

Linköping Studies in Science and Technology
Dissertation No. 2252

Boolean complexes of involutions and smooth intervals in Coxeter groups

Vincent Umutabazi

Linköping Studies in Science and Technology.
Dissertations No. 2252

Boolean complexes of involutions and smooth intervals in Coxeter groups

Vincent Umutabazi



Department of Mathematics
Linköping, 2022



This work is licensed under a Creative Commons Attribution-NonCommercial 4.0 International License.

<https://creativecommons.org/licenses/by-nc/4.0/>

Linköping Studies in Science and Technology. Dissertations No. 2252

Boolean complexes of involutions and smooth intervals in Coxeter groups

Copyright © Vincent Umutabazi, 2022, unless otherwise noted

*Department of Mathematics
Linköping University
SE-581 83 Linköping, Sweden*

<https://doi.org/10.3384/9789179294694>

ISSN 0345-7524

ISBN 978-91-7929-468-7 (print), 978-91-7929-469-4 (PDF)

Printed by LiU-Tryck, Linköping, Sweden, 2022

Abstract

This dissertation is composed of four papers in algebraic combinatorics related to Coxeter groups.

By a *Coxeter group*, we mean a group W generated by a subset $S \subset W$ such that for all $s \in S$, we have $s^2 = e$, and $(ss')^{m(s,s')} = (s's)^{m(s,s')} = e$, where $m(s, s') = m(s's) \geq 2$ for all $s \neq s' \in S$. The condition $m(s, s') = \infty$ is allowed and means that there is no relation between s and s' . There are some partial orders that are associated with every Coxeter group. Among them, the most notable one is the Bruhat order. Coxeter groups and their Bruhat orders have important properties that can be utilised to study Schubert varieties.

In Paper I, we consider Schubert varieties that are indexed by involutions of a finite simply laced Coxeter group. We prove that the Schubert varieties which are indexed by involutions that are not longest elements of some standard parabolic subgroups are not smooth.

Paper II is based on the Boolean complexes of involutions of a Coxeter group. These complexes are analogues of the Boolean complexes invented by Ragnarsson and Tenner. We use discrete Morse theory to compute the homotopy type of the Boolean complexes of involutions of some infinite Coxeter groups together with all finite Coxeter groups.

In Paper III, we prove that the subposet induced by the fixed elements of any automorphism of a pircon is also a pircon. In addition, our main results are applied to the symmetric groups S_{2n} . As a consequence, we prove that the signed fixed point free involutions form a pircon under the dual of the Bruhat order on the hyperoctahedral group.

Let W be a Weyl group and I denote a Bruhat interval in W . In Paper IV, we prove that if the dual of I is a zircon, then I is rationally smooth. After examining when the converse holds, and being influenced from conjectures by Delanoy, we are led to pose two conjectures. Those conjectures imply that for Bruhat intervals in type A , duals of smooth intervals, zircons, and being isomorphic to lower intervals are all equivalent. We have verified our conjectures in types A_n , $n \leq 8$, by using SageMath.

Populärvetenskaplig sammanfattning

Denna avhandling består av fyra artiklar inom algebraisk kombinatorik med kopplingar till *Coxetergrupper*. Informellt kan Coxetergrupper sägas vara ett sätt att beskriva och förklara symmetrier, exempelvis hos geometriska objekt som reguljära polyedrar. En del Coxetergrupper är ändliga, en del oändliga. Permutationsgrupper och hyperoktahedra grupper är klassiska exempel på ändliga Coxetergrupper, medan symmetrierna hos tesselleringar av planet utgör exempel på oändliga.

Coxetergrupper dyker upp inom olika områden av matematiken, som algebraisk geometri, kombinatorik och Lieteori. Inom dessa fält är objekt vid namn *Schubert-varieteter* välstuderade. En del är *släta*, vilket ungefär betyder att de lokalt ser ut som euklidiska rum, en del är det inte; de kallas *singulära*. Inom algebraisk kombinatorik studeras sätt att med hjälp av Coxetergrupperns kombinatoriska egenskaper beskriva singulariteter hos Schubertvarieteter. I denna avhandlingens första artikel visas att för en viss klass av Coxetergrupper gäller att, frånsett en på förhand självklar uppsättning undantag, alla Schubertvarieteter som hör till involutioner i gruppen är singulära.

I den andra artikeln studeras vissa klasser av *booleska cellkomplex*, alltså reguljära cellkomplex vars sidopomängder är simpliciella. Löst uttryckt innebär det att ett booleskt cellkomplex uppstår genom att foga samman simplex längs gemensamma sidor. De komplex som studeras i artikeln associeras med booleska involutioner i Coxetergrupper. Det visas att för alla ändliga, och för många oändliga, Coxetergrupper är dessa komplex homotopiekvivalenta med buketter av sfärer. Det betyder ungefär att de kan deformerar, på ett som inte river upp eller fyller igen hål, till en uppsättning sfärer sammanfogade i en gemensam punkt.

En *speciell matchning* på en pomängd (partiellt ordnad mängd) är ett sätt att para ihop elementen som i en viss bemärkelse respekterar ordningen. En *zirkon* är en pomängd som är sådan att varje delmängd som består av allt som är mindre än ett fixerat element har en speciell matchning. På likartat sätt definieras en *pirkon*, utom att det räcker med speciella *partiella* matchningar (vissa element behöver inte paras ihop med något). Viktiga exempel på båda dessa klasser av pomängder har att göra med så kallade *Bruhatordningar* på Coxetergrupper. I avhandlingens tredje artikel visas att för vilken som helst pomängdssymmetri hos en pirkon bildar de av symmetrin fixerade elementen själva en pirkon; motsvarande resultat var förut endast känt för zirkoner. Som en följd beskrivs tidigare okända pirkoner i hyperoktahedra grupper.

I den fjärde artikeln betraktas intervall i Bruhatordningen på en Coxetergrupp. Det visas att om dualen till ett intervall (alltså intervallets element med omvänd ordning) är en zirkon, så är intervallet rationellt slätt. I vissa Coxetergrupper har omvändningen förmodats av Delanoy. I artikeln generaliseras hans förmodan för permutationsgrupper och generaliseringen styrks av datorberäkningar.

Acknowledgements

Firstly, I would like to thank my sincere supervisor, Axel Hultman, for supporting me in my PhD studies. He introduced me to the area of algebraic combinatorics, and guided me in my research. In fact, my supervisor gave me a foundation in combinatorics which helped me achieve my learning objectives.

I would also like to thank my second supervisor, Jan Snellman, for his discussions about my papers and thesis. He introduced me to some mathematical software that I needed for coursework and for the research that led to the fourth paper of this thesis.

In addition, many thanks go to the SIDA bilateral program through ISP which gave me a stipend. Without it, I would not be at the level of completing my studies.

I would like to thank the University of Rwanda for giving me a study leave during all of my studies.

I also appreciate Mikael Hansson for fruitful discussions about paper three of this thesis.

To my fellow PhD students in the Department of Mathematics, I would like to extend thanks for more encouragement and good cooperation.

Finally, I express my gratitude to Vincent Murenzi, my son, for allowing me to do my studies while being away from him. His patience and love are of great value.

List of included papers

This dissertation includes four papers:

- (I) A. Hultman and V. Umutabazi, *Smoothness of Schubert varieties indexed by involutions in finite simply laced types*, Sém. Lothar. Combin. 84 (2022), Art. B84b, 8pp.
- (II) A. Hultman and V. Umutabazi, *Boolean complexes of involutions*, preprint submitted for publication (2021).
- (III) V. Umutabazi, *Fixed elements of automorphisms of pircons*, preprint (2022).
- (IV) V. Umutabazi, *Zircons and smooth Bruhat intervals in symmetric groups*, preprint submitted for publication (2022).

Papers I and II are joint work. The thoughts about these two papers were elaborated and expanded and the results produced by the authors. The authors contributed equally. I also conducted the writing of these papers.

Contents

Abstract	i
Populärvetenskaplig sammanfattning	iii
Acknowledgements	v
List of included papers	vii
Contents	ix
I Introduction	1
1 Partially ordered sets	3
2 Coxeter groups and Schubert varieties	5
2.1 Coxeter groups	5
2.1.1 Classification of finite Coxeter groups	6
2.1.2 Bruhat order and Bruhat graphs	6
2.2 Reflection subgroups of a Coxeter group	8
2.3 Schubert varieties	9
3 Zircons and pircons	11
3.1 Zircons	11
3.2 Pircons	12
4 Boolean complexes	15
4.1 Boolean cell complexes	15
4.1.1 Discrete Morse theory	16
4.2 Boolean complexes of Coxeter systems	17
5 Summary of papers	21
5.1 Paper I: Smoothness of Schubert varieties indexed by involutions in finite simply laced types	21
5.2 Paper II: Boolean complexes of involutions	21
5.3 Paper III: Fixed elements of automorphisms of pircons	21
5.4 Paper IV: Zircons and smooth Bruhat intervals in symmetric groups	22
II Papers	25
Paper I	27
Paper II	37
Paper III	57
Paper IV	65

Introduction

1 – Partially ordered sets

We recall some preliminaries on partially ordered sets that shall be used in other chapters of this thesis. For more details about partially ordered sets, we recommend the reader to consult [22].

By a *partially ordered set* or (*poset*), we mean a set P , together with a binary operation “ \leq ” such that for all $x, y, z \in P$:

- (1) $x \leq x$ (reflexive),
- (2) If $x \leq y$ and $y \leq x$ then $x = y$ (antisymmetric),
- (3) If $x \leq y$ and $y \leq z$ then $x \leq z$ (transitive).

Let $x, y \in P$. If $x \leq y$ and $x \neq y$, we write $x < y$.

Definition 1.0.1 *Let $x < y$ in P . Then, if there is no $z \in P$ such that $x < z < y$, we say that x is covered by y (or y covers x), and write $x \triangleleft y$ (or $y \triangleright x$).*

Definition 1.0.2 *Let P be a finite poset. The Hasse diagram of P is the directed graph having P as vertex set and the cover relation as edge set, drawn in such a way that if $x \triangleleft y$, then x is below y .*

A *maximum* $\hat{1} \in P$ is a unique element that satisfies $x \leq \hat{1}$ for all $x \in P$. Similarly, a *minimum* $\hat{0} \in P$ is a unique element that satisfies $\hat{0} \leq x$ for all $x \in P$. By an *induced subposet* of P , we mean an ordered subset $R \subseteq P$ so that for all $x, y \in R$, $x \leq y$ in R if and only if $x \leq y$ in P . The subposet of P induced by $[x, y] := \{t \in P \mid x \leq t \leq y\}$ is called a (*closed*) *interval*.

Definition 1.0.3 *An induced subposet I of P satisfying the property that, for each $t \in I$, all elements x below t (i.e., $x \leq t$) are also in I , is called an *order ideal*.*

An order ideal having a maximum is said to be *principal*.

Definition 1.0.4 *Let P_1 and P_2 be posets. A function $\phi : P_1 \rightarrow P_2$ is called an *order-preserving map* if for all $x_1, x_2 \in P_1$ with $x_1 \leq x_2$ in P_1 it holds that $\phi(x_1) \leq \phi(x_2)$ in P_2 .*

A bijective order-preserving map $\phi : P_1 \rightarrow P_2$ whose inverse $\phi^{-1} : P_2 \rightarrow P_1$ is order-preserving is called an *isomorphism* of posets. An *automorphism* of a poset P is an isomorphism from P to itself.

2 – Coxeter groups and Schubert varieties

In this chapter we recall some properties of Coxeter groups and Schubert varieties. For more on these subjects, see [4], [17] and [3].

2.1 Coxeter groups

A *Coxeter group* is a group W generated by a set of *simple reflections* $S \subset W$, under relations of the form $s^2 = e$ for all $s \in S$, and

$$(ss')^{m(s,s')} = (s's)^{m(s,s')} = e,$$

where $m(s, s') \in \{2, 3, \dots\} \cup \{\infty\}$ for all $s \neq s' \in S$. If it happens that $m(s', s) = \infty$, then it means that there is no relation between s and s' . Here, e stands for the identity element in W . The pair (W, S) is called a *Coxeter system* and the cardinality $|S|$ is called the *rank* of (W, S) . Every $w \in W$ is a product of simple reflections from S . This means that $w = s_1 s_2 \cdots s_j$ for some $s_i \in S$. Among all such expressions for w , let $s_1 s_2 \cdots s_j$ be some expression for which j is minimal. Then j is the *length* of w (denoted by $\ell(w) = j$) and the expression $s_1 s_2 \cdots s_j$ is *reduced*. If W is finite, then there is an element $w_0 \in W$, called the *longest element*, with the property that $\ell(w_0 s) < \ell(w_0)$ for all $s \in S$ and $\ell(w_0) \geq \ell(w_0 w)$ for all $w \in W$.

Definition 2.1.1 *A subgroup of W generated by a subset $J \subseteq S$ is called a standard parabolic subgroup of W .*

Let $W_J = \langle J \rangle$ be a standard parabolic subgroup of W . Then, (W_J, J) is a Coxeter system. If W_J is finite, it has a longest element denoted by $w_0(J)$.

Definition 2.1.2 *A Coxeter system (W, S) for which $m(s, s') \leq 3$ for all $s, s' \in S$ is called simply laced (or (W, S) is said to be of simply laced type). Otherwise, it is called a multiply laced Coxeter system.*

We often abuse notation and refer to Coxeter *groups* even though we really have an entire Coxeter *system* in mind.

For example, the symmetric group S_n generated by simple transpositions $s_i = (i, i + 1)$ for all $i = 1, 2, \dots, n - 1$ is of finite simply laced type (this is a Coxeter group of type A_{n-1} ; see e.g. [4]). Other important examples of finite simply laced Coxeter groups are of types D_n , E_6 , E_7 , and E_8 as described in the classification below.

2.1.1 Classification of finite Coxeter groups

Coxeter groups can be represented by their *Coxeter graphs* (*Coxeter diagrams*). By the *Coxeter graph* of W , we mean the simple graph whose set of vertices is S and whose edges are unordered pairs $\{s, s'\}$ if $m(s, s') \geq 3$. If $m(s, s') \geq 4$, we label the edge $\{s, s'\}$ by that number, and if $m(s, s') = 3$ the edge has no label. Note that if $m(s, s') = 2$ (i.e., s and s' commute), then there is no edge between s and s' . A Coxeter group whose Coxeter graph is connected is called *irreducible*. Finite irreducible Coxeter groups have been classified (see [4] and [17]). In that classification we have:

- (1) Three classical families of types A_n ($n \geq 1$), B_n ($n \geq 2$), D_n ($n \geq 4$),
- (2) Six exceptional groups of types E_6, E_7, E_8, F_4, H_3 and H_4 ,
- (3) One family of dihedral groups of type $I_2(m)$, $m \geq 3$.

The Coxeter graphs are recorded in Figure 2.1. Note that $I_2(3) = A_2$ and $I_2(4) = B_2$.

2.1.2 Bruhat order and Bruhat graphs

Let $T := \{wsw^{-1} \mid s \in S, w \in W\}$ be the set of *reflections* in W . For $u, w \in W$, write $u \rightarrow w$ if there is $t \in T$ such that $w = tu$ and $\ell(w) > \ell(u)$.

Definition 2.1.3 *The partial order relation on W defined by $u \leq w$ if there is a sequence $u = w_0 \rightarrow w_1 \rightarrow \cdots \rightarrow w_m = w$, is called the Bruhat order on W .*

Let $\text{Br}(W)$ denote the Bruhat order on W .

Definition 2.1.4 *The directed graph with vertex set W and with edge set*

$$\{(v, w) \mid v \rightarrow w\}$$

is called the Bruhat graph of W . We denote the Bruhat graph of W by $\text{Bg}_S(W)$.

Note that in Paper IV of this thesis, we think of Bruhat graphs as being undirected.

Definition 2.1.5 *Let $u \leq w$ in $\text{Br}(W)$. Then $[u, w] := \{v \in W \mid u \leq v \leq w\}$ is called a Bruhat interval.*

The identity element is the minimum of $\text{Br}(W)$. If W is finite, w_0 is the maximum.

Definition 2.1.6 *For any $w \in W$, $[e, w] := \{v \in W \mid e \leq v \leq w\}$ is called a lower Bruhat interval.*

Theorem 2.1.7 (Subword property) *Let $u, w \in W$, and $s_1 s_2 \cdots s_n$ be a reduced expression for w . Then $u \leq w$ if and only if $s_{j_1} s_{j_2} \cdots s_{j_k}$ is a reduced expression for u , for some $1 \leq j_1 < j_2 < \cdots < j_k \leq n$.*

For any Coxeter group W , the map given by $x \mapsto x^{-1}$ is an automorphism of the Bruhat order of W . For example, this follows from Theorem 2.1.7 since if we reverse an expression for x , we get an expression for its inverse. That is,

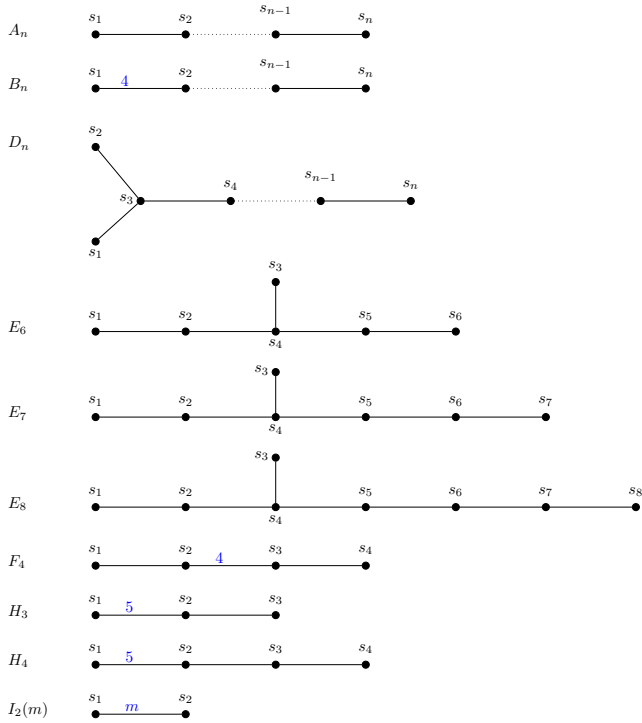


Figure 2.1: The finite, irreducible Coxeter groups.

Lemma 2.1.8 For all $x, y \in W$, $x < y$ if and only if $x^{-1} < y^{-1}$.

Theorem 2.1.9 (Chain property) Let $u < w$. Then, there exists a chain

$$u = u_0 < u_1 < \cdots < u_n = w$$

such that $\ell(u_i) = \ell(u_{i-1}) + 1$ for all $1 \leq i \leq n$.

For $w \in W$, let $D_L(w) := \{s \in S \mid \ell(sw) < \ell(w)\}$, and $D_R(w) := \{s \in S \mid \ell(ws) < \ell(w)\}$. We call $D_L(w)$ the *left descent set* of w , and $D_R(w)$ the *right descent set* of w respectively.

The following lemma due to Deodhar is known as the *lifting property*.

Lemma 2.1.10 ([10]) Let $v < w$ and $s \in D_R(w) \setminus D_R(v)$. Then $v \leq ws$ and $vs \leq w$.

Note that the left hand version of Lemma 2.1.10 holds too. Using the above lemma, one can show that for any u and v in W , there exists $w \in W$ such that $u \leq w$ and $v \leq w$. This implies that $\text{Br}(W)$ is always a directed poset, even if W is infinite.

2.2 Reflection subgroups of a Coxeter group

In this section we recall important results due to Dyer [12] on subgroups generated by reflections. Suppose that W' is a subgroup of W . Recall that T is the set of reflections of W .

Definition 2.2.1 If $W' = \langle W' \cap T \rangle$, then W' is a reflection subgroup of W .

Definition 2.2.2 If $W' = \langle t, t' \rangle$ for $t \neq t' \in T$, W' is called a dihedral reflection subgroup of W .

From Definitions 2.2.1 and 2.2.2, it is clear that every dihedral reflection subgroup of W is also a reflection subgroup of W . However a reflection subgroup need not be dihedral.

Theorem 2.2.3 ([12]) Let $t_1, t_2, t_3, t_4 \in T$ and $t_1 t_2 = t_3 t_4 \neq e$. Then, $\langle t_1, t_2, t_3, t_4 \rangle$ is a dihedral reflection subgroup of W .

Let $N(w) := \{t \in T \mid \ell(tw) < \ell(w)\}$, and $Y := \{t \in T \mid N(t) \cap W' = \{t\}\}$. Then Y is a set of simple reflections for W' :

Theorem 2.2.4 ([12]) If W' is a reflection subgroup of W , then (W', Y) is a Coxeter system.

If R is an arbitrary subset of W , we let $\text{Bg}_S(R)$ denote the directed subgraph of $\text{Bg}_S(W)$ induced by R . Being a Coxeter system, (W', Y) has a Bruhat graph of its own, while W' also induces a subgraph of $\text{Bg}_S(W)$. However, the two graphs coincide:

Theorem 2.2.5 ([12]) If W' is a reflection subgroup of W , then $\text{Bg}_S(W') = \text{Bg}_Y(W')$.

2.3 Schubert varieties

We now describe how Bruhat graphs of Coxeter groups can be utilized to study the geometry of Schubert varieties. For more preliminaries and background on Schubert varieties, one can consult [3].

Consider a semi-simple, simply connected algebraic group G defined over the field of complex numbers \mathbb{C} . Let \mathcal{T} be a maximal torus contained in a Borel subgroup \mathcal{B} of G . Let $W = \mathcal{N}(\mathcal{T})/\mathcal{T}$ be the *Weyl group*, where $\mathcal{N}(\mathcal{T})$ is the normalizer of \mathcal{T} in G . In fact, a Weyl group is a finite Coxeter group. For example, if $G = \mathrm{SL}_n(\mathbb{C})$, W is of type A_{n-1} . The *flag variety* is G/\mathcal{B} , and it decomposes as

$$G/\mathcal{B} = \bigsqcup_{w \in W} \mathcal{B}w\mathcal{B}/\mathcal{B} \text{ (i.e., disjoint union).}$$

Each set $\mathcal{B}w\mathcal{B}/\mathcal{B}$ is a *Schubert cell*. The closure $\overline{\mathcal{B}w\mathcal{B}/\mathcal{B}}$ is a *Schubert variety*. Let $X(w)$ denote the Schubert variety corresponding to $w \in W$ (i.e., $X(w) = \overline{\mathcal{B}w\mathcal{B}/\mathcal{B}}$). The Bruhat order on W controls containment of these varieties. That is, $v \leq w$ if and only if $X(v) \subseteq X(w)$.

Let $\mathbf{Bg}_S(v) := \mathbf{Bg}_S([e, v])$ for some lower interval $[e, v]$ in $\mathrm{Br}(W)$, and let y be a vertex in $\mathbf{Bg}_S(v)$.

Definition 2.3.1 *The degree of y in $\mathbf{Bg}_S(v)$ is the number of edges that are incident to y (regardless of the directions).*

Let $\mathrm{deg}_{e,v}(y)$ denote the degree of the vertex y in $\mathbf{Bg}_S(v)$. The following theorem which is a result from [13] bounds this degree. For special classes of Coxeter groups, this theorem can also be found in [20, 7].

Theorem 2.3.2 ([13]) *For every $y \leq v$, $\mathrm{deg}_{e,v}(y) \geq \ell(v)$.*

We say that $\mathbf{Bg}_S(v)$ is *regular*, if for every vertex y in $\mathbf{Bg}_S(v)$, $\mathrm{deg}_{e,v}(y) = \mathrm{deg}_{e,v}(v)$. In fact, $\mathrm{deg}_{e,v}(v) = \ell(v)$ for every $v \in W$.

Rational smoothness is a weaker notion than smoothness which informally means that a variety looks like a smooth variety up to local cohomology. See e.g. [3] for details. In this thesis, we take Theorem 2.3.3 as the definition of rational smoothness of a Schubert variety.

Theorem 2.3.3 (Carrell-Peterson [7]) *Let $v \in W$. Then $X(v)$ is rationally smooth if and only if $\mathbf{Bg}_S(v)$ is regular.*

Theorem 2.3.4 ([8]) *Let W be a finite simply laced Coxeter group and $w \in W$. Then, $X(w)$ is rationally smooth if and only if it is smooth.*

Note that in general a rationally smooth Schubert variety need not be smooth. For example, if W is of type $B_2 = \langle s_1, s_2 \rangle$ where s_2 corresponds to the short root, then $X(s_2s_1s_2)$ is rationally smooth but not smooth. We now have the following corollary.

Corollary 2.3.5 *Suppose that W is a finite simply laced Coxeter group, and let $v \in W$. Then, $X(v)$ is smooth if and only if $\mathbf{Bg}_S(v)$ is regular.*

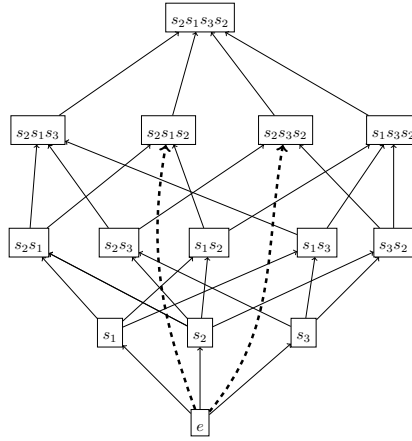


Figure 2.2: $\mathbf{Bg}_S(s_2 s_1 s_3 s_2)$ for $s_2 s_1 s_3 s_2 \in A_3$

Example 2.3.6 Consider a Coxeter group of type A_3 whose Coxeter graph is as in Figure 2.1 for $n = 3$. The directed graph $\mathbf{Bg}_S(s_2 s_1 s_3 s_2)$ is not regular (see Figure 2.2). Using Corollary 2.3.5, we have that $X(s_2 s_1 s_3 s_2)$ is not smooth. However $X(v)$ is smooth for every $v < s_2 s_1 s_3 s_2$. Note also that, if we ignore the dashed edges and arrowheads in $\mathbf{Bg}_S(s_2 s_1 s_3 s_2)$, we get the Hasse diagram of the lower Bruhat interval $[e, s_2 s_1 s_3 s_2]$.

3 – Zircons and pircons

In this chapter, we recall the definitions of zircons and pircons and some of their properties.

3.1 Zircons

Let P be a poset and $M : P \rightarrow P$ be an involution. Then, M is called a *matching* of P if $M(x) \triangleleft x$ or $x \triangleleft M(x)$ for all $x \in P$. Definition 3.1.1 is due to Brenti, and can be found in [5, 6].

Definition 3.1.1 *Let M be a matching of P . Then, M is called special if for all $x, y \in P$ with $x \triangleleft y$, either $M(x) = y$ or $M(x) \triangleleft M(y)$.*

For $x \in P$, let $P_{\leq x} := \{q \in P \mid q \leq x\}$.

Proposition 3.1.2 ([6]) *Let M be a special matching of a poset P , and $M(x) \triangleleft x$ for some $x \in P$. Then M restricts to a special matching of $P_{\leq x}$.*

If a poset is *Eulerian*, a special matching is equivalent to a *compression labelling* as was independently invented by du Cloux [11]. See e.g. [22] for the definition of the Eulerian property. Note that Bruhat orders are examples of Eulerian posets.

Lemma 3.1.3 below is called the *Lifting property for special matchings*. It is essentially due to Brenti [6] who stated it under a gradedness assumption. A proof without this assumption appears in [16].

Lemma 3.1.3 *Let M be a special matching of a locally finite poset P . Let also $y, z \in P$ be such that $y \triangleleft z$ and $M(z) \triangleleft z$. The following conditions are satisfied.*

- (1) $M(y) \leq z$,
- (2) $M(y) \triangleleft y \Rightarrow M(y) \triangleleft M(z)$.

Definition 3.1.4 ([16]) *A zircon is a poset P , such that for every non-minimal element x , the principal order ideal $P_{\leq x}$ is finite and has a special matching.*

Originally, zircons were introduced by Marietti in [18] in a different way. However, as was proved in [16], those two definitions of zircons are equivalent.

For example, let W be a Coxeter group. Every lower Bruhat interval $[e, w]$ is finite and, if $e \neq w$, has a special matching given by multiplication by any descent element of w . Hence, the Bruhat order of any Coxeter group is a zircon.

The following lemma is one of the main results from [16].

Lemma 3.1.5 *If Z is a zircon with an automorphism, then the subposet of Z induced by the fixed points of the automorphism is itself a zircon.*

Example 3.1.6 Consider the Coxeter group of type A_3 . We have that $\text{Br}(A_3)$ is a zircon, and that the map $x \mapsto x^{-1}$ is an automorphism of $\text{Br}(A_3)$. Hence, the subposet $F_1(A_3)$ induced by the fixed points of that automorphism is a zircon. Note that the fixed points are the involutions. The Hasse diagram of $F_1(A_3)$ is presented in Figure 3.1 where the dashed lines indicate a special matching on $F_1(A_3)$.

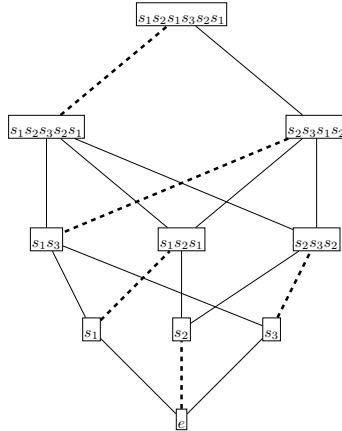


Figure 3.1: The Bruhat order on the involutions of A_3 .

3.2 Pircons

The following definition is taken from [2].

Definition 3.2.1 Let P be a finite poset with $\hat{1}$. An involution $M : P \rightarrow P$ is called a special partial matching if:

- (1) $M(\hat{1}) < \hat{1}$
- (2) For all $x \in P$, we have $x < M(x)$, or $M(x) < x$, or $M(x) = x$, and
- (3) If $x < y$ and $M(x) \neq y$, then $M(x) < M(y)$.

Note that a special partial matching without fixed points is a special matching. We have the following analogue of Proposition 3.1.2.

Proposition 3.2.2 ([2]) Let M be a special partial matching of P and $M(y) \leq y$. If $x \in P_{\leq y}$, then $M(x) \in P_{\leq y}$. In particular, if $M(y) < y$, then M restricts to a special partial matching of $P_{\leq y}$.

The following lemma is the *Lifting property for special partial matchings*. It can be found in [1].

Lemma 3.2.3 *Let P be a finite poset with $\hat{1}$, and with a special partial matching M . If $x, y \in P$ are such that $x < y$ and $M(y) \leq y$, then:*

- (1) $M(x) \leq y$,
- (2) $M(x) \leq x \Rightarrow M(x) < M(y)$,
- (3) $x \leq M(x) \Rightarrow x \leq M(y)$.

Definition 3.2.4 below is from [1].

Definition 3.2.4 *A pircon is a poset P such that for every non-minimal $x \in P$, the principal order ideal $P_{\leq x}$ is finite and admits a special partial matching.*

Clearly, every zircon is a pircon. The following are examples of pircons which are not zircons.

Example 3.2.5 *Let (W, S) be a Coxeter system and $J \subseteq S$. Then*

$$W^J := \{w \in W \mid w < ws \text{ for all } s \in J\}$$

is a set of minimal length representatives of the cosets in the parabolic quotient W/W_J . The Bruhat order $\text{Br}(W^J)$ is a pircon; see [1].

Example 3.2.6 *Consider a Coxeter group of type A_{2n-1} . Let $C(w_0)$ be the conjugacy class of the longest element $w_0 \in A_{2n-1}$. By [2, Theorem 4.3], $C(w_0)$ with the dual of the Bruhat order inherited from A_{2n-1} is a pircon.*

Example 3.2.7 *Figure 3.2 illustrates another example of a pircon. A special partial matching is marked by the dashed lines. Note that the minimum is a fixed point.*

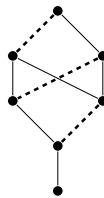


Figure 3.2: A pircon which is not a zircon.

4 – Boolean complexes

We now recall some aspects of CW complexes and regular cell complexes that are needed in this thesis. More on such complexes can be found in [19] and [23].

4.1 Boolean cell complexes

Consider a point $x := (x_1, \dots, x_n) \in \mathbb{R}^n$, and let:

- (1) $\|x\| = (\sum_{i=1}^n x_i^2)^{1/2}$,
- (2) $\mathbf{B}^n := \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$ be the *unit n -ball* in \mathbb{R}^n ,
- (3) $\text{int } \mathbf{B}^n := \{x \in \mathbb{R}^n \mid \|x\| < 1\}$ be the *interior* of \mathbf{B}^n ,
- (4) $\mathbf{S}^{n-1} := \{x \in \mathbb{R}^n \mid \|x\| = 1\}$ for all $n \geq 1$, be the *unit $(n-1)$ -sphere*.

Note that the *zero sphere* is $\mathbf{S}^0 = \{\text{two points}\}$, $\mathbf{S}^{-1} = \emptyset$ (i.e., the empty set), and the *zero ball* is $\mathbf{B}^0 = \{\text{a single point}\}$.

Definition 4.1.1 *Let X be a Hausdorff space. An open n -cell (or an open cell of dimension n) is a subspace of X that is homeomorphic to $\text{int } \mathbf{B}^n$.*

Let also $\bar{\sigma}$ be the closure of σ in X , and $\dot{\sigma} = \bar{\sigma} \setminus \sigma$. Let us denote the dimension of σ by $\dim(\sigma)$. If $\dim(\sigma) = 0$, we have that $\sigma = \bar{\sigma} = \{\text{a point}\}$.

Definition 4.1.2 *A finite cell complex (or a finite CW complex) Δ is a finite collection of disjoint open cells σ_i for which $\|\Delta\| := \cup_{\sigma_i \in \Delta} \sigma_i$ is a Hausdorff space such that:*

- (1) *For each open n -cell $\sigma_i \in \Delta$, there exists a continuous map $f_i : \mathbf{B}^n \rightarrow \|\Delta\|$ whose restriction $f_i|_{\text{int } \mathbf{B}^n} : \text{int } \mathbf{B}^n \rightarrow \sigma_i$ is a homeomorphism.*
- (2) *f_i takes the boundary $\partial \mathbf{B}^n$ into the union of cells whose dimension is less than n .*

In fact, f_i carries \mathbf{B}^n onto $\bar{\sigma}_i$; however \mathbf{B}^n and $\bar{\sigma}_i$ are not necessarily homeomorphic. The map f_i is the *characteristic map* for σ_i .

By a *finite regular cell complex*, we mean a finite cell complex Δ for which every characteristic map $f_i : \mathbf{B}^n \rightarrow \bar{\sigma}_i$ is a homeomorphism, and every $\dot{\sigma}$ is the union of cells of Δ .

Definition 4.1.3 *Let Δ be a finite regular cell complex. The face poset of Δ is the poset $P(\Delta)$ of all cells of Δ ordered by set inclusion of their closures together with a minimum element which we refer to as the empty cell.*

By a *Boolean algebra*, we mean a finite poset consisting of all subsets of $\{1, 2, \dots, m\}$, for some m , ordered by set inclusion.

Definition 4.1.4 A simplicial poset is a poset in which every principal order ideal is isomorphic to a Boolean algebra.

Definition 4.1.5 A Boolean cell complex is a finite regular cell complex Δ whose face poset $P(\Delta)$ is a simplicial poset.

Notice that simplicial complexes are special cases of Boolean cell complexes. Note also that if Q is a simplicial poset, then there is a Boolean cell complex Δ such that $Q = P(\Delta)$, and Δ is unique up to cellular isomorphism. For an example of the construction of Δ with $Q = P(\Delta)$, we refer to the pictures in Figures 4.1, 4.2, 4.3, and 4.4.

4.1.1 Discrete Morse theory

Let P be a poset with cover relation denoted by \triangleleft . In this chapter, we allow a *matching* M to be an involution $P \rightarrow P$ such that for all $x \in P$, either

- (1) $M(x) = x$, or
- (2) $M(x) \triangleleft x$, or
- (3) $x \triangleleft M(x)$.

If M fixes an element $x \in P$ such as in item 1 above, we say that x is *critical*.

Let $H(P)$ denote the Hasse diagram of P . Let also M be a matching on P , and $H_M(P)$ be the directed graph constructed from the Hasse diagram of P by reversing every arrow that belongs to M . In other words, the edge set of $H_M(P)$ is $\{p \rightarrow q \mid q \triangleleft p \text{ and } M(q) \neq p\} \cup \{p \rightarrow q \mid p \triangleleft q \text{ and } M(q) = p\}$.

Definition 4.1.6 A matching $M : P \rightarrow P$ is said to be *acyclic* if there are no directed cycles in $H_M(P)$.

The reversed arrows from $H(P)$ are said to be *upward* in $H_M(P)$, whereas the non reversed arrows are said to be *downward* in $H_M(P)$.

Definition 4.1.7 Let $\{S_j\}_{j \in J}$ be some collection of spheres, where J is an indexing set. Let $\bigvee_{j \in J} S_j$ denote the space that arises by selecting a point in every sphere in the collection, taking the disjoint union of the spheres, and identifying all the selected points. Then, $\bigvee_{j \in J} S_j$ is called a *wedge of spheres*.

The following theorem is due to Forman, and is valid for all regular cell complexes. It is a very useful theorem in discrete Morse theory.

Theorem 4.1.8 [14, Theorem 6.3] *Let Δ be a Boolean cell complex and let M be an acyclic matching on the face poset $P(\Delta)$. If there are n critical cells, all of the same dimension m , then Δ is homotopy equivalent to a wedge of n spheres of dimension m .*

Corollary 4.1.9 *If M is an acyclic matching with no critical cells on the face poset $P(\Delta)$, then Δ is contractible.*

4.2 Boolean complexes of Coxeter systems

In this section, we recall some properties of Boolean complexes of Coxeter systems. For more, we recommend the reader to check in [21].

Let (W, S) be a Coxeter system. Recall that the principal order ideals in the Bruhat order $\text{Br}(W)$ are the lower Bruhat intervals $[e, w]$, $w \in W$.

Definition 4.2.1 *If $[e, w]$ is isomorphic to a Boolean algebra, then $w \in W$ is called Boolean.*

The subposet of $\text{Br}(W)$ induced by the Boolean elements is called the *Boolean ideal* of W . Let $\mathbb{B}(W)$ denote the Boolean ideal of W .

Definition 4.2.2 ([21]) *The Boolean complex of (W, S) is the Boolean cell complex $\Delta(W)$ whose face poset is $\mathbb{B}(W)$.*

Let C be the Coxeter graph of (W, S) . Let also d be an edge in C . Define:

- (1) C/d to be the graph obtained by contraction of the edge d ,
- (2) $C - d$ to be the subgraph obtained after deletion of d ,
- (3) $C - [d]$ to be the subgraph obtained after taking away d with its incident vertices and edges.

Theorem 4.2.3 ([21]) *For every Coxeter system of rank n whose Coxeter graph is C , there exists a non-negative integer $\beta(C)$ such that $\Delta(W)$ is homotopy equivalent to the wedge of $\beta(C)$ spheres, all of dimension $n - 1$. The following equations can be recursively used to determine the values of $\beta(C)$.*

- (1) *If d is an edge in C , then $\beta(C) = \beta(C - d) + \beta(C/d) + \beta(C - [d])$,*
- (2) *If C is the graph without edges and without vertices, then $\beta(C) = 1$,*
- (3) *If C is a graph with some vertices but no edges, then $\beta(C) = 0$.*

The integer $\beta(C)$ is called the *Boolean number* of C . Observe that it does not depend on the edge labels of C .

Let I , $\text{Br}(I)$, and $B(w)$ denote the set of all involutions of W , the subposet of $\text{Br}(W)$ induced by I , and the principal order ideal of $\text{Br}(I)$ generated by $w \in \text{Br}(I)$, respectively.

Definition 4.2.4 *We call $w \in I$ a Boolean involution if $B(w)$ is isomorphic to a Boolean algebra.*

Let B_I be the set of Boolean involutions, and $P(\Delta_{\text{inv}}(W))$ be the subposet of $\text{Br}(I)$ induced by B_I .

Definition 4.2.5 *The poset $P(\Delta_{\text{inv}}(W))$ is called the Boolean involution ideal.*

The poset $P(\Delta_{\text{inv}}(W))$ is a simplicial poset. Hence there is a Boolean cell complex, denoted by $\Delta_{\text{inv}}(W)$, whose face poset is $P(\Delta_{\text{inv}}(W))$.

Example 4.2.6 Consider a Coxeter group of type A_2 whose Coxeter graph is as in Figure 2.1 for $n = 2$.

- (1) The Boolean ideal $\mathbb{B}(A_2)$ is depicted in Figure 4.1. Moreover, $\beta(\bullet\bullet) = 1$, and hence $\Delta(A_2)$ (see Figure 4.2) is homotopy equivalent (actually, homeomorphic) to the circle S^1 .
- (2) The set of Boolean involutions is $B_I = \{s_1s_2s_1, s_1, s_2, e\}$, and hence the Boolean involution ideal is as depicted in Figure 4.3. The cell complex $\Delta_{\text{inv}}(A_2)$ is contractible (i.e., homotopy equivalent to a point); see Figure 4.4.

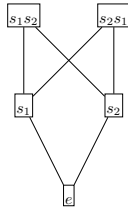


Figure 4.1: The Boolean ideal of A_2

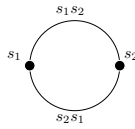


Figure 4.2: The Boolean complex of A_2

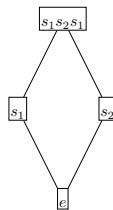


Figure 4.3: The Boolean involution ideal $P(\Delta_{\text{inv}}(A_2))$

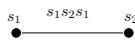


Figure 4.4: The cell complex $\Delta_{\text{inv}}(A_2)$

5 – Summary of papers

5.1 Paper I: Smoothness of Schubert varieties indexed by involutions in finite simply laced types

Let (W, S) be a finite simply laced Coxeter system, and $J \subseteq S$. In this paper, the principal result is about Schubert varieties $X(w)$ where w is an involution from W . The main result generalizes a result of Hohlweg [15] to all finite simply laced Coxeter groups. In brief, we prove that the Schubert variety $X(w)$ is singular if w is an involution which is not the longest element in some standard parabolic subgroup of W . Notice that if W is not simply laced, there are counterexamples.

5.2 Paper II: Boolean complexes of involutions

Suppose that (W, S) is a Coxeter system of finite rank m . Ragnarsson and Tenner in [21] introduced the Boolean complex, denoted by $\Delta(W)$. They showed that $\Delta(W)$ is homotopy equivalent to a wedge of spheres of dimension $m - 1$. In this paper, Boolean complexes of involutions are introduced. These complexes are analogues of the Boolean complexes introduced by Ragnarsson and Tenner. Let $\Delta_{\text{inv}}(W)$ denote the Boolean complex of involutions in W . We calculate the homotopy type of $\Delta_{\text{inv}}(W)$ for all finite Coxeter groups. We also extend our computation to many Coxeter groups that are not finite. Our proofs are based on Theorem 4.1.8.

5.3 Paper III: Fixed elements of automorphisms of pircons

Let P be a partially ordered set with a maximum. In this paper, we generalize the main results of [16] from special matchings to special partial matchings. We show that if P is finite with a special partial matching, then the subposet of P induced by the fixed points of any poset automorphism of P also admits a special partial matching. We also prove that if P is a pircon, then the subposet of P induced by the fixed elements of any automorphism is also a pircon. We finally apply our results to the dual of the Bruhat order on the fixed point free involutions in the symmetric group, leading to the conclusion that the fixed point free signed involutions form a pircon.

5.4 Paper IV: Zircons and smooth Bruhat intervals in symmetric groups

Let (W, S) be a finite simply laced Coxeter system, and $[u, w]$ be a Bruhat interval in the Bruhat order on W . In [9], Delanoy conjectured that if $[u, w]$ is a zircon, then it is isomorphic to some lower interval $[e, x]$ where the two intervals are potentially in different types. In this paper, we prove that if $[u, w]$ is a Bruhat interval in a Weyl group W and the dual of that Bruhat interval is a zircon, then $X(w)$ is rationally smooth at the points corresponding to u . We also conjecture that in type A , the converse of the previous statement holds. In addition, we pose a stronger conjecture (see Conjecture 4.5 in Paper IV) which generalizes Delanoy's conjecture in type A . Using SageMath, we confirm it in types A_n , $n \leq 8$.

Bibliography

- [1] N. Abdallah, M. Hansson, and A. Hultman, *Topology of posets with special partial matchings*, *Advances in Mathematics* **348** (2019), 255–276.
- [2] N. Abdallah and A. Hultman, *Combinatorial invariance of Kazhdan-Lusztig-Vogan polynomials for fixed point free involutions*, *J. Algebraic Combin.* **47** (2018), 543–560.
- [3] S. Billey and V. Lakshmibai, *Singular loci of Schubert varieties*, *Progress in Mathematics*, vol. 182, Birkhäuser Boston, Inc., Boston, MA, 2000.
- [4] A. Björner and F. Brenti, *Combinatorics of Coxeter groups*, *Graduate Texts in Mathematics*, vol. 231, Springer, New York, 2005.
- [5] F. Brenti, *Kazhdan-Lusztig polynomials: History, problems, and combinatorial invariance*, *Sém. Lothar. Combin.* **49** (2002/04), B49b, 30pp.
- [6] ———, *The intersection cohomology of Schubert varieties is a combinatorial invariant*, *European J. Combin.* **25** (2004), 1151–1167.
- [7] J. B. Carrell, *The Bruhat graph of a Coxeter group, a conjecture of Deodhar, and rational smoothness of Schubert varieties*, *Algebraic groups and their generalizations: classical methods* (University Park, PA, 1991), *Proc. Sympos. Pure Math.*, vol. 56, Amer. Math. Soc., Providence, RI, 1994, pp. 53–61.
- [8] J. B. Carrell and J. Kuttler, *Smooth points of T -stable varieties in G/B and the Peterson map*, *Invent. Math.* **151** (2003), 353–379.
- [9] E. Delanoy, *Completely compressible Bruhat intervals and Kazhdan-Lusztig polynomials*, *European J. Combin.* **29** (2008), 746–759.
- [10] V. V. Deodhar, *Some characterizations of Bruhat ordering on a Coxeter group and determination of the relative Möbius function*, *Invent. Math.* **39** (1977), 187–198.

-
- [11] F. du Cloux, *An abstract model for Bruhat intervals*, European J. Combin. **21** (2000), 197–222.
- [12] M. Dyer, *On the “Bruhat graph” of a Coxeter system*, Compositio Math. **78** (1991), 185–191.
- [13] M. J. Dyer, *The nil Hecke ring and Deodhar’s conjecture on Bruhat intervals*, Invent. Math. **111** (1993), 571–574.
- [14] R. Forman, *A user’s guide to discrete Morse theory*, Sémin. Lothar. Combin. **48** (2002), Art. B48c, 35.
- [15] C. Hohlweg, *Minimal and maximal elements in two-sided cells of S_n and Robinson-Schensted correspondence*, Discrete Math. **304** (2005), 79–87.
- [16] A. Hultman, *Fixed points of zircon automorphisms*, Order **25** (2008), 85–90.
- [17] J. E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Studies in Advanced Mathematics, vol. 29, Cambridge University Press, Cambridge, 1990.
- [18] M. Marietti, *Algebraic and combinatorial properties of zircons*, J. Algebraic Combin. **26** (2007), 363–382.
- [19] J. R. Munkres, *Elements of algebraic topology*, Addison-Wesley Publishing Company, Menlo Park, CA, 1984.
- [20] Patrick Polo, *On Zariski tangent spaces of Schubert varieties, and a proof of a conjecture of Deodhar*, Indag. Math. (N.S.) **5** (1994), 483–493.
- [21] K. Ragnarsson and B. E. Tenner, *Homotopy type of the Boolean complex of a Coxeter system*, Adv. Math. **222** (2009), 409–430.
- [22] R. P. Stanley, *Enumerative combinatorics. Volume 1*, second ed., Cambridge Studies in Advanced Mathematics, vol. 49, Cambridge University Press, Cambridge, 2012.
- [23] J. H. C. Whitehead, *Combinatorial homotopy. I*, Bull. Amer. Math. Soc. **55** (1949), 213–245. MR 30759

Papers

Papers

The papers associated with this thesis have been removed for copyright reasons. For more details about these see:

<https://doi.org/10.3384/9789179294694>

FACULTY OF SCIENCE AND ENGINEERING

Linköping Studies in Science and Technology, Dissertation No. 2252, 2022
Department of Mathematics

Linköping University
SE-581 83 Linköping, Sweden

www.liu.se