an injective module need not be injective.

Example 1.8. Take $R = \mathbb{Z}$, $M = \mathbb{Q}$ which is injective as it is divisible, and consider the submodule $\mathbb{Z}$ of $\mathbb{Q}$. It is not injective since it is not divisible.

Proposition 1. (Lam, 1999, Proposition 3.4)

1. The direct product $I = \prod_\alpha I_\alpha$ of right R-modules is injective if and only if each I_α is.
2. A right R-module I is injective if and only if any monomorphism $I_R \to M_R$ splits in $\mathfrak{M}_R$.

Proof:

1. The proof follows from the natural equivalence of functors.
$$\mathrm{Hom}_R(-, \prod_\alpha I_\alpha) \cong \prod_\alpha \mathrm{Hom}_R(-, I_\alpha).$$
2. The "only if" follows by extending the identity map $I \to I$ to a map $M \to I$. For the "if" part, suppose we are given the maps f and h in Definition 9. We form the "pushout": $\frac{I \oplus M_2}{\{(f(a), -h(a)) : a \in M_2\}}$ and let $j : I \to M$, $k : M_2 \to M$ be any two maps. Then, we have a commutative diagram in Figure 1.1:

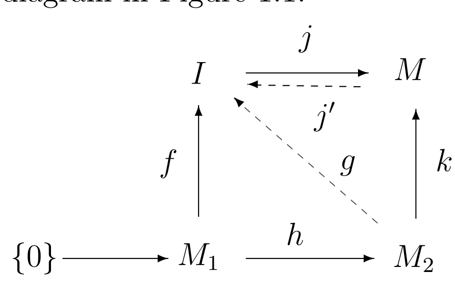

Figure 1.1

The map j is injective. For if $i \in \operatorname{Ker}(j)$, then $(i,0) = (f(a), -h(a))$ for some $a \in M_1$. The injectivity of h implies that $a = 0$, and so $i = f(a) = 0$. By assumption, there exists a splitting $j' : M \to I$ for the monomorphism j. Taking $g = j' \circ k : M_2 \to I$, we have $g \circ h = j' \circ k \circ h = j' \circ j \circ f = f$, as desired.

□

It is known that the direct sum of injective modules need not be injective, but one may ask the conditions under which the direct sum of injective modules is also injective.

Example 1.9. (Lam, 2007, Exercise 19.19.(4)). Let k be a field, and S be a commutative k-algebra[1] $k \oplus \bigoplus_{i \geq 1} ke_i$, where e_i are basis vectors, with $e_i e_j = \delta_{ij} e_i$ ($\delta_{ij} = 1$, if $i = j$ and 0 if $i \neq j$). Let $V_i = kv_i \cong ke_i$ ($i \geq 1$) be the simple right S-modules with the S-action $v_i e_j = \delta_{ij} v_i$ ($i \geq 0, j \geq 1$). Let M be the right S-module $\bigoplus_{i \geq 0} V_i = kv_0 \oplus kv_1 \oplus \cdots$, and define a left S-action on M by $e_i v_j = \delta_{j-1,i} v_i$ ($i \geq 0, j \geq 1$). Define a ring $R = S \bigoplus M$. For any right ideal A in R and any $i, j \geq 1$, any R-homomorphism $f : A \to e_i R$ can be extended to an R-homomorphism $g : A + e_i R \to e_i R$. This shows that $e_i R$ is injective, but $I = e_1 R \oplus e_2 R \oplus \cdots = \bigoplus_{j \geq 1} (ke_j + kv_{j-1})$ is not an injective module.

1.1.1 Simple and Semi-simple modules

Simple modules over a ring R are the right modules over R that have no non-zero proper submodules. Equivalently, a module M_R is simple if and only if every cyclic submodule generated by a non-zero element of M equals M. Simple modules form building blocks for the modules of finite length, and they are analogous to the simple groups in group theory.

Example 1.10. Let R be a division ring. Then R_R is simple.

Example 1.11. Let R be a ring and $\overline{L}$ be a minimal[2] right ideal of R. Then $\overline{L}$ is a simple right R-module.

Definition 10. An R-module M is semi-simple if it is the direct sum of its simple submodules.

Theorem 1.1. Let M be an R-module. Then the following conditions are equivalent:

1. M is semi-simple.

[1] Commutative k-algebra is a vector space where multiplication of elements is commutative.
[2] A minimal right ideal is a non-zero right ideal which contains no other non-zero right ideal.

2. There is a collection $(S_i)_{i\in I}$ of simple R-modules together with an R-isomorphism $M \cong \bigoplus_{i\in I} S_i$.

3. There is a collection $(S_i)_{i\in I}$ of simple R-modules together with an R-linear surjection $\bigoplus_{i\in I} S_i \to M$.

Proof:

$1 \Leftrightarrow 2$. M is semi-simple if and only it is a direct sum of its simple submodule. Thus, $M = \oplus_{i\in I} X_i$ for all semi-simple submodules X_i of M. Taking $\oplus_{i\in I} X_i = \oplus_{i\in I} S_i$, then $M \cong \oplus_{i\in I} S_i$.

$2 \Rightarrow 3$. Suppose that $f : \oplus_{i\in I} S_i \to M$ is an isomorphism and M is semi-simple, then f is a surjective and $\oplus_{i\in I} S_i$ is semi-simple.

$3 \Rightarrow 1$. This is true because a surjective image of a semi-simple module $\oplus_{i\in I} S_i$ is semi-simple

□

Example 1.12. Let $R = \mathbb{Z}$. Then $\mathbb{Z}/p\mathbb{Z}$ is simple for some prime $p \in \mathbb{Z}$. By the previous theorem, $\bigoplus_{i\in I} \mathbb{Z}/p_i\mathbb{Z}$ is a semi-simple module for some indexing set I.

Definition 11. A maximal submodule $\overline{X}$ of an R-module M is a submodule $\overline{X} \neq M$ for which for any other submodule X of M, if $\overline{X} \leq X \leq M$ then $X = M$ or $X = \overline{X}$. Equivalently, $\overline{X}$ is a maximal submodule if and only if the quotient module $M/\overline{X}$ is a simple module.

1.1.2 The socle of a module

Simple modules are the main building blocks of semi-simple modules. In general, not every module can be built from simple modules, but for many modules its semi-simple submodules and semi-simple factor modules play important roles in understanding the module.

Definition 12. Let M be an R-module. The socle of M is the submodule

$$\mathrm{Soc}(M) = \sum_{i\in I} \{S_i \leq M \mid S_i \textit{ is a simple submodule in } M\} \text{ where } I \text{ is an indexing set.}$$

In fact, the socle of M is the largest submodule of M generated by simple modules, or equivalently, it is the largest semi-simple submodule of M. If there are no minimal submodules in M, $\mathrm{Soc}(M) = \{0\}$. If M is semi-simple, $\mathrm{Soc}(M) = M$.

Lemma 1. Let $f : M \to N$ be an R-homomorphism. Then

$$f(\text{Soc}(M)) \subseteq \text{Soc}(N).$$

Proof: Suppose an R-homomorphism $f : M \to N$. If $M = \{0\}$, then $f(M) = f(\text{Soc}(M)) = f(\{0\}) = \{0\} \subseteq \text{Soc}(N)$. Otherwise, $f(\text{Soc}(M)) \subseteq \text{Soc}(N)$ because a homomorphic image of a semi-simple module is semi-simple, and $\text{Soc}(N)$ is the largest semi-simple submodule of N. □

1.1.3 Semi-reduced modules

Definition 13. Over a commutative ring R, a module M is *reduced* if for all $r \in R$ and $m \in M$, $mr^2 = 0$ implies that $mr = 0$.

For a not necessarily commutative ring Lee and Zhou in (2004) defined an R-module M to be reduced if for all $r \in R$ and $m \in M$ $mr = 0$ implies that $Mr \cap mR = \{0\}$. This definition is equivalent to saying that for all $r \in R$ and $m \in M$, $mr^2 = 0$ implies that $mRr = \{0\}$. A submodule is reduced if it is reduced as a module. A submodule of a reduced module is reduced but a factor module of a reduced need not be reduced. The $\mathbb{Z}$-module $\mathbb{Z}$ is reduced but its factor module $\mathbb{Z}/n\mathbb{Z}$ is not reduced for a non-square free integer n.

Let $\text{Red}(M)$ denote the sum of all reduced submodules of M i.e.,

$$\text{Red}(M) := \sum_{i \in I} \{N_i \mid N_i \text{ is a reduced submodule of } M\}$$

where I is an indexing set.

Definition 14. An R-module M is *semi-reduced* if $\text{Red}(M) = M$.

Lemma 2. Over a commutative ring R for any R-module M the following implications hold:

$$\text{Simple} \Rightarrow \text{semi-simple} \Rightarrow \text{reduced} \Rightarrow \text{semi-reduced}.$$

Proof: We prove that a semi-simple module is reduced. The other implications follow from the definition of semi-simple and semi-reduced modules respectively. Since a simple R-module is prime[3] and every prime module is reduced, a simple module is reduced. Suppose that M is a semi-simple R-module where $m \in M$, $r \in R$ such that $mr^2 = 0$. Then,

[3]An R-module M for which $RM \neq \{0\}$ is prime if for all $a \in R$ and every $m \in M$, $am = 0$ implies that $m = 0$ or $aM = \{0\}$

$(m_1, m_2, \cdots, m_i, \cdots)r^2 = 0$ where $(m_1, m_2, \cdots, m_i, \cdots) = m \in M = \bigoplus_{i\in I} M_i$ for some simple modules M_i. Since every simple module is reduced, then $m_i r^2 = 0 \Rightarrow m_i r = 0 \; \forall \; i \in I$. Hence $mr = 0$, thus M is reduced. In particular $\mathrm{Soc}(M) \subseteq \mathrm{Red}(M)$ because $\mathrm{Soc}(M)$ is a semi-simple module and $\mathrm{Red}(M)$ is the sum of all reduced submodules of M. □

A reduced module need not be semi-simple, e.g., $\mathbb{Z}$ and $\mathbb{Q}$ are reduced but they are not semi-simple.

1.1.4 Some definitions on generalizations of injective modules

A1. Max-injective (or maximal injective) modules

Definition 15. An R-module M over a ring R is said to be max-injective if for any maximal right ideal L of R, every R-homomorphism $f : L \to M$ can be extended to an R-homomorphism $f' : R \to M$. A ring R is said to be right max-injective if the right regular module R_R is max-injective.

Example 1.13. Any injective module is max-injective.

Max-injective (or maximal injective) modules were introduced and investigated by Wang and Zhao, (2005). They first constructed an example to show that maximal injectivity is a proper generalization of injectivity. Then, they proved that any right R-module over a left perfect ring R is maximally injective if and only if it is injective. They also gave a partial affirmative answer to Faith's conjecture by further investigating the property of maximally injective rings. The concept of maximal injectivity helped them to get an approximation to Faith's conjecture, which asserts that every injective right R-module over any left perfect right self-injective ring R is the injective hull of a projective submodule.

A2. Simple-injective modules

Let M be any R-module. Any R-module N is generated by M or M-generated if there exists an epimorphism $\eta : \prod M_{j\in J} \to N$ for some indexing set J. An R-module N is said to be subgenerated by M if N is isomorphic to a submodule of an M-generated module. Injective modules in terms of Soc-injective, *simple*-injective and *min*-injective modules have been generalized by Özcan, *et al.*, (2008). According to Harada and Manabu, in 1965, if M and N are R-modules, M is called simple-N-injective if, for every submodule X of N, every homomorphism $\gamma : X \to M$ with $\gamma(X)$ simple extends to N. If $N = R$, M is called *simple*-injective, and if $M = N$, M is called *simple*-quasi-injective. Let $\overline{M} \in \sigma[M]$ be the full subcategory of the right R-modules whose objects are all right R-modules subgenerated by M. $\overline{M}$ is called strongly *simple*-injective in $\sigma[M]$ if $\overline{M}$ is *simple*-N-injective for all $N \in \sigma[M]$.

A3. Min-injective modules

An R-module M is called *min*-N-injective if, for every simple submodule S of N, every homomorphism $\gamma : S \to M$ extends to N. If $N = R$, M is called *min*-injective, and if $M = N$, M is called *min*-quasi-injective. Let $\overline{M} \in \sigma[M]$, then $\overline{M}$ is called strongly *min*-injective in $\sigma[M]$, if $\overline{M}$ is *min*-N-injective for all $N \in \sigma[M]$.

In 2008, Özcan *et al.*, showed that the notion of strongly *min*-injectivity and strongly *simple*-injectivity coincide. They proved that any module M is locally Noetherian if and only if every strongly *simple*-injective module in $\sigma[M]$ is strongly Soc-injective, and that if M is finitely generated self-projective, then M is Noetherian QF-module if and only if every strongly *simple*-injective module in $\sigma[M]$ is projective in $\sigma[M]$.

A4. Soc-injective rings and Soc-injective modules

I emphasize the notion of Soc-injectivity because it shares some properties with Red-injectivity which is defined later.

Definition 16. (Amin, *et al.*, 2005, Definition 1.1).

Let M and N be R-modules. M is called *socle-N-injective* (Soc-N-injective) if any R-homomorphism $f : \text{Soc}(N) \to M$ extends to N. Equivalently, for any semi-simple submodule K of N, any R-homomorphism $f : K \to M$ extends to N. An R-module M is called Soc-quasi-injective if M is Soc-M-injective. M is called Soc-injective, if M is Soc-R-injective. R is called right (self-) Soc-injective, if the module R_R is Soc-injective (equivalently, if R_R is Soc-quasi-injective).

Definition 17. (Amin, *et al.*, 2005, Definition 1.2).

An R-module M is called *strongly Soc-injective*, if M is Soc-N-injective for all R-modules N. A ring R is called strongly Soc-injective, if the module R_R is strongly Soc-injective.

In 2005, Amin, *et al.*, generalized injective modules in terms of Soc-injective modules. They showed that every injective module is strongly Soc-injective. They proved that:

1. most of the basic results on (quasi-) injective modules hold on (quasi-) Soc-injective modules;

2. (strongly) Soc-injectivity is a Morita invariant property of rings;

3. if M_R is a projective module, then every quotient of a Soc-M- injective right R-module is Soc-M-injective;

4. if M_R is finitely generated, then direct sums of Soc-M-injective right R-modules are Soc-M-injective if and only if Soc(M) is finitely generated, this is used to characterize the right quotient finite dimensional rings;

5. direct sums of Soc-M-injective right R modules are Soc-M-injective, for each cyclic R-module M if and only if R is right quotient finite dimensional ring;

6. an R-module M is strongly Soc-injective if and only if M can be decomposed as a direct sum of an injective module and a module with zero socle.

After extending the known results on right Noetherian rings for injective modules, they proved that a ring is right Noetherian if and only if every direct sum of strongly Soc-injective modules is again strongly Soc-injective. After obtaining the preceding result, they proved that a ring R is a right Noetherian V-ring if and only if every right R-module is strongly Soc-injective.

They also proved that, if N is a right R-module, then every strongly Soc-injective right R-module is simple-N-injective; and gave examples showing that the converse is not true. They characterized semi-Artinian, pseudo-Frobenius and quasi-Frobenius rings in terms of strongly strongly-Soc-injective modules. They also proved that every projective right R-module is strongly Soc-injective if and only if $R = E \bigoplus T$, where E and T are right ideals of R, E_R is $\sum$-injective (arbitrary direct sums of copies of E_R are injective) and Soc$(T_R) = \{0\}$. They showed also that a ring R is right pseudo-Frobenius if and only if R is right Kash strongly right Soc-injective ring extending a result of Osofsky on self-injective rings Osofsky, (1966).

Definition 18. (Pardo & Asensio, 1998). A ring R is called right CF-ring (FGF-ring) if every cyclic (finitely generated) right R-module embeds in a free module.

It was not known whether right CF-rings (FGF-rings) are right Artinian (quasi-Frobenius). The positive answer to this question was given in 2005 by Amin, *et al.*, by considering the ring R as strongly right Soc-injective. But if the strongly right Soc-injective ring is replaced by right simple-injective ring, the problem is yet to be solved.

The work on Soc-injective modules led to the definitions 19 and 20. Let M and N be right R-modules. Then:

Definition 19. An R-module M is called *Red-N-injective* if any R-homomorphism $f : K \to M$ extends to N for any semi-reduced submodule K of N. M is called *Red-quasi-injective* if it is Red-M-injective. M is called *Red-injective* if it is Red-R-injective. The ring R is *Red-injective* if the module R_R is Red-injective.

Definition 20. An R-module M is called *strongly-Red-injective*, if M is Red-N-injective for all R-modules N. A ring R is called strongly-Red-injective if the module R_R is strongly-Red-injective.

The following implications will be shown to hold:

$$M \text{ is injective} \Rightarrow\ M \text{ is strongly-Red-injective} \Rightarrow\ M \text{ is strongly-Soc-injective}$$

$$\Rightarrow\ M \text{ is simple-}N\text{-injective} \Rightarrow\ M \text{ is min-}N\text{-injective.}$$

These implications motivate me to study Red-injective modules since strongly-Red-injective modules carry more properties of injective modules than strongly Soc-injective modules which were done by Amin, *et al.*, (2005).

1.2 Statement of the problem

Generalizations of injective modules have been studied in many papers like Johnson and Wong, (1961), Jain and Singh, (1975), Azumaya, *et al.*, (1975) and Nicholson and Yousif, (2003). Recently, in 2005, Amin, *et al.*, generalized them in terms of Soc-injective modules. In this work, I have come up with Red-injective and strongly Red-injective modules to lie between injective and Soc-injective modules.

1.3 Objectives of the study

1.3.1 General objective

To define and characterize properties of Red-injective and strongly Red-injective modules which carry many properties of injective modules than other generalizations of injective modules.

1.3.2 Specific objectives

1. To extend definitions and basic properties of Soc-injectivity to Red-injectivity.

2. To characterize semi-Artinian and quasi-Frobenius rings in terms of strongly Red-injective modules.

3. To establish whether the notion of (strongly) Red-injectivity is different from that of min-injectivity, simple-injectivity and injectivity.

1.4 Significance of the study

The findings of this research will help to formulate other generalizations of injective modules which have many properties of injective modules than known generalizations of injective modules in literature. This leads to a weaker notion namely, "Red-injective" than "injective" which carries more properties of injective modules than the known generalizations in literature.

1.5 Methodology

1.2.1. The first specific objective has been achieved by using the known definitions like (Definition 16 and Definition 17) and properties like (Amin, *et al.*, 2005, Theorem 2.2) from Soc-injectivity in order to establish properties for Red-injective modules.

1.2.2. The second specific objective has been achieved by using definitions and some properties of semi-Artinian and quasi-Frobenius rings. Theorem 1.2 is a characterization of semi-Artinian rings in terms of strongly Soc-injective modules.

Theorem 1.2. (Amin, *et al.*, 2005, Theorem 3.6). The following conditions are equivalent:

1. R is right semi-Artinian.
2. Every strongly Soc-injective R-module is injective.
3. Every strongly Soc-injective R-module is quasi-continuous.

In particular, over a left perfect ring R, every strongly Soc-injective R-module is injective.

Proposition 2 is a characterization of quasi-Frobenius rings in terms of strongly Soc-injective modules.

Proposition 2. (Amin, *et al.*, 2005, Property 3.7). A ring R is quasi-Frobenius if and only if every strongly Soc-injective right R-module is projective.

Theorem 1.2 and Proposition 2 have been extended to strongly Red-injective modules.

1.2.3. The third specific objective has been achieved by constructing counter-examples to show that a strongly simple-injective module is not always strongly Soc-injective, a strongly Soc-injective module is not always strongly Red-injective, and a strongly Red-injective module is not always injective.

1.6 Literature Review

The concept of injective modules, which has greatly enriched the study of ring theory have been introduced. In view of Baer's Criterion, a series of weakened injectivity of rings has been intensively investigated by various authors. The properties of finitely injective rings have been studied. Using these, the work of Ikeda and Nakayama, (1954) was simplified. In 1970, Stenström extended the notion of injective module to that of FP-injective module, showing that a left FP-injective left coherent and right perfect ring are quasi-Frobenius. In 1995, Nicholson and Yousif studied principally injective rings, answering a question of Camillo. In 1997, Nicholson and Yousif introduced the notion of minimal injectivity. Replacing self-injectivity by minimal injectivity, they extended many properties of quasi-Frobenius and pseudo-Frobenius rings, see Nicholson and Yousif,(1997). Chen and Ding investigated general principally injective rings satisfying additional conditions, extending various known results, Chen and Ding, (1999).

In 1954, Ikeda and Nakayama characterized Nakayama's quasi-Frobenius rings as the self-injective Artinian rings. Later, Faith conjectured that every left (or right) perfect, right self-injective ring is quasi- Frobenius, which is fairly known as Faith's conjecture,see Faith and Hynh, (2002). This outstanding open problem about quasi-Frobenius rings has been extensively investigated by various authors under the assumption of weakened injectivity. Osofsky showed that if R is a left perfect ring and if J/J^2 ($J = rad(R)$ denotes the Jacobson radical for a ring R) is finitely generated, then R is right Artinian, Faith, (1976). From this, it follows that any one-sided perfect right self-injective ring is quasi-Frobenius if J/J^2 is finitely generated. Herbera and Shamsuddin showed that a left and right perfect, right self-injective ring R is quasi-Frobenius if J/J^2 is countably generated as left R-module, see Herbera and Shamsuddin, (1996). It has been proved that a left and right perfect self-injective ring R is quasi-Frobenius if and only if the second right socle of R is finitely generated as a right ideal. Xue showed that a right pseudo-Frobenius and one-sided perfect ring R is quasi-Frobenius if its second left socle is finitely generated as a left ideal (Xue, 1996).

It has been showed that finding a counterexample to Faith's conjecture depends on the existence of a vector space over a ring satisfying certain topological conditions, see Ara, *et al.*, (2000).

1.6.1 M-injective and strongly M-injective modules

Definition 21. (Anderson & Fuller, 1992, p.184) Let M, N and K be R-modules. A module N is said to be M-injective (or N is injective relative to M) if for each monomorphism $g : K \to M$ and each homomorphism $f : K \to N$ there is an R-homomorphism $\overline{f} : M \to N$ such that $f = \overline{f} \circ g$. M is called self-injective (or quasi-injective) if it is M-injective.

Necessary and sufficient conditions were obtained for a direct sum $\bigoplus_{\alpha\in J} A_\alpha$ of R-modules to be M- injective in the sense of Azumaya (Azumaya, *et.al.*, 1975). Using this result, it has been shown that if $(A_\alpha)_{\alpha\in J}$ is a family of R-modules with the property that $\bigoplus_{\alpha\in K} A_\alpha$ is M-injective for every countable subset K of J, then $\bigoplus_{\alpha\in J} A_\alpha$ is itself M-injective. Also in 1975 Azumaya, *et al.*, proved that arbitrary direct sums of M-injective modules are M-injective if and only if M is locally Noetherian, in the sense that every cyclic submodule of M is Noetherian. An R-module Q is strongly M-injective if every homomorphism of any submodule of M into Q can be extended to a homomorphism of $\prod M_{j\in J}$ into Q for any indexing J (Morimoto, 1983, p.165). Every injective module is strongly M-injective and every strongly M-injective module is M-injective. But the converse is not necessary true. It has been proved that a direct product of modules is strongly M-injective if and only if so are all its factors, which is the same case as M-injective modules (Morimoto, 1983).

1.6.2 Quasi-injective and Pseudo-injective modules

Definition 22. (Johnson & Wong, 1961)

Let R be a ring with identity not equal to zero. A right R-module is said to be quasi-injective (pseudo-injective) if for every submodule N of M, every R-homomorphism (R-monomorphism) of N into M can be extended to an R-endomorphism of M.

The direct sum of finitely many copies of quasi-injective modules is quasi-injective (Harada & Manabu, 1965). The pseudo-injective modules over a P.I.D (Principal Ideal Domain) are quasi-injective (Singh, 1968). If the direct sum of two copies of a pseudo-injective module M is pseudo-injective then M is quasi-injective (Jain & Singh, 1975). In 1975, Jain and Singh, showed that any pseudo-injective module over a generalized uniserial ring is quasi-injective and used this result to show that any torsion pseudo-injective module over a bounded hereditary Noetherian prime ring is quasi-hereditary. They gave the example showing that a pseudo-injective module over a hereditary Noetherian prime ring need not be quasi-injective. They also showed that torsion free pseudo-injective modules over prime Goldie rings are injective and this extends an early result of Singh in (1968). After, they showed that if R is a commutative ring and M a pseudo-injective R-module, then M is quasi-injective module; but the problem left open in the case when R is non-commutative. Later, it has been given a construction for pseudo-injective modules which are not quasi-injective (Teply, 1975).

1.6.3 S-injective modules and rings

Definition 23. An R-module M is called s-N-injective if every R-homomorphism $f : K \to M$ extends to N , where K is a submodule of the singular submodule $Z(N)$. M is called s-injective if M is s-R-injective. M is called strongly s-injective, if M is s-N-injective for all right R-modules N.

For example every nonsingular R-module is strongly s-injective. In particular, the ring of integers $\mathbb{Z}$ is s-injective, but not injective. The notions of s-injective modules and rings were introduced by Zeyada, (2014). Several properties characterizing s-injective modules have been investigated. Onwards, Zeyada showed that being right strong s-injectivity is a Morita invariant property of rings. Zeyada showed also the connection between s-injectivity condition and other injectivity conditions and established examples distinguishing s-injectivity from the other injective concepts such as min-injectivity and Soc-injectivity.

1.6.4 N-injective modules

n-injective modules have been discussed by Campos and Smith, (2012). Given a positive integer n, an R-module X is n-injective provided, for each n-generated right ideal A of R, every homomorphism $\theta : A \to X$ lifts to R. 1-injective modules are also called principally injective or simply P-injective. In addition, an R-module X is called F-injective if, for each finitely generated right ideal B of R, every homomorphism $\chi : B \to X$ lifts to R. Clearly a module is F-injective if and only if it is n-injective for every positive integer n. Next, an R-module X is called C-injective provided, for each countably generated right ideal C of R every homomorphism $\mu : C \to X$ can be lifted to R. In 2012, Campos and Smith proved that the following implications hold for a module X:

$$X \text{ is injective} \Rightarrow X \text{ is } C\text{-injective} \Rightarrow X \text{ is } F\text{-injective} \Rightarrow X \text{ is } n\text{-injective}$$

and

$$X \text{ is } (n+1) \text{ -injective} \Rightarrow X \text{ is } n \text{ -injective.}$$

for every positive integer n.

Lemma 3. Let R be a ring, let X be an R-module, let G be a finitely generated submodule of a free R-module F and let $\psi : G \to X$ be a homomorphism. Then ψ lifts to F if and only if ψ lifts to H for every finitely generated $(free)$ submodule H of F containing G.

According to Nicholson and Yousif, (2003) in their book of *Quasi-Frobenius Rings*, a module M over a ring R is called finitely presented provided there exists a finitely generated free R-module F and a finitely generated submodule K of F such that $M \cong F/K$. In addition, an R-module X is called FP-injective (*or absolutely pure*) if, for every finitely generated free R-module F and finitely generated submodule K of F, every homomorphism $\psi : K \to X$ can be lifted to F. It is has been proved that an R-module X is FP-injective if and only if for every R-module M and submodule L of M such that the module M/L is finitely presented, every homomorphism $\alpha : L \to X$ can be lifted to M (W. K. Nicholson & Yousif, 2003). The following implications hold for a module X:

$$X \text{ is injective} \Rightarrow\ X \text{ is } FP\text{-injective} \Rightarrow\ X \text{ is } F\text{-injective.}$$

Let n be a positive integer. A module X over a ring R is nP-injective provided for every free R-module F and n-generated submodule G of F, every homomorphism $\psi : G \to X$ can be lifted to F (Campos & Smith, 2012). A module is FP-injective if and only if it is nP -injective for every positive integer n. Moreover, for any module X, Campos and Smith proved the following implications:

$$X \text{ is } FP\text{-injective} \Rightarrow\ X \text{ is } (n+1)P\text{-injective} \Rightarrow\ X \text{ is } nP\text{-injective}$$

and

$$X \text{ is } nP\text{-injective} \Rightarrow\ X \text{ is } n\text{-injective}$$

for every positive integer n.

For a right semi-hereditary ring R, it has been proved that a right R-module X is F-injective if and only if it is FP-injective. Until now it is not yet known whether there is an example of a ring R and an F-injective R-module X such that X is not FP-injective (Campos & Smith, 2012).

Chapter 2

Injectivity and Related Concepts

I have already introduced some generalizations of injective modules in Chapter 1. In this chapter, I discuss more generalizations of injective modules, which will be used in Chapters 3 and 4.

2.1 Injective modules

Injective module was defined in Chapter 1, in this section some properties of injective modules are given and it is proved that every module can be embedded in an injective module.

Firstly, a useful test for injectivity in Lemma 4 which is called Injective Test Lemma or Baer's Criterion is given. Recall that a right R-module M is injective relative to R if M is R-injective, i.e., for every ideal I of R any R-homomorphism $f : I \to M$ extends to R.

Lemma 4. (Anderson & Fuller, 1992, Injective Test Lemma)

The following statements about an R-module M are equivalent:

1. M is injective;
2. M is injective relative to R;
3. For every right ideal $I \leq R_R$ and every R-homomorphism $h : I \longrightarrow M$ there exists an $x \in M$ such that h is a left multiplication by x, i.e., $h(a) = xa$ for $a \in I$.

Proof:

$1 \Rightarrow 2$. Suppose that M_R is an injective module. Then, it is R-injective. Hence, M_R is injective relative to R.

$2 \Rightarrow 1$. Suppose that M_R is injective module relative to R. Then, M_R is R_R injective. Hence M_R is injective.

$2 \Rightarrow 3$. If M is R_R-injective and $I \leq R_R$ with $h : I \longrightarrow M$, then there is an $\overline{h} : R \longrightarrow M$ such that $(\overline{h}|_I) = h$. Let $x = h(1)$. Then $h(a) = \overline{h}(a) = \overline{h}(1)a = xa$ for all $a \in I$.

$3 \Rightarrow 2$. If $I \leq R_R$, $x \in M$ and $h(a) = xa$ for all $a \in I$, then left multiplication by x, $\rho(x) : R \longrightarrow M$, extends h, thus 3 implies M is R_R-injective.

□

Definition 24. A non-zero element a in a ring R is said to have a left (resp. right) inverse b if $ba = 1$ (resp. $ab = 1$). An element a is invertible or a unit in R if it has a left and a right inverse. If $1 \neq 0$ in R, and all non-zero elements are invertible, then R is called a division ring. A commutative division ring is called a field.

Definition 25. An element a of a commutative ring R is called a zero divisor if there is a non-zero element $b \in R$ such that $ab = 0$. An element $a \in R$ that is not a zero divisor is called a non-zero divisor. If all non-zero elements of a commutative ring are non-zero divisors, then R is called an integral domain. A nonempty subset S of a ring R is called a sub-ring of R if S is a ring with respect to the addition and multiplication in R.

Definition 26. Let R be an integral domain and let M be an R-module. M is a divisible R-module if $Mr = M$, for all $0 \neq r \in R$.

Lemma 5 below shows that injective modules and divisible modules are the same over the ring $\mathbb{Z}$.

Lemma 5. (Anderson & Fuller, 1992) A right R-module M is divisible if and only if M is injective as a $\mathbb{Z}$-module.

Proof:

$(\Rightarrow)$. Every non-zero ideal of $\mathbb{Z}$ is of the form, $n\mathbb{Z}$, $n \neq 0 \in \mathbb{Z}$. If M is divisible and $h : n\mathbb{Z} \rightarrow M$, then there exist an element $b \in M$ with $h(n) = bn$ and $h(jn) = h(n)j = b(jn)$ $\forall\ jn \in n\mathbb{Z}$. Then, by Injective Test Lemma, M is injective relative to $\mathbb{Z}$.

(⇐). If $M_{\mathbb{Z}}$ is injective, $a \in M_{\mathbb{Z}}$ and $0 \neq n \in \mathbb{Z}$, then $h : jn \to ja$ defines a homomorphism $h : n\mathbb{Z} \to \mathbb{Z}$ which, by the Injective Test Lemma 4, must be multiplication by some $b \in M$. But, then $a = h(n) = bn$.

□

Example 2.1. $\mathbb{Q}_{\mathbb{Z}}$ and $\mathbb{R}_{\mathbb{Z}}$ are injective modules.

Definition 27. A ring R is called a Principal Ideal Domain (P.I.D) if it is commutative without zero divisors and every ideal in it is principal, i.e., generated by one element.

Theorem 2.1. Over a Principal Ideal Domain R, a module M is injective if and only if it is divisible.

Proof: Consider the problem of extending a map of a principal ideal $aR \longrightarrow M$ to all of R where $a \in R$. If $a = 0$ the map is 0 and the 0 map can be used as the required extension. If $a \neq 0$, then since $aR \cong R$ is free on the generator a, the homomorphism to be extended might take any value $e \in M$ on a. To extend the homomorphism, we must specify the value e' of the extended homomorphism on 1 in such a way that the extended homomorphism takes a to e; the condition that e' must satisfy is precisely that $ae' = e$. Thus, M is divisible if and only if every homomorphism of a principal ideal of R to M extends to a homomorphism of R to M. The result is now obvious, considering that in a principal ideal domain every ideal is principal. □

Theorem 2.2. (Lam, 1999, Theorem 3.20). If R is any ring, every R-module embeds in an injective R-module.

Proof: It is enough to show that every $\mathbb{Z}$-module embeds in an injective $\mathbb{Z}$-module. Let A be an Abelian group. Then, $A \cong F/K$ for some free Abelian group F. Now F embeds in the divisible Abelian group $Q = \mathbb{Q} \otimes_{\mathbb{Z}} F$. Note that Q/K is still divisible and that A embeds in the divisible Abelian group Q/K. By Lemma 5, divisible Abelian groups are injective $\mathbb{Z}$-modules. □

Corollary 1. If L is a right R-module, then L is injective if and only if every short exact sequence of right R-modules $\{0\} \to L \to M \to N \to \{0\}$ splits.

Proof: Suppose that every exact sequence $\{0\} \to L \to M \to N \to \{0\}$ splits. By Theorem 2.2, we can find such a sequence with M injective. By hypothesis, $M \cong L \oplus N$, hence L is injective. Conversely, suppose that L is injective and let $\{0\} \to L \xrightarrow{i} M \to N \to \{0\}$ be a short exact sequence. By Proposition 1, there is a map $g : M \longrightarrow L$ with $g \circ i = id_L$. Hence, the sequence splits. □

A ring R is said to satisfy the descending chain condition (dcc) on left ideals if every descending chain of left ideals $L_1 \supset L_2 \supset L_3 \supset \cdots$ becomes stationary after a finite number of steps, i.e., for some $k \in \mathbb{N}$ we get $L_k = L_{k+1} = L_{k+2} = \cdots$. If this condition is satisfied, R is called a left Artinian ring. Similarly right Artinian rings are defined. An Artinian ring is a ring which is both left and right Artinian. A ring R is left (right) Noetherian if R satisfies the ascending chain condition (acc) on left (right) ideals. If R is commutative notions of right and left Noetherian coincide, and R is said to be Noetherian.

Proposition 3 characterize right Noetherian rings.

Proposition 3. (Anderson & Fuller, 1992, Proposition 18.13). A ring R is right Noetherian if and only if every direct sum of injective right R-modules is injective.

Proof:

($\Rightarrow$). Suppose that every direct sum of injective right R-modules is injective and that I_1, $I_2, \cdots$ is an ascending chain of right ideals in R. Let $I = \cup_{i=1}^{\infty} I_i$. If $a \in I$, then $a \in I_i$ for all but finitely many $i \in \mathbb{N}$. Then, there is an $f : I \to \bigoplus_{i=1}^{\infty} E(R/I_i)$ defined via $\pi_i f(a) = a + I_i$ $(a \in I)$. By the Injective Test Lemma, there is an $x \in \bigoplus_{i=1}^{\infty} E(R/I_i)$ such that $f(a) = xa$ for all $a \in I$. We choose n such that $\pi_{n+k}(x) = 0$, $k = 0,\ 1, \cdots$. So $I/I_{n+k} = \pi_{n+k}(f(I)) = \pi_{n+k}(Ix) = I\pi_{n+k}(x) = \{0\}$ or equivalently, $I_n = I_{n+k}$ for all $k = 0,\ 1,\ 2, \cdots$.

($\Leftarrow$). If R is right Noetherian, $I \leq R_R$ and $f : I \to \bigoplus_A E_\alpha$, then since I is finitely generated, Imf is contained in $\bigoplus_F E_\alpha$. Now, applying the fact that direct products and direct summands of injective modules are injective, and the Injective Test Lemma, the proof is complete.

□

A ring R is left(resp. right) hereditary in case each of its left (resp. right) ideals is projective. R is hereditary if it is both left and right hereditary. For example, every P.I.D is left hereditary.

Theorem 2.3. (Matlis, 1958, Theorem 1.4). Let R be any ring. The sum of two injective submodules of an R-module is always injective if and only if R is left-hereditary.

Proof: If R is left-hereditary and N_1, N_2 are injective submodules of an R-module N, then $N_1 + N_2$ is a homomorphic image of the injective R-module $N_1 \bigoplus N_2$, and hence is injective. Conversely, assume that the sum of two injective submodules of any R-module is injective. Let M be any injective R-module and H a submodule of M. We show that

M/H is injective, and this proves that R is left-hereditary. Let M_1, M_2 be two copies of M, $N = M_1 \bigoplus M_2$ and D the submodule of N consisting of the elements (h, h), where $h \in H$. The canonical homomorphism $N \to N/D$ maps M_1, M_2 isomorphically onto submodules $\overline{M}_1$, $\overline{M}_2$ of N/D, respectively. Since $N/D = \overline{M_1} \bigoplus \overline{M_2}$, N/D is injective and, therefore, $(N/D)/\overline{M_1}$ is injective. The composite mapping $M \to M_2 \to \overline{M}_2 \to (N/D)/\overline{M}_1$ defines a homomorphism of M onto $(N/D)/\overline{M}_1$ with kernel H. Therefore, M/H is injective. $\square$

Definition 28. A right R-module $E \geq M$ is called an essential extension of M (written $M \subseteq^e E$), if $M \cap N \neq \{0\}$ $\forall$ $N \leq E$; $N \neq \{0\}$. An essential extension $E \geq M$ is said to be maximal if no module containing E can be an essential extension of M. If $E \geq M$ is an essential extension; M is called an essential or (large) submodule of E.

Remark 3. (Lam, 1999, Remark 3.27)

1. $M \subseteq^e E$ if and only if, for any non-zero $a \in E$, $\exists$ $r \in R$ such that $0 \neq ar \in E$.
2. If $M \subseteq^e E$ and $E \subseteq^e E'$, then $M \subseteq^e E'$.

Lemma 6. (Lam, 1999, Lemma 3.28) A right R-module M is injective if and only if it has no proper essential extensions.

Proof: Assume that M is injective, and consider any proper essential extension $E > M$. By Corollary 1, we have $E = M \oplus N$ for some submodule $N \neq \{0\}$, so $E \geq M$ is not an essential extension. Conversely, assume that M has no proper essential extensions, and embed M in an injective right R-module I. By Zorn's Lemma, there exists a submodule $S \leq I$ maximal with respects to the property that $S \cap M = \{0\}$. Then in the quotient I/S, any non-zero submodule S'/S intersects the image of M nontrivially, so $\mathrm{Im}(M) \subseteq^e I/S$. By assumption, we must have $\mathrm{Im}(M) = I/S$. This means that $I = M \oplus S$, so M is an injective module by Proposition 1. $\square$

Lemma 7. (Lam, 1999, Lemma 3.29) Any right R-module M has an essential extension.

Proof: Fix an injective module $I \geq M$, and consider any family of essential extensions of M in I that are linearly ordered by inclusion. By Remark 1, the union of the family is also essential over M. By Zorn's Lemma, it follows that we can find a submodule E maximal with respect to the property that $M \subseteq^e E \leq I$. We claim that E is a maximal essential extension of M. Indeed, if this is false, we could be able to find an embedding $E < E'$ such that $M \subseteq^e E'$. By the injectivity of I, the inclusion map $E \subseteq I$ can be extended to some $g : E' \to I$. Then, $\mathrm{Ker}\ g \cap M = \{0\}$, so $M \subseteq^e E'$ implies that $\mathrm{Ker}\ g = \{0\}$. We can therefore identify E' with $g(E')$. But then, $M \subseteq^e E'$ contradicts the maximality of E. $\square$

Theorem 2.4. (Lam, 1999, Theorem 3.30) For modules $M \leq I$, the following conditions are equivalent:

1. I is maximal essential over M.
2. I is injective, and is essential over M.
3. I is minimal injective over M.

Proof:

$1 \Rightarrow 2$. By Remark 1, I being maximal essential over M implies that I has no proper essential extension. Therefore, I is injective by Lemma 6.

$2 \Rightarrow 3$. Let I' be an injective module such that $M \leq I' \leq I$. By Corollary 1 page 20, $I = I' \oplus N$ for some submodule $N \leq I$. Since $N \cap M = \{0\}$, we must have $N = \{0\}$ (since $M \subseteq^e I$), so $I = I'$.

$3 \Rightarrow 1$. Assume that I is a minimal injective module over M. The proof of Lemma 7 shows that there exists a submodule $E \leq I$ that is maximal essential over M. Using $1 \Rightarrow 2$, we know that E is injective, and therefore $E = I$, which proves $3 \Rightarrow 1$.

□

Definition 29. If the module $M \leq I$ satisfies the (equivalent) conditions 1, 2, 3 in Theorem 2.4, then I is an injective hull (or injective envelope) of M.

Corollary 2. Any R-module M has an injective hull.

Proof: By Lemma 7, M has a maximal essential extension I. By Theorem 2.4, I is an injective hull of M. □

Corollary 3. (Lam, 1999, Corollary 3.32) Any two injective hulls, I, I' of M are isomorphic over M; that is, there exists an isomorphism $g : I' \to I$ which is an identity over M.

Proof: By the injectivity of I, we can find $g : I' \to I$ extending the inclusion map $M \to I$. As in the proof of Lemma 7, Ker $g = \{0\}$, since $M \subseteq^e I'$. Therefore, $g(I')$ is an injective submodule of I containing M. Now the condition 3 in Theorem 2.4 implies that $g(I') = I$, so $g : I' \longrightarrow I$ is the desired isomorphism. □

Example 2.2. (Lam, 1999, Example 3.36) Let the ring $R = \mathbb{Z}$. Let C_n denote the cyclic group of order n. For any prime p, let C_{p^∞} (the "Prufer p-group") be the ascending union of the groups $C_p \subset C_{p^2} \subset C_{p^3} \subset ....$ Then, C_{p^∞} is p-divisible. (It is isomorphic to p-primary part of $\mathbb{Q}/\mathbb{Z}$). By Lemma 5, C_{p^∞} is $\mathbb{Z}$-injective, and by Remark 3, C_{p^∞} is essential over any C_{p^i} $(i \geq 1)$. Therefore, the injective hull of C_{p^i} is C_{p^∞} $\forall\ i \geq 1$.

2.2 M-injective modules

In this section, it is shown that M injectivity is closed under direct sums. If N_R is M_R-injective module and $B \leq M_R$, then N_R is B-injective and M_R/B-injective. Furthermore, it is shown that for a module N_R to be M_R-injective is a Morita invariant property.

Theorem 2.5. (Anderson & Fuller, 1992, Theorem 3.6). Let M, M', and N be R-modules and let $f : M \to N$ be an R homomorphism. If $g : M \to M'$ is an epimorphism with $\mathrm{Ker}(g) \subseteq \mathrm{Ker}(f)$, then there exists a unique homomorphism $h : M' \to N$ such that $f = h \circ g$. Moreover, $\mathrm{Ker}(h) = g(\mathrm{Ker}(f))$ and $\mathrm{Im}(h) = \mathrm{Im}(f)$, so that h is monic if and only if $\mathrm{Ker}(g) = \mathrm{Ker}(f)$ and h is epic if and only if f is epic.

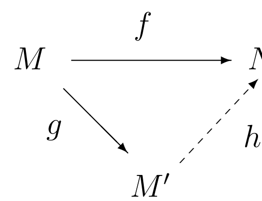

Figure 2.1

Proof: Since $g : M \to M'$ is epic, for each $m' \in M'$ there is at least one $m \in M$ with $g(m) = m'$. If also $l \in M$ with $g(l) = m'$, then $m - l \in \mathrm{Ker}(g)$. But since $\mathrm{Ker}(g) \subseteq \mathrm{Ker}(f)$, we have that $f(m) = f(l)$. Thus there is a well defined function $h : M' \to N$ such that $f = h \circ g$. To see actually if h is an R-homomorphism, let $x', y' \in M'$ and $x, y \in M$ with $g(x) = x', g(y) = y'$. Then, for each $a, b \in R$, $g(xa + yb) = x'a + y'b$, so that

$$h(x'a + y'b) = f(xa + yb) = f(x)a + f(y)b = h(x')a = h(y')b.$$

Since g is an epimorphism, then h is unique. If $\mathrm{Ker}(h) = g(\mathrm{Ker}(f))$, then h is monic if and only if $\mathrm{Ker}(g) = \mathrm{Ker}(f)$. If $\mathrm{Im}(h) = \mathrm{Im}(f)$, then h is epic if and only if f is epic. □

Proposition 4. (Anderson & Fuller, 1992, Proposition 16.8). Let U and M be R-modules. Then, U is M-injective if and only if for each submodule K of M every R-homomorphism

$h : K \to U$ can be extended to an R-homomorphism $\overline{h} : M \to U$ (i.e., every $h : K \to U$ factors through the natural monomorphism $i_K : K \to M$).

Proof:

($\Rightarrow$) This implication holds because every submodule $K \leq M$ is an R-module.

($\Leftarrow$) This follows from Theorem 2.5.

□

In fact, an R-module N is said to be injective if it is M-injective for all R-modules M.

Lemma 8. If N is M-injective, then any monomorphism $g : N \to M$ splits.

Proof: Let $\overline{f} : M \to N$ be an extension of a homomorphism $f : X \to N$ where $X \leq M$, and $g : N \to M$ be a monomorphism. We show that g splits at M. Since both N and M are unital, then $\overline{f}(1_M) = 1_N$ and $g(1_N) = 1_M$. Hence $g \circ \overline{f}(1_M) = g(1_N) = 1_M$, thus g splits at M. □

Proposition 5. (Mohamed & Müller, 1990, Proposition 1.3) Let N be an M-injective R-module. If $B \leq M$, then N is B-injective and M/B-injective.

Proof: Consider the diagram in Figure 2.2 below

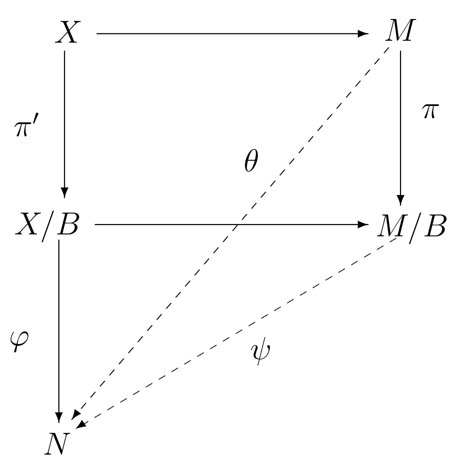

Figure 2.2

N is B-injective, because, there exists an inclusion map $i : B \to M$ such that $\overline{f} \circ i : B \to N$ is an extension of $f : X \to N$, where $\overline{f}$ is a homomorphism from M to N. Let X/B be a submodule of M/B, and $\varphi : X/B \to N$ be a homomorphism. Let π denote the natural homomorphism of M onto M/B and $\pi' = \pi|_X$. Since N is M-injective, there exist a homomorphism $\theta : M \to N$ that extends $\varphi \circ \pi'$. Now $\theta(B) = \varphi \circ \pi'(B) = \varphi(\{0\}) = \{0\}$. Hence Ker $\pi \leq$ Ker θ, and consequently there exists $\psi : M/B \to N$ such that $\psi \circ \pi = \theta$. For every $x \in X$, $\psi(x + B) = \psi\pi(x) = \theta(x) = \varphi\pi'(x) = \varphi(x + B)$. Thus ψ extends φ, and therefore N is M/B injective. □

Proposition 6 is a generalization of Baer's Criterion.

Proposition 6. (Mohamed & Müller, 1990, Proposition 1.4) An R-module N is M-injective if and only if N is aR-injective for every $a \in M$.

Proof: The only if part follows by the preceding proposition. Conversely, assume that N is aR-injective for every $a \in M$. Let $X \leq M$ and $\varphi : X \to N$ be a homomorphism. By Zorn's Lemma, we can find a pair $(B,\ \psi)$ maximal with the properties $X \leq B \leq M$ and $\psi : B \to N$ is a homomorphism which extends φ. It is clear that $B \subseteq^e M$. Suppose that $B \neq M$ and consider an element $a \in M - B$. Let $K = \{r \in R : ar \in B\}$; then it is clear that $aK \neq \{0\}$. Define $\mu : aK \to N$ by $\mu(ak) = \psi(ak)$. Then, by assumption μ can be extended to $\nu : aR \to N$. Now, define $\chi : B + aR \to N$ by $\chi(b + ar) = \psi(b) + \nu(ar)$. Then, χ is well defined, since if $b + ar = 0$, then $r \in K$ and so $\psi(b) + \nu(ar) = \psi(b) + \mu(ar) = \psi(b) + \psi(ar) = \psi(b + ar) = 0$. But then the pair $(B + aR,\ \chi)$ contradicts the maximality of $(B,\ \psi)$. Hence $B = M$, and $\psi : M \to N$ extends φ. □

Proposition 7. (Mohamed & Müller, 1990, Proposition 1.5) An R-module N is $\bigoplus_{i \in I} M_i$-injective if and only if N is M_i-injective for every $i \in I$, and M_i an R-module.

Proof: Assume that N is M_i-injective for all $i \in I$. Let $M = \bigoplus_{i \in I} M_i$, $X \leq M$ and consider a homomorphism $\varphi : X \to N$. We may assume, by Zorn's Lemma, that φ cannot be extended to a homomorphism $X' \to N$ for any submodule X' of M which contains X properly. Then, $X \subseteq^e M$. We claim that $X = M$. Suppose not. Then, there exist $j \in I$ and $a \in M_j$ such that $a \notin X$. Since N is M_j-injective, N is aR-injective by Proposition 5. By an argument similar to that given in Proposition 6, we can extended φ to a homomorphism $\psi : X + aR \to N$, which contradicts the maximality of φ. This proves our claim, and hence N is M-injective. The converse follows by Proposition 6. □

As a direct product of injective R-modules is injective if and only if every component is injective, the same is for M-injective modules.

Proposition 8. (Mohamed & Müller, 1990, Proposition 1.6). $\Pi_{\alpha\in\Lambda}K_\alpha$ is M-injective if and only if K_α is M-injective for every $\alpha \in \Lambda$.

Proof: Suppose that all K_α are M-injective. Then, for any $\mu \in \Lambda$ and any $N \leq M$, a diagram in Figure 2.3

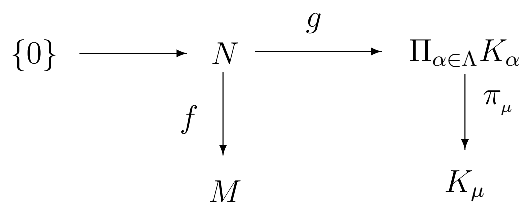

Figure 2.3

can be extended commutatively by $h_\mu : M \to K_\mu$. Hence by the universal property of direct products, we have an $h : M \to \Pi_{\alpha\in\Lambda}K_\alpha$ with $\pi_\mu h = h_\mu$ and $\pi_\mu hf = h_\mu f = \pi_\mu g$, this implies that $hf = g$. Thus $\Pi_{\alpha\in\Lambda}K_\alpha$ is M-injective. Conversely, suppose that $\Pi_{\alpha\in\Lambda}K_\alpha$ is M-injective, then a diagram in Figure 2.4

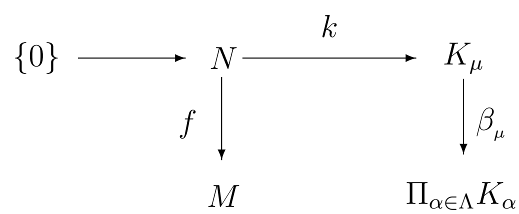

Figure 2.4

can be extended commutatively by $\delta : M \to \Pi_{\alpha\in\Lambda}K_\alpha$ and $\beta_\mu k = \delta f$ immediately yields $k = \pi_\mu \beta_\mu k = \pi_\mu \delta f$. Thus K_μ is M-injective. □

Furthermore, in Proposition 9, it is shown that for an R-module M, the property of M-injectivity is a Morita invariant property.

2.2.1 Categories

Categories of modules, functors, and equivalent rings are defined since they are used in Proposition 9.
The following are some properties of morphisms.

Let $Obj(\mathcal{C})$ be class of objects. Then:

1. For every ordered pair (A, B) of objects in $\mathcal{C}$ there exists a set $Mor_{\mathcal{C}}(A, B)$, the morphisms from A to B, such that $Mor_{\mathcal{C}}(A, B) \cap Mor_{\mathcal{C}}(A', B') = \emptyset$ for $(A, B) \neq (A', B')$.

2. A composition of morphisms, i.e., a map $Mor_{\mathcal{C}}(A, B) \times Mor_{\mathcal{C}}(B, C) \to Mor_{\mathcal{C}}(A, C)$, $(f, g) \mapsto g \circ f$, for every triple (A, B, C) of objects in $\mathcal{C}$, with the properties:

   (1) It is associative: For A, B, C, D in $Obj(\mathcal{C})$ and $f \in Mor_{\mathcal{C}}(A, B)$, $g \in Mor_{\mathcal{C}}(B, C)$, $h \in Mor_{\mathcal{C}}(C, D)$ we have $(h \circ g \circ f = h \circ (g \circ f)$;

   (2) There are identities: For every $A \in Obj(\mathcal{C})$ there are unique morphisms $1_A \in Mor_{\mathcal{C}}(A, A)$, the identity of A, and $1_B \in Mor_{\mathcal{C}}(B, B)$, the identity of B with $f \circ 1_A = 1_B \circ f = f$ for every $f \in Mor_{\mathcal{C}}(A, B)$, $B \in Obj(\mathcal{C})$.

A morphism $f : A \to B$ in $\mathcal{C}$ is called an isomorphism in case there is a morphism $f^{-1} : B \to A$ in $\mathcal{C}$ such that $f^{-1} \circ f = 1_A$ and $f \circ f^{-1} = 1_B$. An example of a category is a category of rings, where objects are all (associative) rings, morphisms are exactly ring homomorphisms and compositions are compositions of homomorphisms.

2.2.2 Categories of modules

Given two R-modules M and N, every R-homomorphism $f : M \to N$ is an element of the set of homomorphisms from $M \to N$. In particular, these homomorphisms form a set denoted by $\mathrm{Hom}_R(M, N)$. In fact, if M and N are right R-modules, then $\mathrm{Hom}_R(M, N)$ is an Abelian group with respect to the operation of addition $(f, g) \mapsto f + g$ defined by $(f + g)(x) = f(x) + g(x)$ for all $x \in M$. Given a ring R, the category of right R-modules is the system $\mathfrak{M}_R = (mod_R,\ \mathrm{Hom}_R,\ \circ)$ where mod_R is the class of all right R-modules, $\mathrm{Hom}_R : (M, N) \mapsto \mathrm{Hom}_R(M, N)$, and $\circ$ is the usual composition of functions.

Definition 30. Let $\mathfrak{M}_R$ be a category. A homomorphism $f : A \longrightarrow B$ in $\mathfrak{M}_R$ is called :

1. a monomorphism if, for $g,\ h \in \mathrm{Hom}_R(C, A)$, $C \in mod_R$ then, $f \circ g = f \circ h$ implies $g = h$;

2. an epimorphism if, for $g,\ h \in \mathrm{Hom}_R(B, D)$, $D \in mod_R$ then, $g \circ f = h \circ f$ implies $g = h$;

3. a bimorphism if f is both a monomorphism and an epimorphism;

4. a retraction if there exists $g \in \mathrm{Hom}_R(B, A)$ with $f \circ g = 1_B$;

5. a coretraction if there exists $g \in \mathrm{Hom}_R(B, A)$ with $g \circ f = 1_A$;

6. an isomorphism if f is both a retraction and a coretraction;

7. a (left and right) zero morphism if, for any $g,\ h \in \mathrm{Hom}_R(D, A)$, $D \in mod_R$, $f \circ g = f \circ h$, and, for any $g',\ h' \in \mathrm{Hom}_R(B, D')$, $D' \in mod_R$, $g' \circ f = h' \circ f$.

(a) Functors between module categories

Let R and S be ring and, $\mathfrak{M}_R$ and $\mathfrak{M}_S$ be module categories. The functor $F : \mathfrak{M}_R \to \mathfrak{M}_S$ is additive in case for each M, N in $\mathfrak{M}_R$, and each pair $f,\ g : M \to N$ in $\mathfrak{M}_R$, $F(f + g) = F(f) + F(g)$. If F is additive and covariant, then the restriction $F : \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_S(F(M), F(N))$ is an Abelian group homomorphism, whereas if F is additive and contravariant, then the restriction $F : \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_S(F(N), F(M))$ is an Abelian group homomorphism.

2.2.3 Equivalent rings

Let $\mathcal{C}$ and $\mathcal{D}$ be arbitrary categories. Then a covariant functor $F : \mathcal{C} \to \mathcal{D}$ is a categorical equivalence in case there is a functor (necessarily covariant) $G : \mathcal{D} \to \mathcal{C}$ and natural isomorphisms $GF \cong 1_{\mathcal{C}}$ and $FG \cong 1_{\mathcal{D}}$. A functor G with this property (also a category equivalence) is called an *inverse equivalence* of F. Two categories are equivalent in case there exists a category equivalence from one to the other. In case $\mathcal{C}$ and $\mathcal{D}$ are equivalent we write $\mathcal{C} \approx \mathcal{D}$. In fact, this defines an equivalence relation on the class of all categories.

Definition 31. (Anderson & Fuller, 1992, p.250) Two rings R and S are (Morita) equivalent abbreviated $R \approx S$ in case $\mathfrak{M}_R \approx \mathfrak{M}_S$ (that is to say there are two additive category equivalences F and its inverse G between these categories of modules).

Definition 32. (Anderson & Fuller, 1992, p.251) Let R and S be a pair of equivalent rings. Specifically, assume that $F : \mathfrak{M}_R \to \mathfrak{M}_S$ and $G : \mathfrak{M}_S \to \mathfrak{M}_R$ are inverse (additive) equivalences. In particular, $GF \cong 1_{\mathfrak{M}_R}$ and $FG \cong 1_{\mathfrak{M}_S}$; that is, there exists natural isomorphisms $\eta : GF \to 1_{\mathfrak{M}_R}$ and $\zeta : FG \to 1_{\mathfrak{M}_S}$. This means that for each right R-module M there is an isomorphism $\eta_M : GF(M) \to M$ in $\mathfrak{M}_R$ such that for each M, M' in $\mathfrak{M}_R$ and each $f : M \to M'$ in $\mathfrak{M}_R$, the diagram in Figure 2.5 commutes.

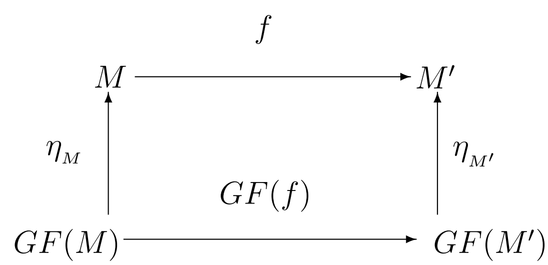

Figure 2.5

(Of course parallel remarks apply to ζ). Now for each module M in $\mathfrak{M}_R$ and each module N in $\mathfrak{M}_S$, there are $\mathbb{Z}$-homomorphisms

$$\phi = \phi_{MN} : \mathrm{Hom}_S(N, F(M)) \to \mathrm{Hom}_R(G(N), M)$$

$$\theta = \theta_{MN} : \mathrm{Hom}_S(F(M), N) \to \mathrm{Hom}_R(M, G(N))$$

defined via

$$\phi_{MN} : \gamma \mapsto \eta_M \circ G(\gamma)$$
$$\theta_{MN} : \delta \mapsto G(\delta) \circ \eta_M^{-1}.$$

Lemma 9. (Anderson & Fuller, 1992, Proposition 21.2) Let $F : \mathfrak{M}_R \to \mathfrak{M}_S$ be a category equivalence. Then for each M, M' in $\mathfrak{M}_R$ the restriction of F to $\mathrm{Hom}_R(M, M')$ is an Abelian group isomorphism $F : \mathrm{Hom}_R(M, M') \to \mathrm{Hom}_S(F(M), F(M'))$ such that $F(f)$ is an epimorphism (resp. monomorphism) in $\mathfrak{M}_S$ if and only if f is an epimorphism (resp. monomorphism) in $\mathfrak{M}_R$. Moreover, if $M \neq \{0\}$, then, this restriction $F : End(M) \to End(F(M))$ is a ring isomorphism.

Proof: Since F is additive, these restrictions are Abelian group homomorphisms. To show that $F : End(M) \to End(F(M))$ is a ring homomorphism, let $f \in End(M)$. Then, $f : M \to M$ belongs to $\mathfrak{M}_R$. Hence, $F(f) : F(M) \to F(M)$ is an endomorphism of $F(M)$. i.e., for $f,\ g \in End(M)$, $F(g \circ f) = F(g) \circ F(f)$. Since F is additive and preserves an identity map, it restricts to the ring homomorphism. To finish the proof, we shall adopt the notation of Definition 32. Then for each M and M' in $\mathfrak{M}_R$ $H : \mathrm{Hom}_S(F(M), F(M')) \to \mathrm{Hom}_R(M, M')$ defined by

$$H : g \mapsto \eta_{M'} G(g) \eta_M^{-1}$$

is a $\mathbb{Z}$-homomorphism. Moreover, it is monic, for if $H(g) = 0$, then $G(g) = 0$, so $g = \zeta_{F(M')} FG(g) \zeta_{F(M)}^{-1} = 0$. But now, for all $f \in \mathrm{Hom}_R(M, M')$

$$HF(f) = \eta_{M,} GF(f) \eta_M^{-1} = f.$$

It follows that H is an epimorphism. Thus H is an isomorphism with inverse F. Therefore, F is an isomorphism. Now, from Definition 32, f is monic (epic) if and only if $GF(f)$ is monic (epic). So suppose that f is monic and that for some h in $\mathfrak{M}_R$, $F(f)h = 0$. Then since G is an additive functor and $GF(f)$ is monic, $GF(f)G(h) = 0$, and hence $G(h) = 0$. But then, $FG(h) = 0$, so from the version of Definition 32 for ζ, $h = 0$, whence $F(f)$ is monic. The remainder of the proof is entirely similar and will be omitted. □

Lemma 10. (Anderson & Fuller, 1992, Lemma 21.3) Let R and S be equivalent rings. Then, in the notation of Definition 32, the homomorphisms

$$\phi : \mathrm{Hom}_S(N, F(M)) \to \mathrm{Hom}_R(G(N), M)$$

$$\theta : \mathrm{Hom}_S(F(M), N) \to \mathrm{Hom}_R(M, G(N))$$

are natural isomorphisms in each variable. In particular, for each

$$\gamma \in \mathrm{Hom}_S(N_1, F(M_1)),\ \ \delta \in \mathrm{Hom}_S(F(M_2), N_2)$$

$$\overline{\gamma} \in \mathrm{Hom}_R(G(N_1), M_1),\ \ \overline{\delta} \in \mathrm{Hom}_R(M_2, G(N_2))$$

and for each $h : M_1 \to M_2,\ k : N_2 \to N_1$ we have:

1. $\phi(F(h)\gamma k) = h\phi(\gamma)G(k)$,
2. $\theta(k\delta F(h)) = G(k)\theta(\delta)h$,
3. $\phi^{-1}(h\overline{\gamma}G(k)) = F(h)\phi^{-1}(\overline{\gamma})k$,
4. $\theta^{-1}(G(k)\overline{\delta}h) = k\theta^{-1}(\overline{\delta})F(h)$.

Finally, $\phi(\gamma)$ is a monomorphism (resp. epimorphism) if and only if γ is a monomorphism (resp. epimorphism), and $\theta(\delta)$ is a monomorphism (resp. epimorphism) if and only if δ is a monomorphism (resp. epimorphism).

Proof: The $\mathbb{Z}$-homomorphism induced by G

$$G : \mathrm{Hom}_S(N, F(M)) \to \mathrm{Hom}_R(G(N), GF(M))$$

is an isomorphism by Lemma 9. Since $\eta_{_M} : GF(M) \to M$ is an isomorphism, so is $\mathrm{Hom}_R(G(N), \eta_{_M}) : \mathrm{Hom}_R(G(N), GF(M)) \to \mathrm{Hom}_R(G(N), M)$. Thus, since it is the composite of these two maps, $\phi : \mathrm{Hom}_S(N, F(M)) \to \mathrm{Hom}_R(G(N), M)$ is a $\mathbb{Z}$-isomorphism. Also with h, k, and γ as given in the hypothesis,

$$\phi(F(h)\gamma k) = \eta_{_{M_2}} GF(h)G(\gamma)G(k)$$

$$= \eta_{M_2} GF(h)\eta_{M_1}^{-1}\eta_{M_1} G(\gamma)G(k)$$

$$= h\phi(\gamma)G(k).$$

That θ is an isomorphism and that the identities 2, 3, and 4 hold are proved similarly and therefore will be omitted. The equations 1 and 2 mean that ϕ and θ are natural in both M and N. For instance, taking $k = id_N$ we see from 1 that for each $h : M_1 \to M_2$ in $\mathfrak{M}_R$ the diagram in Figure 2.6 commutes.

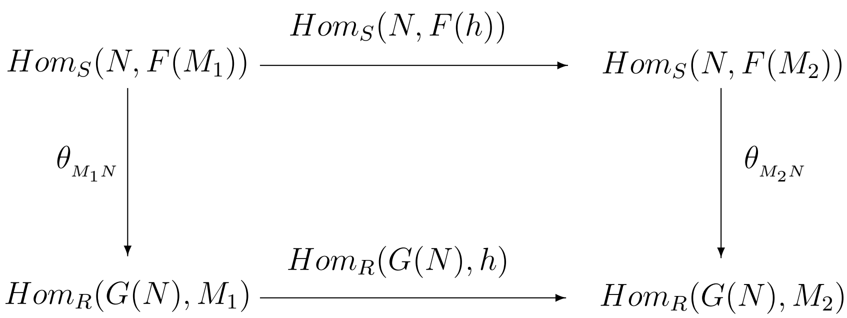

Figure 2.6

For the final assertion, let $\gamma \in \mathrm{Hom}_S(N, F(M))$. Then, $\phi(\gamma) = \eta_M \circ G(\gamma)$. So, since η_M is an isomorphism, $G(\gamma)$ is an monomorphism (epimorphism) if and only if $\phi(\gamma)$ is a monomorphism (epimorphism). But by Lemma 9, $G(\gamma)$ is monic (epic) if and only if γ is. □

Remark 4. It should be observed that we can use ϕ and θ to transform certain diagrams in $\mathfrak{M}_R$ (resp. $\mathfrak{M}_S$) to corresponding diagrams in $\mathfrak{M}_S$ (resp. $\mathfrak{M}_R$). For example part 1 of Lemma 10 asserts that the composite diagram in Figure 2.7

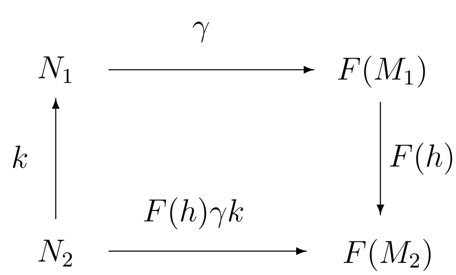

Figure 2.7

is transformed by ϕ to the diagram in Figure 2.8

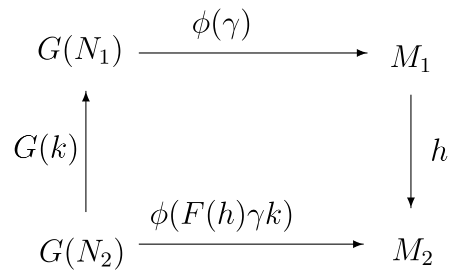

Figure 2.8

Let $\mathbb{P}$ be a property of modules that is preserved by isomorphisms. By Nicholson and Yousif, (2003), $\mathbb{P}$ is called a Morita invariant if, for every additive equivalence $F : \mathfrak{M}_R \to \mathfrak{M}_S$, $F(X)$ has $\mathbb{P}$ whenever X has $\mathbb{P}$. Note that if $F(X)$ has $\mathbb{P}$ then X has $\mathbb{P}$ because $GF(X) \cong X$ for any equivalence inverse G of F. Thus $\mathbb{P}$ is a Morita invariant means that X has $\mathbb{P}$ if and only if $F(X)$ has $\mathbb{P}$.

Proposition 9. (Anderson & Fuller, 1992, Proposition 21.6) Let R and S be equivalent rings via an additive equivalence $F : \mathfrak{M}_R \to \mathfrak{M}_S$. Let M, M', and U be right R-modules. Then:

1. U is M-injective if and only if $F(U)$ is $F(M)$-injective.
2. U is injective if and only if $F(U)$ is injective.

Proof: We adopt the notation of Definition 32.

1. Suppose that U is M-injective, and that in $\mathfrak{M}_S$ there is a diagram in Figure 2.9

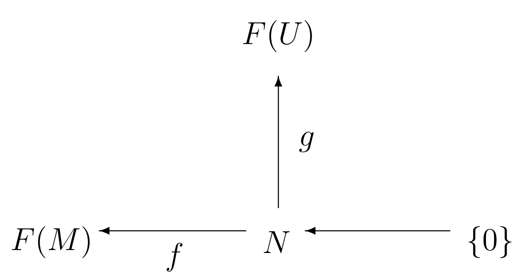

Figure 2.9

with f a monomorphism. Then, $\phi(f)$ is monic in $\mathfrak{M}_R$, so there is an h such that the diagram in Figure 2.10 below commutes

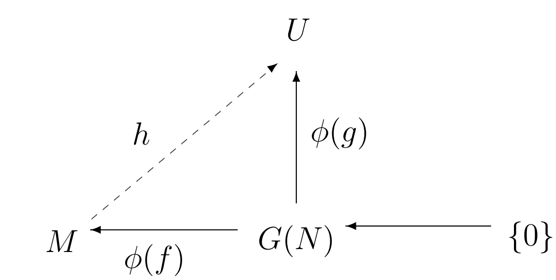

Figure 2.10

Now, by part 3 of Lemma 10, $g = \phi^{-1}(\phi(g)) = \phi^{-1}(h\phi(f)) = F(h)f$, whence $F(U)$ is $F(M)$-injective. Conversely, suppose that $F(U)$ is $F(M)$-injective, and that in $\mathfrak{M}_R$ there is a diagram in Figure 2.11

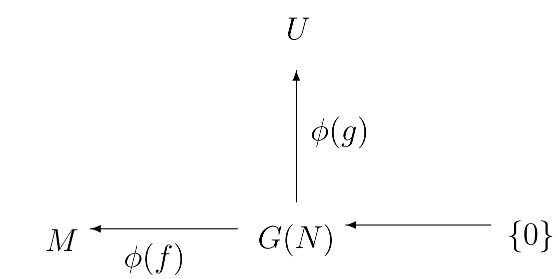

Figure 2.11

with $\phi(f)$ a monomorphism. Then, $\phi^{-1}(\phi(f)) = f$ is monomorphic in $\mathfrak{M}_S$, so there is an $F(h)$ such that the diagram in Figure 2.12 commutes.

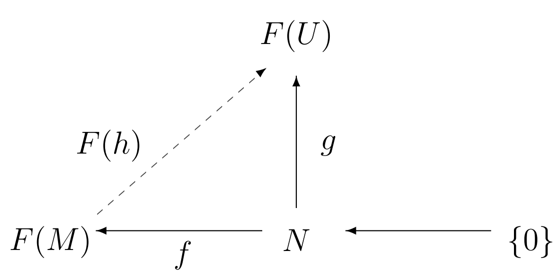

Figure 2.12

Now, by part 1 of Lemma 10, $\phi(g) = \phi(F(h)f) = h\phi(f)$, whence U is M-injective.

2. This is immediate from 1.

□

2.3 Simple-injective modules and min-injective modules

In this section, it is shown that a *simple*-injective module is *min*-injective, strongly *min*-injectivity coincides with strongly *simple*-injectivity, and strongly *min*-injectivity is closed under direct product and direct summand.

Definition 33. Let R be a ring and N an R-module. Then, an R-module M is said to be N-simple-injective if for any submodule L of N, any homomorphism $\theta : L \to M$ with $\theta(L)$ simple, can be extended to a homomorphism $\beta : N \to M$. A ring R is right simple-injective, if R_R is simple injective; equivalently, if I is a right ideal of R and $\gamma : I \to R$ is an R-homomorphism with simple image, then γ is a left multiplication by an element in R. M is called simple-injective if it is simple R-injective. M is called simple-quasi-injective if it is simple M-injective. M is called strongly simple-injective, if M is simple-N-injective for all R-modules N.

Definition 34. Let M and N be R-modules. M is called min-N-injective if, for every simple submodule L of N, every homomorphism $\gamma : L \to M$ extends to N. If $N = R$, M is called min-injective, and if $M = N$, M is called min-quasi-injective. M is called strongly min-injective, if M is min-N-injective for all R-modules N.

Theorem 2.6. (Özcan, *et al.*, 2008, Theorem 5.1) The following statements are equivalent for all R-modules M and N.

1. M is strongly *min*-injective.
2. M is strongly *simple*-injective.
3. Every homomorphism from a finitely generated semi-simple submodule K of any module N into M extends to N.
4. Every homomorphism γ from a submodule K of any module N into M with $\gamma(K)$ finitely generated semi-simple, extends to N.

Proof:

$4 \Rightarrow 3$. Suppose that f is any homomorphism from a finitely generated semi-simple submodule K of any module N into M. Then, $f(K)$ is finitely generated semi-simple submodule of M because the homomorphic image of a finitely generated semi-simple module is finitely generated semi-simple. Then $f : K \to M$ extends to N by 4.

$3 \Rightarrow 1$. Every simple module is finitely generated semi-simple module. Suppose that K is any simple submodule of any module N. Then, by 3, any homomorphism $f : K \to M$ extends to N. From Definition 34, M is strongly *min*-injective.

$1 \Rightarrow 2$. Let K be a submodule of N and $\gamma : K \to M$ a homomorphism with $\gamma(K)$ simple. If $T = \text{Ker}(\gamma)$, then γ induces an embedding $\overline{\gamma} : K/T \to M$ defined by $\overline{\gamma}(x+T) = \gamma(x)$ for all $x \in K$. Since M is strongly *min*-injective and K/T is simple, $\overline{\gamma}$ extends to a homomorphism $\widetilde{\gamma} : N/T \to M$. If $\eta : N \to N/T$ is the natural epimorphism, the homomorphism $\widetilde{\gamma}\eta : N \to M$ is an extension of γ, for if $x \in K$,$(\widetilde{\gamma}\eta)(x) = \widetilde{\gamma}(x+T) = \overline{\gamma}(x+T) = \gamma(x)$, as required.

$2 \Rightarrow 4$. Let N be any module, K a submodule of N, $\gamma : K \to M$ a homomorphism with $\gamma(K)$ finitely generated semi-simple and consider the diagram in Figure 2.13

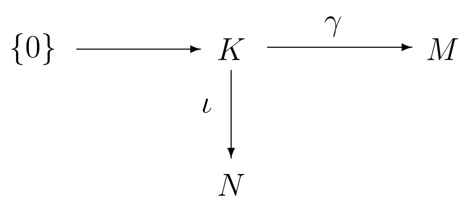

Figure 2.13

Write $\gamma(K) = \oplus_{i=1}^{n} S_i$ where each S_i is simple. Let $\pi_i : \oplus_{i=1}^{n} S_i \to S_i$ be the canonical projection, $1 \le i \le n$, and consider the diagram in Figure 2.14

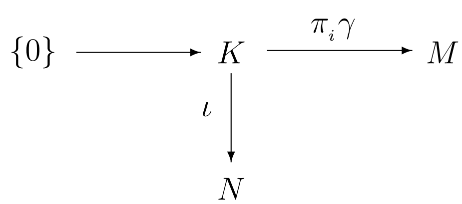

Figure 2.14

Since M is strongly *simple*-injective, for each $i, 1 \le i \le n$, there exists a homomorphism $\gamma_i : N \longrightarrow M$ such that $\gamma_i \iota(x) = \pi_i \gamma(x)$, for all $x \in K$. Now, define the map $\widehat{\gamma} : N \to M$ by $\widehat{\gamma}(x) = \Sigma_{i=1}^{n} \gamma_i(x)$. Then $\widehat{\gamma}(x) = \gamma(x)$ for all $x \in K$. □

Remark 5 shows a connection between *min*-injective modules and *simple*-injective modules.

Remark 5. 1. Since any simple module is finitely generated semi-simple, if we take $\gamma(K)$ to be simple in 4 of Theorem 2.6, and assume that 4 implies 3, then, every simple-N-injective module is min-N-injective.

2. If M is strongly simple-injective module, then any homomorphism f from any simple submodule K of any module N to M extends to N with $f(K)$ a simple submodule of M (because a homomorphic image of a simple module is simple). Then, every strongly simple-injective module is strongly min-injective module. But, from Theorem 2.6, every strongly min-injective module is strongly simple-injective. Thus,

$$M \text{ is a strongly min-injective module} \Leftrightarrow \ M \text{ is a strongly simple-injective module.}$$

Proposition 10. (Özcan *et al.*, 2008, Proposition 5.2)

1. Let N be an R-module and $\{M_i : i \in I\}$ be a family of R-modules. Then, a direct product $\prod_{i\in I} M_i$ is min-N-injective if and only if each M_i is min-N-injective, $i \in I$. In particular, $\prod_{i\in I} M_i$ is strongly simple-injective if and only if each M_i is strongly simple-injective, $i \in I$.

2. A direct summand of a strongly simple-injective module is strongly simple-injective.

Proof:

1. We prove only for $M = M_i \times M_j$ where $i,\ j \in I$. The proof for the general case is analogous. Let M_i and M_j be min-N-injective right R-modules, $h : K \to N$ and $f : K \to M_i \times M_j$ be any R-homomorphisms, where K is any simple submodule of N. Define

$$f_{M_i} : K \to M_i \text{ such that } \pi_{M_i} \circ f = f_{M_i}$$

and

$$f_{M_j} : K \to M_j \text{ such that } \pi_{M_j} \circ f = f_{M_j},$$

where $\pi_{M_i} : M_i \times M_j \to M_i$ and $\pi_{M_j} : M_i \times M_j \to M_j$ are R-projections. Since M_i and M_j are min-N-injective there exist $f'_{M_i} : N \to M_i$ and $f'_{M_j} : N \to M_j$ such that

$$f_{M_i} = f'_{M_i} \circ h \text{ and } f_{M_j} = f'_{M_j} \circ h.$$

By the uniqueness part of the universal property of direct product, there exists an R-homomorphism $f' : N \to M_i \times M_j$ such that $f = f' \circ h$. It follows that $\pi_{M_i} \circ (f' \circ h) = f_{M_i}$ and $\pi_{M_j} \circ (f' \circ h) = f_{M_j}$. By the uniqueness of the universal property we conclude that $f = f' \circ h$. Hence, $f : K \to M_i \times M_j$ extends to N. Thus $M_i \times M_j$ is min-N-injective. Conversely, assume that $M_i \times M_j$ is min-N-injective. Let $h : K \to N$ and $f_{M_i} : K \to M_i$ be any R-homomorphisms, where K is any simple submodule of N. Choose $f_{M_j} : K \to M_j$ to be the zero R-homomorphism. We obtain $f' : N \to M_i \times M_j$ such that $f = f' \circ h$. Finally we obtain $f_{M_i} = \pi_{M_i} \circ f = (\pi_{M_i} \circ f') \circ h$. Hence

$\pi_{M_i} \circ f' : N \to M_i$ is an extension of f_{M_i}. Thus, M_i is min-N-injective. Similarly M_j is min-N-injective. Since a direct product $\prod_{i \in I} M_i$ is strongly min-injective if and only if every M_i is strongly min-injective, then the particular case follows from Remark 5.

2. Let $N \subseteq^{\oplus} M$ and M be strongly simple-injective. We show that N is strongly simple-injective. Since $N \subseteq^{\oplus} M$, there exists an submodule N' of M such that $N \bigoplus N' = M$. Let $\pi_N : N \bigoplus N' \to N$ be the projection map. Since M is strongly simple-injective, then any homomorphism $g_M : K \to M$ with $g_M(K)$ a simple submodule of M, extends to $g'_M : T \to M$, for every right R-module T with $K \leq T$. Suppose that $f = \pi_N \circ g_M$ is a homomorphism from K to N. Clearly $f(K) = \pi_N(g_M(K))$ is a simple submodule of N (because a homomorphic image of a simple submodule $g_M(K)$ is simple). Then, the homomorphism $f' = \pi_N \circ g'_M : T \to N$ is an extension of $f : K \to N$. Hence N is strongly simple-injective.

□

Theorem 2.7. (Özcan *et al.*, 2008, Theorem 5.5)

The following are conditions equivalent for all right R-modules N, M and S.

1. N is strongly simple-injective.
2. N is min-$E(M)$-injective.
3. N is min-$E(S)$-injective for every simple module S.
4. N is min-$E(S)$-injective for every simple submodule S of N.

Proof:

$1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$ follows from Remark 5.

$4 \Rightarrow 1$. Let $K \leq T$ for a right R-module T, and $\gamma : K \to N$ a non-zero homomorphism with $\gamma(K)$ simple, and consider the following diagram in Figure 2.15

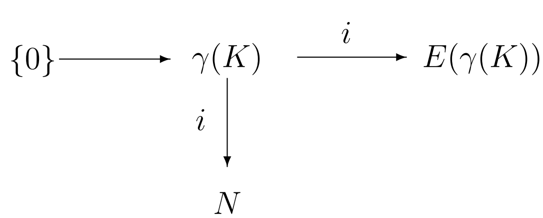

Figure 2.15

where i is the inclusion map. Since N is min-$E(\gamma(K))$-injective, there exists an embedding $\sigma : E(\gamma(K)) \to N$ such that $\sigma\gamma(x) = \gamma(x)$ for every $x \in K$. Now, the map γ may be viewed as a map from K into an M-injective submodule of N, and hence has an extension $\widehat{\gamma} : T \to N$.

□

2.4 Soc-injective and strongly Soc-injective modules

In Chapter 1, definitions of Soc-injective and strongly Soc-injective modules were given. In this section, some properties on Soc-injective modules are given to show that any direct sum of Soc-injective modules is Soc-injective, a direct summand of Soc-injective module is Soc-injective, Soc-injectivity satisfies C_2 and C_3-Conditions.

Example 2.3. (Amin, *et al.*, 2005, Example 2.1). A module M_R with $\text{Soc}(M_R) = \{0\}$ is strongly Soc-injective, because for any module N_R and K_R with $N_R = \text{Soc}(K_R)$, any R-homomorphism $f : N_R \to M_R$ is a zero-homomorphism, and a zero-homomorphism extends to K_R. To see that f is a zero homomorphism, $f(N_R) \subseteq \text{Soc}(M_R) = \{0\}$. Thus, $f(N_R) = \{0\}$. In particular, Polynomial rings and the ring of integers $\mathbb{Z}$ are examples of strongly Soc-injective rings.

Theorem 2.8. (Amin, *et al.*, 2005, Theorem 2.2).

1. Let N be an R-module and $\{M_i : i \in I\}$ a family of R-modules. Then the direct product $\prod_{i\in I} M_i$ is Soc-N-injective if and only if each M_i is Soc-N-injective, $i \in I$.

2. Let M, N, and X be R-modules with $X \leq N$. If M is Soc-N-injective, then M is Soc-X-injective.

3. Let M, N, and $\overline{N}$ be R-modules with $M \cong N$. If M is Soc-$\overline{N}$-injective, then N is Soc-$\overline{N}$-injective.

4. Let N be an R-module and $\{M_i : i \in I\}$ a family of R-modules. Then N is Soc-$\bigoplus_{i\in I} M_i$ -injective if and only if N is Soc-M_i-injective, for all $i \in I$.

5. A R-module M is Soc-injective if and only if M is Soc-P-injective for every projective R-module P.

6. Let M, N, and K be R-modules with $N \subseteq^{\oplus} M$. If M is Soc-K-injective, then N is Soc-K-injective.

7. If N, $\overline{N}$, and M are R-modules, $N_R \cong \overline{N}_R$, and M is Soc-N-injective, then M is Soc-$\overline{N}$-injective.

Proof:

1. Suppose that M_i is Soc-N-injective. Then, for any $j \in I$ $\text{Soc}(N) \leq N$, a diagram in Figure 2.16

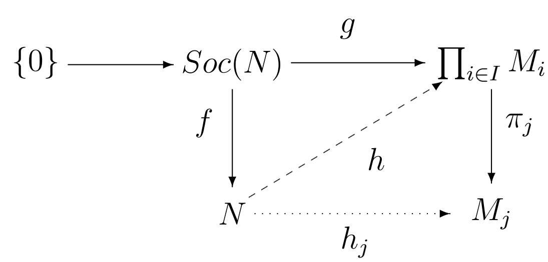

Figure 2.16

can be extended commutatively by $h_j : N \to M_j$. Hence, by the universal property of direct products, we have an $h : N \to \prod_{i\in I} M_i$ with $\pi_j \circ h = h_j$ and $\pi_j \circ h \circ f = h_j \circ f = \pi_j \circ g$, this implies that $h \circ f = g$. Thus, $\prod_{i\in I} M_i$ is Soc-N-injective. Conversely, suppose that $\prod_{i\in I} M_i$ is Soc-N-injective, then a diagram in Figure 2.17 below

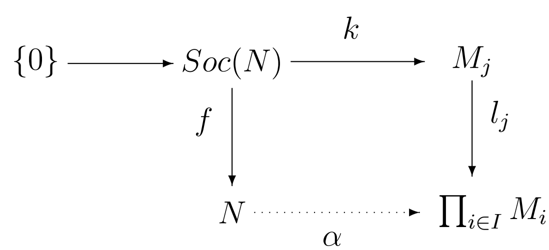

Figure 2.17

can be extended commutatively by $\alpha : N \to \prod_{i\in I} M_i$ and $l_j \circ k = \alpha \circ f$ immediately yields $k = \pi_j \circ l_j \circ k = \pi_j \circ \alpha \circ f$. Thus, M_j is Soc-N-injective.

2. Suppose that M is Soc-N-injective. Let $X \leq N$. Then, $\text{Soc}(X) = X \cap \text{Soc}(N)$, $\text{Soc}(X) \subseteq \text{Soc}(N)$ and $\text{Soc}(X)$ is a semi-simple submodule of N. Since M is Soc-N-injective, then any R-homomorphism $f : \text{Soc}(X) \to M$ extends to N by $\overline{f} : N \to M$.

Clearly, M is Soc-X-injective because there exists an inclusion map $i : X \to N$ such that $\overline{f} \circ i : X \to M$ is an extension of f.

3. Suppose that M is Soc-$\overline{N}$-injective, and $M \cong N$. Let $g : \mathrm{Soc}(\overline{N}) \to N$ be any R-homomorphism and $\theta : M \to N$ an isomorphism. Then, there exists an R-homomorphism $f : \mathrm{Soc}(\overline{N}) \to M$ such that $g = \theta \circ f$. Since M is Soc-$\overline{N}$-injective, then f extends to $\overline{N}$ by $\overline{f} : \overline{N} \to M$. Hence, g extends to $\overline{N}$ by $\overline{g} : \theta \circ \overline{f} : \overline{N} \to N$. Thus, N is Soc-$\overline{N}$-injective.

4. Suppose that M is Soc-$\bigoplus_{i\in I} M_i$-injective. Then, for each $j \in I$, $\overline{M}_j = \{0\} \oplus \{0\} \oplus \cdots \oplus M_j \oplus \{0\} \oplus \cdots < \bigoplus_{i\in I} M_i$ and $\overline{M}_j \cong M_j$. Since M is Soc-$\bigoplus_{i\in I} M_i$-injective, and $\overline{M}_j < \bigoplus_{i\in I} M_i$, then M is Soc-$\overline{M}_j$-injective by 2. Hence, M is Soc-M_j-injective. Since M_j is arbitrary , it follows that M is Soc-M_i-injective for each $i \in I$. Conversely, suppose that M is Soc-M_i-injective for each $i \in I$. Let $\theta_i : M_i \to M$ be an extension of $f_i : \mathrm{Soc}(M_i) \to M$. Let also $g : \mathrm{Soc}(\bigoplus_{i\in I} M_i) \to M$ be any R-homomorphism. Then, by the fundamental property of direct sum of modules, there exists an R-homomorphism $\theta = \langle \theta_i \rangle : \bigoplus_{i\in I} M_i \to M$ such that $\theta \circ \iota_i = \theta_i$ for all $i \in I$, where $\iota_i : M_i \to \bigoplus_{i\in I} M_i$ is an injection. Then, θ is an extension of $g : \mathrm{Soc}(\oplus_{i\in I} M_i) \to M$ because for any $g : \mathrm{Soc}(\oplus_{i\in I} M_i) = \oplus_{i\in I}\mathrm{Soc}(M_i) \to M$ there exists $f_i : \mathrm{Soc}(M_i) \to M$ such that $f_i \circ \iota_i = g$ where $\iota_i : \mathrm{Soc}(M_i) \to \oplus_{i\in I}\mathrm{Soc}(M_i)$ is an injection for each $i \in I$. Hence, M is Soc-$\bigoplus_{i\in I} M_i$-injective.

5. Suppose that M is Soc-injective. Then, M is Soc-R-injective. There exists a free R-module $\overline{M}$ on the set S such that $\overline{M} = \bigoplus_{s\in S} R_s$ where $R_s = R_R$ for $s \in S$. Then, M is *Soc*-$\bigoplus_{s\in S} R_s$-injective by 4. Let P a projective module, then there exists $P'_R \subseteq^{\oplus} \overline{M}$ such that $P \cap P'_R = \{0\}$ and $P \bigoplus P'_R = \overline{M}$. Hence, by 4, M is Soc-P-injective and Soc-P'_R-injective. Conversely, suppose that M is Soc-P-injective for any projective R-module P. Then, by 4, M is Soc-$\overline{M}$-injective, for a free R-module $\overline{M} = P \bigoplus P'_R$; where $P \cap P'_R = \{0\}$. Since $\overline{M} \cong \bigoplus_{s\in S} R_s$ where $R_s = R_R$; it follows that M is Soc-R-injective by 4.

6. Suppose that M is Soc-K-injective. Since $N \subseteq^{\bigoplus} M$, there exists $\overline{N} \subseteq^{\bigoplus} M$ such that $N \cap \overline{N} = \{0\}$ and $N \bigoplus \overline{N} = M$. Then, $N \bigoplus \overline{N}$ is Soc-K-injective. Thus, N is Soc-K-injective by 1.

7. Let $\theta : \overline{N} \to N$ be an isomorphism, then for any $f : \mathrm{Soc}(\overline{N}) \to M$, there is $g : \mathrm{Soc}(N) \to M$ such that $f = g \circ \theta|_{\mathrm{Soc}(\overline{N})}$; where $\theta|_{\mathrm{Soc}(\overline{N})} : \mathrm{Soc}(\overline{N}) \to N$. Since M is Soc-N-injective, then an R-homomorphism $g : \mathrm{Soc}(N) \to M$ extends to N by $\overline{g} : N \to M$. Suppose that $f : \mathrm{Soc}(\overline{N}) \to M$ is any R-homomorphism. Then, f extends to $\overline{N}$ by $\overline{g} \circ \theta : \overline{N} \to M$, and hence M is Soc-$\overline{N}$-injective.

□

Definition 35. Let K and L be submodules of M. M is said to satisfy:

1. C1-condition if every submodule of M is essential in a summand.
2. C2-condition if $K \cong L$ and $K \subseteq^{\oplus} M$, then $L \subseteq^{\oplus} M$.
3. C3-condition if $K \cap L = \{0\}$, $K \subseteq^{\oplus} M$ and $L \subseteq^{\oplus} M$, then $K \bigoplus L$ is a summand of M.

Proposition 11. (Amin, *et al.*, 2005, Proposition 2.5). Suppose that M is a Soc-quasi-injective R-module.

1. (Soc-C2) If K and L are semi-simple submodules of M, $K \cong L$, and $K \subseteq^{\oplus} M$, then $L \subseteq^{\oplus} M$.
2. (Soc-C3) Let K and L be semi-simple submodules of M with $K \cap L = \{0\}$. If $K \subseteq^{\oplus} M$ and $L \subseteq^{\oplus} M$, then $K \bigoplus L$ is a summand of M.

Proof:

1. Since $K \cong L$, and K is Soc-M-injective, being a summand of the Soc-quasi- injective R-module M, L is Soc-M-injective. If $\iota : L \to M$ is the inclusion map, the identity map $id_{_L} : L_R \to L_R$ has an extension $\eta : M \to L$ such that $\iota \circ \eta = id_{_L}$, and so L is a summand of M.
2. Since K and L are summands of M, and M is Soc-quasi-injective, both K and L are Soc-M-injective. Thus the semi-simple module $K \bigoplus L$ is Soc-M-injective, and so is a summand of M.

□

Lemma 11. (Yousif & Yiqiang, 2002, Lemma 2.14). Let $R/\text{Soc}(R_R)$ be semi-simple Artinian. Then an R-module M is Soc-injective if and only if M is injective.

Proof: $R/\text{Soc}(R_R)$ is semi-simple if and only if $\text{Soc}(R_R)$ respects every right ideal of R (Yousif & Yiqiang, 2002, Theorem 2.3). Hence by (Ozcan, *et al.*, 2008, Proposition 2.11), the result holds. □

Theorem 2.9. (Amin, *et al.*, 2005, Theorem 3.1). For an R-module M, the following conditions are equivalent:

1. M is strongly Soc-injective.
2. M is Soc-$E(M)$-injective.
3. $M = \overline{M} \oplus T$, where $\overline{M}$ is injective and T has zero socle.

Moreover, if M has non-zero socle then $\overline{M}$ can be taken to have essential socle.

Proof:

$1 \Rightarrow 2$. Suppose that M is a strongly Soc-injective R-module. Then by Definition 17, M is Soc-$E(M)$-injective for $E(M)$ an injective envelope right R-module of M.

$2 \Rightarrow 3$. If $\mathrm{Soc}(M) = \{0\}$ then, $\overline{M}$ is injective and $\mathrm{Soc}(T) = \{0\}$. Assume $\mathrm{Soc}(M) \neq \{0\}$, and consider the diagram in Figure 2.18:

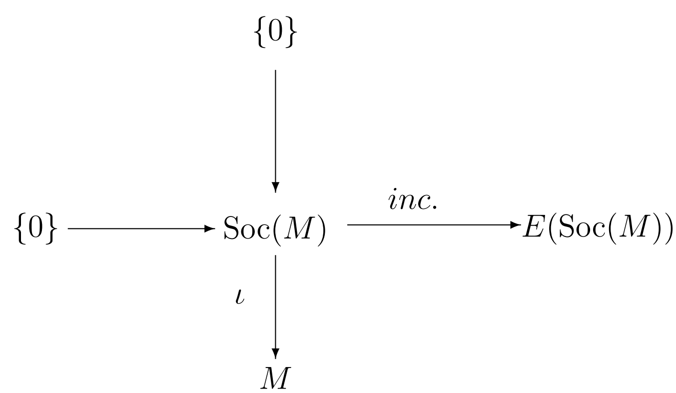

Figure 2.18

where ι is the inclusion map. Since M is Soc-$E(M)$-injective, M is Soc-$E(\mathrm{Soc}(M))$-injective. So, there exists an R-homomorphism $\sigma : E(\mathrm{Soc}(M)) \rightarrow M$, which extends ι. Since $\mathrm{Soc}(M) \subseteq^e E(\mathrm{Soc}(M))$, σ is an embedding of $E(\mathrm{Soc}(M))$ in M. If we write $\overline{M} = \sigma(E(\mathrm{Soc}(M)))$, then $M = \overline{M} \oplus T$ for some submodule T of M. Clearly, $\overline{M}$ is injective and T has zero socle.

$3 \Rightarrow 1$. This is clear, since modules with zero socle are strongly Soc-injective and finite direct sums of strongly Soc-injective modules are strongly Soc-injective.
For the last statement of the Theorem 2.9, $\sigma(\mathrm{Soc}(M)) \subseteq^e \overline{M}$. On the other hand, $\mathrm{Soc}(\overline{M}) = \mathrm{Soc}(M) = \sigma(\mathrm{Soc}(M)) \subseteq^e \overline{M}$ implies that $\mathrm{Soc}(\overline{M}) \subseteq^e \overline{M}$.

□

Corollary 4 is a an immediate consequence of Theorem 2.9.

Corollary 4. Let M be an R-module with essential socle. Then, M is strongly Soc-injective if and only if it is injective.

Proof:

($\Rightarrow$). Suppose that $\text{Soc}(M) \subseteq^e M$. From Theorem 2.9, $\text{Soc}(M) \subseteq^e E(\text{Soc}(M)) \subseteq \sigma(E(\text{Soc}(M))) = \overline{M}$ where σ is an embedding of $E(\text{Soc}(M))$ in M. Then, $\overline{M} \subseteq^e M$. Since $\overline{M}$ is injective, then $\overline{M} = M$ and hence M is injective.

($\Leftarrow$). If M is injective, then it is strongly Soc-injective by Amin *et al.*, (2005).

□

Proposition 12. A ring R is right Noetherian, if and only if $\bigoplus_{i=1}^{\infty} E(K_i)$ is injective for K_1, K_2,... simple R-modules.

Proof: By (Lam, 1999, p.72) each $E(K_i)$ is injective. Then using Proposition 3, a ring R is right Noetherian, if and only if $\bigoplus_{i=1}^{\infty} E(K_i)$ is injective. □

Theorem 2.10. (Amin, *et al.*, 2005, Theorem 3.3). A ring R is right Noetherian if and only if every direct sum of strongly Soc-injective R-modules is strongly Soc-injective.

Proof:

($\Rightarrow$). Let $\{M_i\}_{i\in I}$ be a family of strongly Soc-injective R-modules. By Theorem 2.8, for each $i \in I$, write $M_i = E_i \oplus T_i$ where E_i is injective and $\text{Soc}(T_i) = \{0\}$. If $E = (\bigoplus_{i\in I} E_i)$ and $T = (\bigoplus_{i\in I} T_i)$, then $\bigoplus_{i\in I} M_i = E \oplus T$, with $\text{Soc}(T) = \{0\}$. Since R is right Noetherian, E is injective, and by Theorem 2.8, $\bigoplus_{i\in I} M_i$ is strongly Soc-injective.

($\Leftarrow$). In order to prove that R is right Noetherian we only need to show that if K_1, K_2,... are simple R-modules then $\bigoplus_{i=1}^{\infty} E(K_i)$ is injective, where $E(K_i)$ is the injective hull of K_i. Since $\bigoplus_{i=1}^{\infty} E(K_i)$ is strongly Soc-injective with essential socle, it follows from Corollary 4, that $\bigoplus_{i=1}^{\infty} E(K_i)$ is injective.

□

Let us first recall that, an R-module M is called $\sum$-injective if any direct sum of copies of M is injective. It is well known that (Faith, 1966, Corollary 3) a ring R is quasi-Frobenius if and only if the module R_R is $\sum$-injective.

Proposition 13. (Amin, *et al.*, 2005, Proposition 3.8). The following conditions on a ring R are equivalent: Every projective R-module is strongly Soc-injective if and only if $R = E \oplus T$, where E and T are right ideals of R, E is $\sum$-injective as an R-module and $\text{Soc}(T_R) = \{0\}$.

Proof:

(⇒). This is clear if $\mathrm{Soc}(R_R) = \{0\}$. Assume that $\mathrm{Soc}(R_R) \neq \{0\}$. Since R_R is projective, it follows from Theorem 2.9 that $R = E \oplus T$, where E and T are R-modules, E is injective with essential socle and $\mathrm{Soc}(T) = \{0\}$. Since $E^{(\alpha)}$ is projective for any ordinal number α, $E^{(\alpha)}$ is strongly Soc-injective with essential socle, and by Corollary 4, $E^{(\alpha)}$ is injective.

(⇐). If α is any ordinal number, then $R^{(\alpha)} = E^{(\alpha)} \oplus T^{(\alpha)}$, and since $E^{(\alpha)}$ is injective, $R^{(\alpha)}$ is strongly Soc-injective by Theorem 2.9. Since every summand of a strongly Soc-injective module is again strongly Soc-injective, it follows that every projective R-module is strongly Soc-injective.

□

Lemma 12. (Amin, *et al.*, 2005, Lemma 3.11) Let M be a semi-simple R-module. The following conditions are equivalent:

1. M is injective.
2. M is strongly Soc-injective.
3. M is Soc-N-injective, for every factor module N_R of any module $\overline{M}_R$.

Proof:

$1 \Leftrightarrow 2$ By Corollary 4.

$1 \Rightarrow 3$ Since $1 \Leftrightarrow 2$ and every strongly Soc-injective module is $\mathrm{Soc}(N)$-injective for every right R-module.

$3 \Rightarrow 1$ Consider the diagram in Figure 2.19

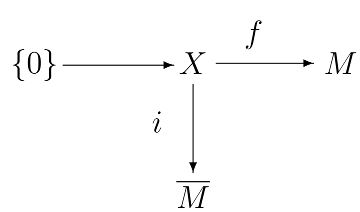

Figure 2.19

where $X \leq \overline{M}$ and $f : X \to M$ is any homomorphism. Then, there is a diagram in Figure 2.20

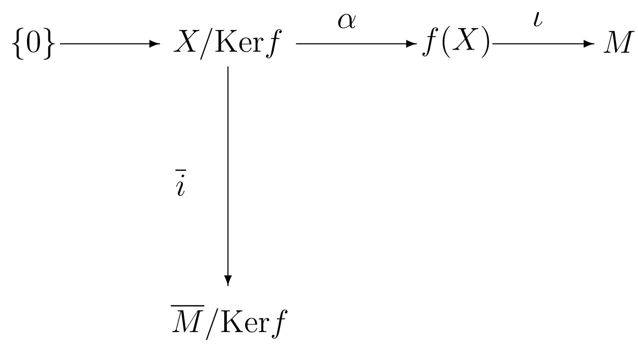

Figure 2.20

where α is an isomorphism and ι is the inclusion map. Since M is Soc-$\overline{M}/\mathrm{Ker}f$-injective and $X/\mathrm{Ker}f$ is semi-simple, there exists a homomorphism $g : \overline{M}/\mathrm{Ker}f \to M$ such that $g \circ \bar{i} = \iota \circ \alpha$. Then, the homomorphism $h = g \circ \pi$ extends f where $\pi : M \to \overline{M}/\mathrm{Ker}f$ is the natural epimorphism.

$\square$

Chapter 3

Red-Injective and Strongly Red-Injective Modules

In this Chapter, Red-injective and strongly Red-injective modules are introduced and studied. Many of the basic results on (quasi-)injective modules are shown to hold for (strongly) Red-injective modules. Examples are given to show that the notions of strongly Red-injectivity, strongly Soc-injectivity and strongly *min*-injectivity are different.

3.1 Red-Injective Modules

In this section, it is proved that the class of (strongly) Red-injective R-modules is closed under isomorphisms, direct products and summands. If P is a projective module over a right Noetherian right self-injective ring, then P is Red-injective. If M_R is a projective module, then every quotient of a Red-M-injective R-module is *Red*-M-injective if and only if Red(M) is projective. Over a Principal Ideal Domain a free module is Red-injective if each of its submodule is Red-injective. Furthermore, we give an example to show that Red-injectivity is not a Morita invariant property of rings.

Proposition 14. Let M and N be R-modules such that M is Red-N-injective (resp., strongly Red-injective, Red-quasi-injective). Then M is Soc-N-injective (resp., strongly-Soc-injective, Soc-quasi-injective).

Proof: Let M be a Red-N-injective module, then any R-homomorphism $f : \text{Red}(N) \to M$ extends to $f' : N \to M$. Assume an R-homomorphism $g : \text{Soc}(N) \to M$. Since $\text{Soc}(N) \subseteq \text{Red}(N)$ then, there is an inclusion R-homomorphism $i : \text{Soc}(M) \to \text{Red}(M)$ such that

$g = f \circ i$. Since M is Red-N-injective, then $f = f' \circ j$ where $j : \text{Red}(N) \to N$ is an inclusion R-homomorphism. The R-homomorphism $g : \text{Soc}(N) \to M$ extends $f' : N \to M$ since $g = f' \circ j \circ i$. Hence M is Soc-N-injective. □

Proposition 15. Every injective module is strongly Red-injective.

Proof: Let M be an injective module. Then M is N-injective for every R-module N. For every submodule K of N, any R-homomorphism $f : K \to M$ extends to N. For every module N, any R-homomorphism $f : \text{Red}(N) \to M$ extends to N. Hence, M is strongly Red-injective. □

From Proposition 14 and 15, the following implication holds:

$$\text{Injective} \Rightarrow \text{ strongly Red-injective} \Rightarrow \text{ strongly Soc-injective}$$

which shows that the generalization of injective modules namely, strongly Red-injective is a closer generalization to injective (i.e., it carries many properties of injective modules) modules than strongly Soc-injective modules.

Example 3.1. Let R be a ring for which each module M has $\text{Red}(M) = \{0\}$. Then, M is strongly Red-injective.

Example 3.2. Every projective module over a right Noetherian right self-injective ring R is strongly Red-injective.

Proof: Every projective R-module P is injective if R is a right Noetherian right self-injective ring (Lam, 1999, p.117). Hence, by Proposition 15, P_R is Red-injective. □

Remark 6. A Red-N-injective module need not be injective, see Example 3.3.

Proposition 16 will help us to construct an example of Red-N-injective module which is not injective.

Proposition 16. (Pierce, 1982, p.28) Let M and N be R-modules. Suppose that $\phi : M \to N$ is a non-zero R-homomorphism.

1. If M is simple, then ϕ is injective.

2. If N is simple, then ϕ is surjective.

Proof: Since $\phi \neq 0$, it follows that Ker $\phi \neq \{0\}$ and Im $\phi \neq \{0\}$. Hence, M simple implies Ker $\phi = \{0\}$, and N simple implies Im $\phi = N$. □

Example 3.3. The module $\mathbb{Z}_{\mathbb{Z}}$ is not injective since a homomorphism $f : 2\mathbb{Z} \to \mathbb{Z}_{\mathbb{Z}}$ defined by $f(2n) = n$ for all $n \in \mathbb{Z}$ cannot be extended to a homomorphism $f' : \mathbb{Z} \to \mathbb{Z}_{\mathbb{Z}}$. But, $\mathbb{Z}_{\mathbb{Z}}$ is Red-$\mathbb{Z}/p\mathbb{Z}$-injective module for any prime integer p, because $\mathbb{Z}/p\mathbb{Z}$ is a simple $\mathbb{Z}$-module and by Proposition 16, any non-zero homomorphism (which is injective) $f : \mathrm{Red}(\mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}_{\mathbb{Z}}$ must extend to $\mathbb{Z}/p\mathbb{Z}$ by $\widehat{f} : \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}_{\mathbb{Z}}$.

Proposition 17. The following statements are equivalent:

1. Any R-homomorphism $f : K \to M$ extends to N for any semi-reduced submodule K of N.

2. Any R-homomorphism $f : \mathrm{Red}(N) \to M$ extends to N.

Proof:

$1 \Rightarrow 2$. Suppose that $f : \mathrm{Red}(N) \to N$ is an R-homomorphism. $\mathrm{Red}(N)$ is a semi-reduced submodule of N. By 1, $f : K \to M$ extends to N.

$2 \Leftarrow 1$. Suppose $f : K \to M$ is an R-homomorphism and K is a semi-reduced submodule of N. Then $\mathrm{Red}(K) = K$. We get the R-homomorphism $f : \mathrm{Red}(K) \to M$. 2 implies f extends to N.

□

Remark 7. Strongly Soc-injective $\not\Rightarrow$ Strongly Red-injective, because $\mathbb{Z}_{\mathbb{Z}}$ is Strongly Soc-injective but not strongly Red-injective.

Proposition 18. Let M and N be R-modules such that N is semi-simple. Then M is Soc-N-injective if and only if it is Red-N-injective.

Proof:

($\Rightarrow$). Let M be a Soc-N-injective module. Since N is semi-simple, $N = \mathrm{Soc}(N) = \mathrm{Red}(N)$. Then, any R-homomorphism $f : \mathrm{Red}(N) \to M$ extends to N. Hence, M is Red-N-injective.

($\Leftarrow$). By Proposition 14, every Red-injective module is Soc-injective. Thus M is Soc-N-injective.

□

Example 3.4. The $\mathbb{Z}$-module $\mathbb{Z}_{\mathbb{Z}} \oplus \mathbb{Z}_{\mathbb{Z}}$ is Red-$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$-injective.

$\mathbb{Z}_{\mathbb{Z}} \oplus \mathbb{Z}_{\mathbb{Z}}$ is Soc-$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$-injective since $\text{Soc}(\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$, i.e., $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ is semi-simple. But $\text{Red}(\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$. Thus, $\mathbb{Z}_{\mathbb{Z}} \oplus \mathbb{Z}_{\mathbb{Z}}$ is Red-$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$-injective.

Theorem 3.1. Let $\{M_i : i \in I\}$ be a family of R-modules and N, M, A, C, S and K be R-modules. Then, the following conditions hold:

1. A direct product $\prod_{i \in I} M_i$ is Red-N-injective if and only if each M_i is Red-N-injective.
2. Let $S \leq N$ (S a submodule of N). If M is Red-N-injective, then M is Red-S-injective.
3. Let $M \cong N$; M is Red-S-injective if and only if N is Red-S-injective.
4. Let $A \cong B$; C is Red-A-injective if and only if it is Red-B-injective.
5. Let $N \subseteq^{\oplus} M$ (N be a direct summand of M). If M is Red-K-injective, then N is Red-K-injective.

Proof:

1. We prove only for $M = M_i \times M_j$ where $i,\ j \in I$. The proof for the general case is analogous. Let M_i and M_j be Red-N-injective R-modules, $h : \text{Red}(N) \to N$ and $f : \text{Red}(N) \to M_i \times M_j$ be any R-homomorphisms.
   Define
   $$f_{M_i} : \text{Red}(N) \to M_i \text{ such that } \pi_{M_i} \circ f = f_{M_i}$$
   and
   $$f_{M_j} : \text{Red}(N) \to M_j \text{ such that } \pi_{M_j} \circ f = f_{M_j},$$
   where $\pi_{M_i} : M_i \times M_j \to M_i$ and $\pi_{M_j} : M_i \times M_j \to M_j$ are R-homomorphisms. Since M_i and M_j are Red-N-injective there exists $f'_{M_i} : N \to M_i$ and $f'_{M_j} : N \to M_j$ such that
   $$f_{M_i} = f'_{M_i} \circ h \text{ and } f_{M_j} = f'_{M_j} \circ h.$$
   By the uniqueness part of the universal property of direct product there exists an R-homomorphism $f' : N \to M_i \times M_j$ such that $f = f' \circ h$. It follows that $\pi_{M_i} \circ (f' \circ h) = f_{M_i}$ and $\pi_{M_j} \circ (f' \circ h) = f_{M_j}$. Hence, $f : \text{Red}(N) \to M_i \times M_j$ extends to N. Thus $M_i \times M_j$ is Red-N-injective.

   Conversely, assume that $M_i \times M_j$ is Red-N-injective. Let $h : \text{Red}(N) \to N$ and $f_{M_i} : \text{Red}(N) \to M_i$ be any R-homomorphisms. Choose $f_{M_j} : \text{Red}(N) \to M_j$ to be the zero R-homomorphism. We obtain $f' : N \to M_i \times M_j$ such that $f = f' \circ h$. Finally we obtain $f_{M_i} = \pi_{M_i} \circ f = (\pi_{M_i} \circ f') \circ h$. Hence $\pi_{M_i} \circ f' : N \to M_i$ is an extension of f_{M_i}. Thus, M_i is Red-N-injective. Similarly M_j is Red-N-injective.

2. Consider the diagram in Figure 3.1, where M is Red-N-injective, i.e., any R-homomorphism $f : \text{Red}(N) \to M$ extends to $f' : N \to M$.

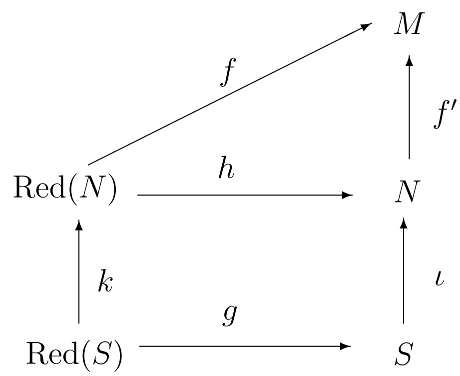

Figure 3.1

Since $S \leq N$ then $\text{Red}(S) \leq \text{Red}(N)$. Consider the injections

$$k : \text{Red}(S) \to \text{Red}(N);\; g : \text{Red}(S) \to S;\; h : \text{Red}(N) \to N \text{ and } \iota : S \to N.$$

$f' \circ l : S \to M$ is an extension for any R-homomorphism $q : \text{Red}(S) \to M$. Thus M is Red-S-injective.

3. Let $N \cong M$ where $\theta : N \to M$ is an R-isomorphism between them. Let $f_N : \text{Red}(S) \to N$ be any R-homomorphism. Since M is Red-S-injective, then any R-homomorphism $f_M : \text{Red}(S) \to M$ extends to $f'_M : S \to M$. It implies that for any R-homomorphism $h : \text{Red}(S) \to S$, $f_M = f'_M \circ h$. Since M and N are isomorphic there exists an inverse homomorphism $\theta^{-1} : M \to N$ such that $\theta^{-1} \circ f'_M : S \to N$ is an R-homomorphism. Define $f'_N = \theta^{-1} \circ f'_M : S \to N$. Then f'_N is an extension of f_N since $f_N = \theta^{-1} \circ f_M = \theta^{-1} \circ (f'_M \circ h) = (\theta^{-1} \circ f'_M) \circ h = f'_N \circ h$. Thus N is Red-S-injective. Similarly, if N is Red-S-injective then M is Red-S-injective.

4. Suppose that $A \cong B$ and C is Red-A-injective. We show that C is Red-B-injective.

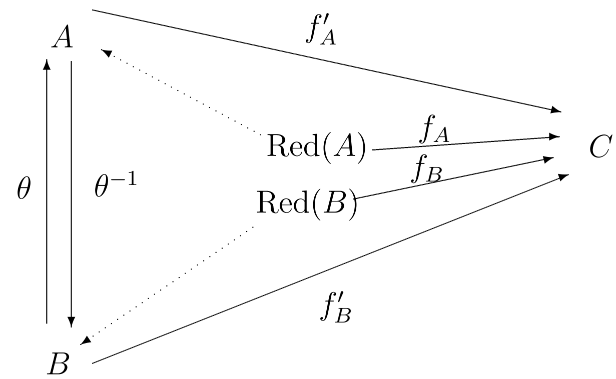

Figure 3.2

Consider the diagram in Figure 3.2 above, where $f'_A : A \to C$ is the extension of $f_A : \mathrm{Red}(A) \to C$. Let also $f_B : \mathrm{Red}(B) \to C$ be an R-homomorphism. Define $f'_B = f'_A \circ \theta : B \to C$. Then $f'_B : B \to C$ is the extension of f_B. Thus C is Red-B-injective. Conversely, let $f'_B : B \to C$ be the extension of $f_B : \mathrm{Red}(B) \to C$. Define $f'_A = f'_B \circ \theta^{-1} : A \to C$. Then f'_A is the extension of $f_A : \mathrm{Red}(A) \to C$. Hence C is Red-A-injective.

5. Let $N \subseteq^{\oplus} M$ and M be Red-K-injective. We show that N is Red-K-injective. Since $N \subseteq^{\oplus} M$, there exists an R-submodule N' of M such that $N \bigoplus N' = M$. Let $\pi_{_N} : N \bigoplus N' \to N$ be the projection R-homomorphism. Since M is Red-K-injective, then any R-homomorphism $f_M : \mathrm{Red}(K) \to M$ extends to $f'_M : K \to M$. Suppose that $f_N = \pi_{_N} \circ f_M : \mathrm{Red}(K) \to N$ is the R-homomorphism. Define $f'_N = \pi_{_N} \circ f'_M : K \to N$. Then $f'_N : K \to N$ is the extension of f_N. Hence, N is Red-K-injective.

$\square$

Corollary 5. Let N be an R-module, then :

1. A finite direct sum of Red-N-injective modules is again Red-N-injective. In particular, a finite direct sum of Red-injective (resp., strongly Red-injective) modules is again Red-injective (resp., strongly Red-injective).

2. A direct summand of Red-quasi-injective (resp., Red-injective, strongly Red-injective) module is again Red-quasi-injective (resp., Red-injective, strongly Red-injective) module.

Proof:

1. Consider the finite family of R-modules M_j, $j = 1, \cdots, n$. Let N be a right R-module. Since $\prod_{j=1}^{n} M_j$ and $\bigoplus_{j=1}^{n} M_j$ are identical and $\prod_{j=1}^{n} M_j$ is Red-N-injective by Theorem 3.1, it follows that $\bigoplus_{j=1}^{n} M_j$ is Red-N-injective.

2. Follows from Theorem 3.1. (5).

□

Proposition 19. Let $\{M_i : i \in I\}$ be a family of R-modules and N be an R-module. Then, N is Red-$\bigoplus_{i\in I} M_i$-injective if it is Red-M_i-injective for each i.

Proof: Suppose that N is Red-$(\bigoplus_{i\in I} M_i)$-injective. We show that N is Red-M_i-injective for each $i \in I$. Let $f : \text{Red}(M_i) \to N$ be any R-homomorphism. By hypothesis, any R-homomorphism $g : \text{Red}(\bigoplus_{i\in I} M_i) \to N$ extends to $\bar{g} : \bigoplus_{i\in I} M_i \to N$. The required extension of f is $\bar{g} \circ \iota$ where ι is the injection $\iota : M_i \to \bigoplus_{i\in I} M_i$. This is because $f = (\bar{g} \circ i) \circ j$ for any inclusion $j : \text{Red}(M_{_i}) \to M_{_i}$. □

Example 3.5. Consider the $\mathbb{Z}$-modules $N = \mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Q}$, $N_1 = \mathbb{Z}/p\mathbb{Z}$ and $N_2 = \mathbb{Q}$ for an integer $n > 1$ and a prime integer p. Then, N is Red-$N_1 \oplus N_2$-injective since $N_1 \oplus N_2$ is reduced. N is Red-N_1-injective and Red-N_2-injective since both N_1 and N_2 are reduced modules.

Corollary 6. If A, B, C, and Q are R-modules and the short exact sequence $\{0\} \to A \xrightarrow{\mu} B \xrightarrow{\varepsilon} C \to \{0\}$ splits, then the following conditions hold:

1. Q is Red-C-injective if and only if it is Red-$(B/\mu(A))$-injective.

2. If Q is Red-A-injective and Red(C) injective then it is Red-B-injective.

Proof:

1. This follows from the fact that $B/\mu(A) \cong C$.

2. Since $\{0\} \to A \xrightarrow{\mu} B \xrightarrow{\varepsilon} C \to \{0\}$ splits, then $B \cong A \oplus C$. Now, if Q is Red-B-injective, then it is Red-$A \oplus C$-injective by Theorem 3.1. By Proposition 19, Q is Red-A-injective and Red-C-injective.

□

Proposition 20. Let N and M be R-modules. Then the following conditions hold:

1. M is injective $\Rightarrow$ M is N-injective $\Rightarrow$ M is Red-N-injective $\Rightarrow$ M is Soc-N-injective $\Rightarrow$ M is *min*-N-injective.

2. M is injective $\Rightarrow$ M is strongly Red-injective $\Rightarrow$ M is strongly Soc-injective $\Rightarrow$ M is strongly *min*-injective $\Leftrightarrow$ M is strongly *simple*-injective.

Proof:

1. M is injective $\Rightarrow$ M is N-injective: Using Definition 9, suppose that $K \leq M$, $f : K \to M$ a homomorphism, and $h : K \to N$ a monomorphism. Since M is injective, there exists a homomorphism $g : N \to M$ such that $g \circ h = f$. Hence M is N-injective. M is N-injective $\Rightarrow$ M is Red-N-injective: This follows from Definition 21 on page 14 and the fact that $\text{Red}(N) \leq N$. M is Red-N-injective $\Rightarrow$ M is Soc-N-injective: by Proposition 14. M is Soc-N-injective $\Rightarrow$ M is *min*-N-injective: Since M is $\text{Soc}(N)$-injective, then by Definition 16, any R-homomorphism $f : K \to M$ extends to N for any semi-simple submodule K of N. Then, for any simple submodule $\overline{K}$ of N, any R-homomorphism $\overline{f} : \overline{K} \to M$ extends to N. Since a homomorphic image of a simple module is simple, then $\overline{f}(\overline{K})$ is a simple submodule of M. Thus M is *min*-N-injective.

2. M is injective $\Rightarrow$ M is strongly Red-injective $\Rightarrow$ M is strongly Soc-injective: This follows from Propositions 14 and 15. M is strongly Soc-injective $\Rightarrow$ M is strongly *min*-injective $\Leftrightarrow$ M is strongly *simple*-injective: If M is strongly Soc-injective, then it is Soc-N-injective for all R-modules N. But from 1, every Soc-N-injective module is *min*-N-injective for any right R-module N. Hence, M is *min*-N-injective for all right R-modules N. Thus M is strongly *min*-injective. M is strongly *min*-injective $\Leftrightarrow$ M is strongly *simple*-injective follows from Remark 5.

$\square$

Proposition 21. For any R-module M, if $\text{Red}(M)$ is a direct summand of M, then every R-module is Red-M-injective.

Proof: Suppose that K is an R-module and $\text{Red}(M) \subseteq^{\oplus} M$. We show that K is Red-M-injective. Let $f : \text{Red}(M) \to K$ be any R-homomorphism. Since $\text{Red}(M)$ is a direct summand of M, there exists a proper R-submodule P of M such that $M = \text{Red}(M) \oplus P$. There exists an R-homomorphism $f' : M \to \text{Red}(M)$ such that $f'(n + p) = n$, for all $n \in \text{Red}(M)$ and $p \in P$. Then, the R-homomorphism $f \circ f' : M \to K$ is an extension of f

because $(f \circ f')(n+p) = f(f'(n+p)) = f(n)$ for all $n+p \in M$. Hence K is Red-M-injective. □

Lemma 13 is used to prove Theorem 3.2.

Lemma 13. (Eilenberg & Cartan, 1956, Proposition 5.1). In order that a module P be projective, it is necessary and sufficient that every diagram in Figure 3.3

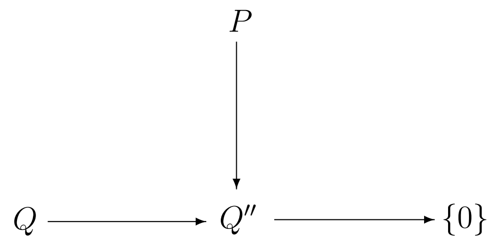

Figure 3.3

in which the row is exact and Q is injective, can be imbedded in a commutative diagram in Figure 3.4.

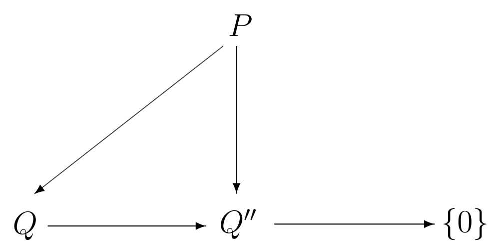

Figure 3.4

Proof: The necessity of the condition is true since P is projective and the row is exact in Figure 3.3, then the diagram in Figure 3.4 is commutative. To prove sufficiency, consider a module A, a submodule A' with $A'' = A/A'$ and a homomorphism $f : P \to A''$. We may regard A as a submodule of an injective module Q. Then, A'' is a submodule of $Q'' = Q/A'$. By the condition above there is then a homomorphism $g : P \to Q$ which when combined with $Q \to Q''$ yields $P \to A'' \to Q''$. It follows that the values of g lie in A. This yields $g' : P \to A$ which when composed with $A \to A''$ yields $f : P \to A''$. Thus, P is projective. □

Theorem 3.2. For a projective R-module M, the following conditions are equivalent:

1. Every quotient of a Red-M-injective R-module is Red-M-injective.

2. Every quotient of an injective R-module is Red-M-injective.

3. Red(M) is a projective R-module.

Proof:

(1) $\Rightarrow$ (2). This is due to the fact that every injective R-module is Red-M-injective.

(2) $\Rightarrow$ (3). Consider Figure 3.5 below:

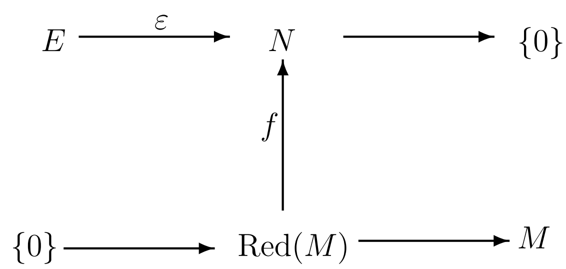

Figure 3.5

where E and N are R-modules, ε an R-epimorphism, and f an R-homomorphism. By Lemma 13, assume that E is injective. Since N is Red-M-injective f can be extended to an R-homomorphism $g : M \to N$. Since M is projective, g can be lifted to an R-homomorphism $\widetilde{g} : M \to E$ such that $\varepsilon \circ \widetilde{g} = g$. Define $\widetilde{f} : \text{Red}(M) \to E$ by $\widetilde{f} = \widetilde{g}|_{\text{Red}(M)}$. Then $\varepsilon \circ \widetilde{f} = \varepsilon \circ \widetilde{g}|_{\text{Red}(M)} = f$. Hence Red($M$) is projective.

(3) $\Rightarrow$ (1). Let N and L be R-modules with $\varepsilon : N \to L$ an R-epimorphism and N is Red-M-injective. Consider the diagram in Figure 3.6.

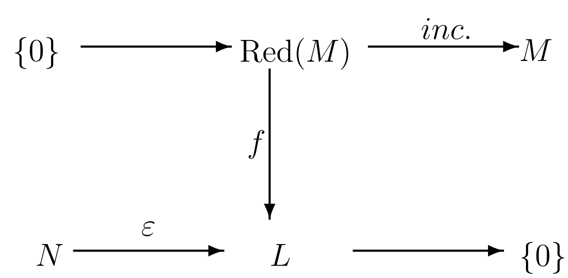

Figure 3.6

Since Red(M) is projective, f can be lifted to an R-homomorphism $g : \text{Red}(M) \to N$ such that $\varepsilon \circ g(m) = f(m)$, for all $m \in \text{Red}(M)$. Since N is Red-M-injective, g can be extended to an R-homomorphism $\widetilde{g} : M \to N$. Hence, $\varepsilon \circ \widetilde{g} : M \to L$ extends f.

□

Proposition 22. Let N be a semi-simple R-module, then the following are equivalent:

1. Every R-module is Soc-N-injective.

2. Every cyclic R-module is Soc-N-injective.

3. Every R-module is Red-N-injective.

Proof:

(1) ⇒ (2). Because every cyclic R-module is an R-module.

(2) ⇒ (1). Suppose that S is a cyclic R-module which is Soc-N-injective and P is any R-module. We show that P is Soc-N-injective. Since S is Soc-N-injective, any R-homomorphism $g : \mathrm{Soc}(N) \to S$ extends to $\overline{g} : N \to S$. Since S and P are any two R-modules, there must exist at least one R-homomorphism $f : S \to P$ (say $f = 0$). The R-homomorphism $f \circ g : \mathrm{Soc}(N) \to P$ has an extension $f \circ \overline{g} : N \to P$; and hence P is Soc-N-injective.

(1) ⇒ (3). Let M be Soc-N-injective. Then, any R-homomorphism $f : \mathrm{Soc}(N) \to M$ extends to $f' : N \to M$. Since N is semi-simple, it follows that $\mathrm{Red}(N) \leq N = \mathrm{Soc}(N)$. But $\mathrm{Soc}(N) \subseteq \mathrm{Red}(N)$. Hence $\mathrm{Soc}(N) = \mathrm{Red}(N)$. Then, every R-homomorphism $f : \mathrm{Red}(N) \to M$ extends to $f' : N \to M$. Thus M is Red-N-injective.

(3) ⇒ (1). as a result of the fact that every Red-injective module is Soc-injective.

□

Proposition 23. Let R be a Principal Ideal Domain (P.I.D) and N be an R-module. Then, the following statements hold:

1. If every free R-module is Red-N-injective then each of its submodules is Red-N-injective.

2. If every projective R-module is Red-N-injective then each of its submodules is Red-N-injective.

3. Every projective R-module is Red-N-injective if and only if every free R-module is Red-N-injective.

Proof:

1. Suppose that every free R-module M is Red-N-injective, and $L \leq M$. Since over a P.I.D a submodule of a free module is free, L is free. By hypothesis, L is Red-N-injective.
2. Suppose that every projective R-module P is Red-N-injective, and $K \leq P$. Since over a P.I.D a submodule of a projective R-module is projective, K is projective. By hypothesis, K is Red-N-injective.
3. Over a P.I.D every projective module is free. The converse holds since any free module is a projective.

□

An R-module is called Noetherian if every submodule is finitely generated. For a field F, a finite-dimensional F-vector space V is a Noetherian F-module, since the submodules of V are its subspaces and they are all finite-dimensional by standard linear algebra.

Proposition 24. Let M be a Noetherian right R-module. Then, a direct sum of Red-M-injective modules is Red-M-injective.

Proof: For $D = \bigoplus_{i \in I} D_i$ a direct sum of Red-M-injective modules, let $f : K \to D$ be an R-homomorphism, where K is any semi-reduced submodule of M. Since K is finitely generated, $f(K) \leq \bigoplus_{i=1}^{n} D_i$ for some positive integer n. Since $\bigoplus_{i=1}^{n} D_i$ is Red-M-injective, then f can be extended to an R-homomorphism $\hat{f} : M \to D$. Thus D is Red-M-injective. □

Corollary 7. Let R_R be Noetherian. Then, a direct sum of Red-injective modules is Red-injective.

Proof: The proof follows from Proposition 24. □

Definition 36. (Ozcan, *et.al.*, 2008, Definition 2.10). Let X be a submodule of a module M. We say that Soc(M) respects X if there exists a direct summand A of M contained in X such that $X = A \oplus B$ and $B \leq \text{Soc}(M)$. M is called Soc(M)-lifting if Soc(M) respects every submodule of M.

From Definition 36 one can get the Definition 37:

Definition 37. Let X be a submodule of a module M. We say that $\text{Red}(M)$ respects X if there exists a direct summand A of M contained in X such that $X = A \oplus B$ and $B \leq \text{Red}(M)$. M is called $\text{Red}(M)$-lifting if $\text{Red}(M)$ respects every submodule of M.

Any semi-simple R-module P is $\text{Red}(P)$-lifting.

Proposition 25. Let N be an R-module. If N is $\text{Red}(N)$-lifting, then any R-module K is Red-N-injective if and only if K is N-injective.

Proof:

($\Rightarrow$). Suppose that K is Red-N-injective. Let L be any submodule of N, $\iota : L \to N$ the inclusion map and $f : L \to K$ any R-homomorphism. Since $\text{Red}(N)$ respects L, L has a decomposition $L = A \oplus B$ such that $A \subseteq^{\oplus} N$ and $B \leq \text{Red}(N)$. $N = A \oplus A'$ for some submodule A' of N. Then, $L = A \oplus (L \cap A')$ and $L \cap A'$ is semi-reduced. Let $i : L \cap A' \to L$ be the inclusion map and $f|_{L\cap A'} : L \cap A' \to K$. Since K is Red-N-injective, there exists an R-homomorphism $g : N \to K$ such that $g \circ \iota \circ i = f|_{L\cap A'}$. Now, define $h : N \to K$ by $h(a + a') = f(a) + g(a')$ $(a \in A, a' \in A')$. Then $h \circ \iota = f$, and hence K is N-injective.

($\Leftarrow$). Every N-injective module is Red-N-injective because $\text{Red}(N)$ is a submodule of N.

□

Remark 8. If $R/\text{Soc}(R_R)$ is a semi-simple Artinian, then a module M is Red-injective if and only if M is injective.

Proof: Let M be Red-injective, then by Proposition 14, M is Soc-injective. Since $R/\text{Soc}(R_R)$ is semi-simple Artinian and M is Soc-injective, then by Lemma 11, M is injective. Conversely, if M is injective then, by Proposition 15, it is Red-injective. □

Proposition 26, shows that Red-*quasi*-injective modules inherit a weaker version of C2-condition and C3-conditions.

Proposition 26. Suppose that an R-module N is Red-quasi-injective.

1. (Red-C2) If P and Q are semi-reduced submodules of N, $P \cong Q$ and P a direct summand of N, then Q is a direct summand of N.

2. (Red-C3) Let P and Q be semi-reduced submodules of N with $P \cap Q = \{0\}$. If both P and Q are direct summands of N; then $P \oplus Q$ is a direct summand of N.

Proof:

1. Since $P \cong Q$, and P is Red-N-injective, being a direct summand of the Red-quasi-injective module N, Q is Red-N-injective by Theorem 3.1. If $i : Q \to N$ is the inclusion map, the identity $id_Q : Q \to Q$ has an extension $\eta : N \to Q$ such that $\eta \circ i = id_Q$, and hence Q is a direct summand of N.

2. Since both P and Q are direct summands of N; then both P and Q are Red-N-injective. Then the semi-reduced module $P \oplus Q$ is Red-N-injective, and so a direct summand of N.

□

A property $\mathbb{P}$ on objects (resp. morphisms) in a module category $\mathfrak{M}_R$ is a categorical property if for any category equivalence $F : \mathfrak{M}_R \to \mathfrak{M}_S$, whenever $M \in \mathfrak{M}_R$ (resp. $g \in \mathrm{Hom}_R(M, N)$) satisfies $\mathbb{P}$, so does $F(M)$ (resp. $F(g)$).

Amin, *et al.*, (2005), Theorem 2.6 and Theorem 4.2 showed that Soc-injectivity is a Morita invariant property of modules and rings. However, the property of "a module M is reduced" is not a categorical property. Thus for a module M being reduced is not a Morita invariant property. It is known that if $S = M_n(R)$, then $F : \mathfrak{M}_R \to \mathfrak{M}_S$ is a category equivalence. We give an example of a module M_R which is reduced but M_S is not reduced.

Example 3.6. Let $M = M_n(R) = S$ where R is a domain. M_R is reduced. For if $mr^2 = 0$ where $m = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$ and $r \in R$, then $a_{ij}r^2 = 0$ for all $i, j \leq n$. If $m = 0$, then $mr = 0$. If $m \neq 0$, then at least one $a_{kl} \neq 0 \in R$ for some $1 \leq k, l \leq 2$. So,$a_{kl}r^2 = 0$ implies that $a_{kl}r = 0$ because in a domain $r^2 = 0$ if and only if $r = 0$. Hence, $mr = 0$. This shows that M_R is reduced. However, $M_S = M_n(R)_{M_n(R)} = F(M_R)$ is not reduced. Take $n = 2$, $m = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ with $a_{11} \neq 0$ and $s = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then, $ms^2 = 0$ but $ms \neq 0$.

3.2 Strongly Red-Injective Modules

In this section, quasi-Frobenius and right V-rings in terms of strongly Red- injective modules are characterized.

A module is quasi-continuous if it satisfies C1 and C3 conditions.

Proposition 27. For an R-module M with essential socle, the following conditions are equivalent:

1. M is strongly Red-injective.
2. M is strongly Soc-injective.
3. M is Soc-$E(M)$-injective.
4. $M = E \oplus T$, where E is injective and T has zero socle.
5. M is injective.

Proof:

$1 \Rightarrow 2$ by Proposition 14.

$2 \Leftrightarrow 3 \Leftrightarrow 4$ by Theorem 2.8.

$4 \Rightarrow 1$ because a strongly Soc-injective module with essential socle is injective by Corollary 4, and an injective module is strongly Red-injective by Proposition 15.

$2 \Rightarrow 5$ by Corollary 4.

$5 \Rightarrow 1$ by Proposition 15.

□

Corollary 8. If a ring R is right Noetherian, then every direct sum of strongly Red-injective R-modules is strongly Soc-injective.

Proof: Suppose that $D = \bigoplus_{\alpha \in \Lambda} D_\alpha$ is a direct sum of strongly Red-injective modules for each $\alpha \in \Lambda$. Since each D_α is strongly Red-injective, then by Proposition 14, D_α is strongly Soc-injective, and hence D is a direct sum of strongly Soc-injective modules. Since R is right Noetherian and D is a direct sum of strongly Soc-injective modules, by Theorem 2.10, we conclude that D is strongly Soc-injective. □

Definition 38. A module M is called CS if every submodule of M is essential in a summand, equivalently, if every closure of every submodule of M is a summand. M is called quasi-continuous if M is a CS module and M satisfies the C3-condition.

Proposition 28. (Amin, *et.al.*, 2005, Proposition 3.4). If M is a strongly Soc-injective R-module, then every semi-simple submodule K of M is essential in a summand.

Proof: If $\mathrm{Soc}(M) = \{0\}$ then M is strongly Soc-injective module and $K = \mathrm{Soc}(M) = \{0\}$ is essential in a summand. If $\mathrm{Soc}(M) \neq \{0\}$, then by Theorem 2.9, $M = E(\mathrm{Soc}(M)) \oplus T$, with $\mathrm{Soc}(T) = \{0\}$, and so $K \subseteq^e L \subseteq^\oplus M$ for some submodule L of M. □

Corollary 9. If M is a strongly *Red*-injective R-module, then every semi-simple submodule K of M is essential in a summand.

Proof: Suppose that a module M is strongly Red-injective module. Then, M is strongly Soc-injective right R-module. Using Proposition 27, then every semi-simple submodule K of M is essential in a summand. □

Proposition 29. (Amin, *et al.*, 2005, Proposition 5.3) If M is a strongly Soc-injective R-module and N is any R-module, then M is simple-N-injective. In particular, every strongly Soc-injective ring is simple-injective.

Proof: Let L be a submodule of N, and $\gamma : L \to M$ an R-homomorphism with $\gamma(L)$ simple. If $K = ker(\gamma)$, then γ induces an embedding $\widetilde{\gamma} : L/K \to M$ defined by $\widetilde{\gamma}(x + K) = \gamma(x)$, for all $x \in L$. Since M is strongly Soc-injective and L/K is simple, $\widetilde{\gamma}$ extends to an R-homomorphism $\overline{\gamma} : N/K \to M$. If $\eta : N \to N/K$ is the canonical quotient map, then the R-homomorphism $\overline{\gamma} \circ \eta : N \to M$ is an extension of γ, for if $x \in L$, $\overline{\gamma} \circ \eta(x) = \overline{\gamma}(x + K) = \widetilde{\gamma}(x + K) = \gamma(x)$ as required. □

Corollary 10. If M is a strongly Red-injective R-module and N is any R-module, then M is simple-N-injective. In particular, every strongly Red-injective ring is simple-injective.

Proof: Since every strongly *Red*-injective right module is strongly Soc-injective, the proof follows from Proposition 29. □

A ring R is called *right semi-Artinian* if every non-zero R-module has nonzero socle. A submodule $S \leq M$ is small if, for any submodule $N \leq M$, $S + N = E$ implies that $N = E$. The projective cover of an R-module M is a projective module P for which there is an epimorphism $P \to M$ whose kernel is small. R is left perfect ring if and only if every left R-modu1e has a projective cover.

Theorem 3.3. The following implications hold:

R is right semi-Artinian $\Rightarrow$ Every strongly Red-injective R-module is injective $\Rightarrow$ Every strongly Red-injective R-module is quasi-continuous.

In particular, over a left perfect ring R, every strongly Red-injective R-module is injective.

Proof:

For a right semi-Artinian ring R, suppose that a non-zero R-module M is strongly Red-injective. Then, $\{0\} \neq \mathrm{Soc}(M) \subseteq^e M$. A strongly Soc-injective module with essential socle is injective by Corollary 4. Since M has essential socle, it follows that it is injective. Suppose that every strongly Red-injective module M is injective. Then M is quasi-continuous because every injective module is quasi-continuous by Muhamed and Muller, (1990), p.18. The last statement follows from the fact that every left perfect ring is right semi-Artinian (Kasch, 1982, Theorem 11.6.3). □

A ring R is called *quasi-Frobenius* if R is right (or left) Artinian, right (or left) self-injective. Equivalently, R is *quasi-Frobenuis* if and only if every injective R-module is projective if and only if every projective R-module is injective.

Theorem 3.4. A ring R is quasi-Frobenius if and only if every strongly Red-injective module is projective.

Proof: If R is quasi-Frobenius, then R is right Artinian and so by Theorem 3.3 every strongly Red-injective module is injective, and hence projective since R is quasi-Frobenius. Conversely, if every strongly Red-injective module is projective, then in particular every injective R-module is projective. Then, by Faith, (1976), Corollary 2, R is quasi-Frobenius if and only if every injective module is projective. □

A ring whose all simple right R-modules are injective is called a right *V-ring*.

Theorem 3.5. R is a right V-ring if and only if every simple R-module is strongly Red-injective.

Proof: Suppose that M is a simple R-module where R is a right V-ring. Then, by definition of a V-ring, M is injective. Hence, M is strongly Red-injective. Conversely, suppose that any simple module M is strongly Red-injective. Since M is simple, $\mathrm{Soc}(M) = M$ and hence $\{0\} \neq \mathrm{Soc}(M) \subseteq^e M$. Since M has essential socle, it is injective by Corollary 4. Hence R is a right V-ring. □

Remark 9. 1. *Min-N*-injective modules need not be Soc-N-injective and,

2. strongly *simple*-injective modules need not be strongly Soc-injective, see Ozcan *et al.*, (2008), p.326.

Example 3.7. (Amin *et al.*, 2005, Example 4.5). Let $F = \mathbb{Z}_2$ be the field of two elements, $F_n = F$ for $n = 1, 2, 3, \cdots, Q = \prod_{i=1}^{\infty} F_i, S = \bigoplus_{i=1}^{\infty} F_i$. If R is the sub-ring of Q generated by 1 and S, then R is a von Neumann regular ring with $\mathrm{Soc}(R) = S$, and hence every R-homomorphism from a finitely generated ideal of R into R is given by multiplication by an element of R, in particular R is a *min*-injective ring. However, the map $f : S_R \to R_R$, given by $(a_1, a_2, a_3, a_4, \cdots) \mapsto (a_1, 0, a_3, 0, \cdots)$, cannot be extended to an R-homomorphism from R into R, and so R is not a Soc-injective ring.

There are examples (Hajarnavis & Norton, 1985, p. 265) of commutative semi-perfect simple-injective rings with essential socle, that are Kasch(i.e., every simple R-module embeds in R), but are not self-injective, and hence are not strongly Soc-injective.

Example 3.8. (Hajarnavis & Norton, 1985, Example 6.2, p.265) Let I be the set of all non-negative real numbers less than or equal to 1. Let k be a field and x a commuting indeterminate over k. Define T to be the set of all formal sums of the form $\sum_{i \in I} a_i x_i$ such that $a_i \in k$ and all except a finite number of a_i are zero. Putting $x^l = 0$ for each $l > 1$, T can be made into a commutative ring by defining addition and multiplication in the usual way. The ideals of T are of the form:

1. $A_i = x^i T$ for some $i \in I$ and
2. $B_i = \sum_{j \in I, i < j} x^j T$ for some $i \in I$.

$A_0 = T$, $B_0 = J = J^2$, where $J = J(T)$ (the radical of T), $B_1 = \{0\}$ and $A_1 = E(T)$. Further $r(A_i) = B_{1-i}$ and $r(B_i)$[1] $= A_{1-i}$ for each $i \in I$. T is not self-injective. Hence T is not strongly Soc-injective by Amin *et al.*, 2005, (Theorem 5.6).

[1] $r(B_i)$: right annihilator of B_i.

Chapter 4

Conclusion

In this dissertation, new generalizations of injective modules namely Red-injective and strongly Red-injective modules have been introduced. Discussions have been made on the differences between *min*-injective, *simple*-injective, Soc-injective and Red-injective modules. Here is a table summarising some similarities between properties of Red-injective modules versus Soc-injective modules.

SOC-INJECTIVE	**RED-INJECTIVE**
A module M_R is called Soc (socle)-N_R-injective if any R-homomorphism $f : \mathrm{Soc}(N) \to M$ extends to N. M is called Soc-quasi-injective if M is Soc-M-injective. M is called Soc-injective if it is Soc-R-injective. The ring R is called right (self) Soc-injective if the module R_R is Soc-injective (equivalently, if R_R is Soc-quasi-injective).	A module M_R is called Red-N_R-injective if any R-homomorphism $f : \mathrm{Red}(N) \to M$ extends to N. M is called Red-quasi-injective if it is Red-M-injective. M is called Red-injective if it is Red-R-injective. The ring R is right (self) Red-injective if the module R_R is Red-injective (equivalently, if R_R is Red-quasi-injective).
Let N be an R-module and $\{M_i : i \in I\}$ a family of R-modules. Then, the direct product $\prod_{i\in I} M_i$ is Soc-N-injective if and only if each M_i is Soc-N-injective, $i \in I$.	Let N be an R-module and $\{M_i : i \in I\}$ a family of R-modules. Then, the direct product $\prod_{i\in I} M_i$ is Red-N-injective if and only if each M_i is Red-N-injective, $i \in I$
Let M, N, and X be R-modules with $X \leq N$. If M is Soc-N-injective, then M is Soc-X-injective.	Let M, N, and X be R-modules with $X \leq N$. If M is Red-N-injective, then M is Red-X-injective
Let M, N, and $\overline{N}$ be R-modules with $M \cong N$. If M is Soc-$\overline{N}$-injective, then, N is Soc-$\overline{N}$-injective.	Let M, N, and $\overline{N}$ be R-modules with $M \cong N$. If M is Red-$\overline{N}$-injective, then, N is Red-$\overline{N}$-injective.
Let N be an R-module and $\{M_i : i \in I\}$ a family of R-modules. Then, N is Soc-$\bigoplus_{i\in I} M_i$ -injective if it is Soc-M_i-injective, for all $i \in I$.	Let N be an R-module and $\{M_i : i \in I\}$ a family of R-modules. Then N is Red-$\bigoplus_{i\in I} M_i$ -injective if it is Red-M_i-injective, for all $i \in I$.
An R-module M is Soc-injective if and only if M is Soc-P-injective for every projective R-module P.	An R-module M is Red-injective if and only if M is Red-P-injective for every projective R-module P.

Let M, N, and K be R-modules with $N \subseteq^{\oplus} M$. If M is Soc-K-injective, then N is Soc-K-injective. If N, $\overline{N}$, and M are R-modules, $N_R \cong \overline{N}_R$, and M is Soc-N-injective, then M is Soc-$\overline{N}$-injective.	Let M, N, and K be R-modules with $N \subseteq^{\oplus} M$. If M is Red-K-injective, then N is Red-K-injective. If N, $\overline{N}$, and M are R-modules, $N_R \cong \overline{N}_R$, and M is Red-N-injective, then M is Red-$\overline{N}$-injective.
Suppose that M is a Soc-quasi-injective R-module. 1. (Soc-C2) If K and L are semi-simple submodules of M with $K \cong L$, and $K \subseteq^{\oplus} M$, then $L \subseteq^{\oplus} M$. 2. (Soc-C3) Let K and L be semi-simple submodules of M with $K \cap L = \{0\}$. If $K \subseteq^{\oplus} M$ and $L \subseteq^{\oplus} M$, then $K \oplus L$ is a summand of M.	Suppose that an R-module N is Red-quasi-injective. 1. (Red-C2). If K and L are semi-reduced submodules of M with $K \cong L$, and $K \subseteq^{\oplus} M$, then $L \subseteq^{\oplus} M$. 2. (Red-C3). Let K and L be semi-reduced submodules of M with $K \cap L = \{0\}$. If $K \subseteq^{\oplus} M$ and $L \subseteq^{\oplus} M$, then $K \oplus L$ is a summand of M.
(Amin, *et al.*, 2005, Theorem 2.8). For a projective R-module M, the following conditions are equivalent: 1. Every quotient of a Soc-M-injective R-module is Soc-M-injective. 2. Every quotient of an injective R-module is Soc-M-injective. 3. Soc(M) is projective.	For a projective R-module M, the following conditions are equivalent: 1. Every quotient of a Red-M-injective module is Red-M-injective. 2. Every quotient of an injective R-module is Red-M-injective. 3. *Red*(M) is a projective R-module.

STRONGLY SOC-INJECTIVE	**STRONGLY RED-INJECTIVE**
An R-module M is called strongly Soc-injective, if M is Soc-N- injective for all R-modules N. A ring R is called strongly Soc-injective, if the R-module R_R is strongly Soc-injective.	A right R-module M is called strongly-Red-injective, if M is Red-N-injective for all R-modules N. A ring R is called strongly-Red-injective if the module R_R is strongly-Red-injective.
A ring R is quasi-Frobenius if and only if every strongly Soc-injective R-module is projective.	A ring R is quasi-Frobenius if and only if every strongly Red-injective R-module is projective.
The following conditions are equivalent: 1. R is right semi-Artinian. 2. Every strongly Soc-injective R-module is injective. 3. Every strongly Soc-injective R-module is quasi-continuous. In particular, over a left perfect ring R, every strongly Soc-injective R-module is injective.	The following conditions are equivalent: 1. R is right semi-Artinian. 2. Every strongly Red-injective R-module is injective. 3. Every strongly Red-injective R-module is quasi-continuous. In particular, over a left perfect ring R, every strongly Red-injective R-module is injective.

Note that Red-injectivity is a less restricted notion than injectivity but carries many properties of injectivity.

Red-injectivity is much closer to injectivity than Soc-injectivity. It should therefore carry much more properties of injectivity than does Soc-injectivity.

However, the above statement makes sense when modules are defined over commutative rings, which is a fact that allows every Red-injective module to be Soc-injective. Otherwise a semi-simple module need not be semi-reduced.

Example 4.1. Let the ring R be the collection of all 2×2 matrices over the field of real numbers. The module $M = R_R$ is semi-simple but not reduced. For if $m = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in M$ and $r = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \in R$, then $mr \neq 0$ but $mr^2 = 0$.

Since a direct sum of reduced modules is reduced, and M is a direct sum of simple modules which is not reduced, a simple module over a not necessarily commutative ring need not be reduced. Hence, M is not semi-reduced.

Further research

I have introduced and studied (strongly) Red-injective modules over a commutative ring with unity. As an extension to this work, an independent study on (strongly) Red-injective modules (resp. Red-injective rings) over a not necessarily commutative ring with unity should be carried out .

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