APPLICATIONS OF CONFORMAL MAPPINGS TO THE FLUID FLOW

By

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Abstract

The complexity of airfoil shapes has brought difficulties in modeling the fluid flow around an airplane. This difficulty is based on fluid interactions around aircraft and generally on the aspect that one wants a solution of the practical part of aerodynamics. To model this, a lot of simplifications must be done. Conformal mapping technique is widely used to transform complicated fluid flow problems into simpler fluid flow problems. In our thesis, we reviewed and implemented the model used to model the ideal fluid around a circular cylinder. This model was constructed by a superposition of elementary potential fluid flows including uniform flow, doublet flow and vortex flow. The Conformal mapping technique known as Joukowsky transformation was used to map the fluid flow around the circular cylinder into the fluid flow around the airfoil. This transformation was implemented in MATLAB to be able to visualize the streamlines around the circular cylinder and the corresponding airfoil. The lift force was calculated versus the angle of attack using the Kutta-Joukowsky formula. It was found that there is a strong linear dependence between the lift force and the angle of attack. Again, the lift coefficient as a function of the angle of attack of the Joukowsky airfoil was compared to the lift coefficient for the NACA 0012 airfoil and these almost match better as the maximum absolute error in the lift coefficient for the two airfoils is in the range of 0 to 12 degrees was found to be 2.8%. 
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Declaration

I, NYANDWI Bosco, hereby declare that this thesis was conducted at University of Rwanda under the supervision of Prof. Rickard Bőgvad and is my original work and has never been submitted anywhere else for academic purposes.

Student: NYANDWI Bosco

Signature:
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Chapter 1

General Introduction

1.1 Introduction

Conformal mapping have been used in studying the airflow around an airplane. Due to the complexity of the flow around it, a lot of simplifications must be done. This difficulty is based on fluid interactions around aircraft and generally on the aspect that one wants a solution of the practical part of aerodynamics. The most useful technique to transform the complicated airflow problem to a simple problem is a conformal mapping which is used as an intermediate step that gives a way of solving the problem with simpler geometry. An airplane wing is defined as the cross sectional shape of an object designed to generate lift when moving through a fluid. Basically, an airplane wing generates lift by diverting the motion of fluid flowing over its surface in a downward direction, resulting in an upward reaction force by Newton’s third law [14]. A quantitative method of analyzing fluid flow and lift is needed to understand the design of applicable systems.

This method, known as conformal mapping, will be the main emphasis of this thesis. Our thesis goal will be to provide students with a deep understanding of multivariable calculus and an understanding of complex variable mathematics with an important application. Conformal mapping is a mathematical technique in which complicated geometries can be transformed by a mapping function into simpler geometries which still preserve both the angles and orientation of the original geometry [1]. Using this technique, the fluid flow around the geometry of an airplane wing can be analyzed as the flow around a cylinder whose symmetry simplifies the needed computations. Since the functions that describe the fluid flow satisfy the equation of Laplace, the conformal mapping method allows for lift calculations on the cylinder to be equated to those on the corresponding airplane wing.

This thesis focuses on constructing the two-dimensional fluid flow around the airplane wing using the conformal mapping technique. We will now introduce some of the concepts
that will be used in this thesis, so as to give an idea of which techniques will be used to solve our problem. We will provide a description and the characteristics of an airplane wing. Next, we will construct a physical model used to represent the inviscid, incompressible fluid flow around an airplane wing (airfoil), and explain the theory behind conformal mapping. We will then use a special application of conformal mapping called Joukowsky transformation, to map the solution for flow around a circular cylinder to the solution for flow around airplane wings. We will implement this conformal mapping transformation to compute the fluid flow and lift around Joukowsky airfoil.

1.2 Airplanes

1.2.1 Description of Airplane Wings

It is very important to give the geometrical description of the wing of an aircraft in the beginning of our journey of constructing the fluid flow around it.

Definition 1.1 A profile is defined as the section of a wing by a plane parallel to the plane of symmetry. In the case of a cylindrical wing (see figure 1.1), the profiles are the same at every distance.

![Figure 1.1: Profiles for a cylindrical wing](image)

Below is presented the figure that shows the main parameters of an airplane wings that play an important role in its aerodynamic performance.

![Figure 1.2: Diagram of an airfoil with key parameters labeled](image)

Definition 1.2 An airfoil (the shape of the wing) has a round leading edge and sharp trailing edge. The term Camber refers to the curvature of the surfaces of an airfoil. The
The airfoil shown in Figure 1.3 is a positive cambered airfoil because the mean camber line is located above the chord line. The mean camber line (3) is a line drawn halfway between the upper and lower surfaces. The mean camber of an airfoil may be considered as the curvature of the median line (mean camber line) of the airfoil. The shape of the mean camber is important in determining the aerodynamic characteristics of an airfoil section. The chord line (1) is a straight line connecting the leading and trailing edges of the airfoil. The chord line connects the ends of the mean camber line. The chord (2) is the length of the chord line from leading edge to trailing edge and is the characteristic longitudinal dimension of an airfoil. Maximum camber (4) (displacement of the mean camber line from the chord line) and where it is located (expressed as fractions or percentages of the basic chord) help to define the shape of the mean camber line. The maximum thickness (5) of an airfoil and where it is located (expressed as a percentage of the chord) help define the airfoil shape, and hence its performance. The leading edge radius (6) of the airfoil is the radius of curvature given the leading edge shape. We are in particular interested in the length of the chord, the angle of attack and the mean camber line.

Figure 1.4: Angle of attack denoted by $\alpha$. [4]

**Definition 1.3** *The angle of attack* denoted by $\alpha$ is the angle between the direction of motion...
the motion and the direction of the chord line.

**Definition 1.4** *Aerodynamic force* is the resultant force exerted on an aircraft.

![Schematic of forces exerted on an airplane](image1)

Figure 1.5: Schematic illustrating forces exerted on an airplane. [4]

**Definition 1.5** *Lift* is the component of aerodynamic force perpendicular to the direction of motion.

![Forces exerted on an airplane](image2)

Figure 1.6: Forces exerted on an airplane, google image.

**Definition 1.6** *Drag* is the component of aerodynamic force in the opposite of the direction of motion.

The angle of attack has a great influence on the lift generated by a wing. When an airplane takes off, the pilot applies as much as thrust possible to make the airplane roll along the runway. But just before lifting off, the pilot rotates the aircraft. The nose of the airplane rises, increasing the angle of attack and producing the increased lift needed for takeoff.
1.3 Fluid Dynamics

Definition 1.7 A fluid is defined as any material that exhibits deformation under the action of forces.

Definition 1.8 A flow is a deformation of a material that increases continuously without limit under the action of forces, however small.

Actual fluids are classified into two categories: gas and liquids. Since a gas such as atmospheric air fills any closed space to which has access, it is therefore classified as compressible fluid. A liquid is regarded as incompressible because a liquid has a constant pressure and temperature and takes a definite volume and when placed in an open vessel will take under the action of gravity the form of the lower part of the vessel and will be bounded above by a horizontal free surface.

Definition 1.9 Viscous fluid flows refer to real flows which exhibit the effects of transport phenomena (the phenomena of mass diffusion, viscosity and thermal conductivity).

Definition 1.10 A fluid flow is called an inviscid flow if it is assumed to involve no friction, thermal conductivity or diffusion.

But in nature such flows do not exist, however, there are many practical aerodynamic flows where the influence of the transport phenomena is small, and we can model the flow as being inviscid.

1.3.1 Fluid Velocity

One can define a fluid particle as consisting of the fluid contained in an infinitesimal volume, that is to say, a volume whose size may be considered so small that for the particular purpose
in hand its linear dimensions are negligible. We can then treat a fluid particle as a geometrical point for the particular purpose of discussing its velocity and acceleration. If we consider, the particle which at time \( t \) is at the point \( P \), defined by the vector \( \vec{s} = \overrightarrow{OP} \) at time \( t_1 \), this particle will have moved to the point \( Q \), defined by the vector \( \vec{s}_1 = \overrightarrow{OQ} \) (see figure1.8).

![Figure 1.8: The particle at the point P at time t moves the point Q at time t1](image)

**Definition 1.11** The velocity of the particle at \( P \) is then defined by the vector

\[
\vec{V} = \lim_{t_1 \to t} \frac{s_1 - s}{t_1 - t} = \frac{d\vec{s}}{dt}.
\]

Thus the velocity \( \vec{V} \) is a function of \( s \) and \( t \), say \( \vec{V} = g(s, t) \). If the form of the function \( g \) is known then we know the motion of the fluid.

**Definition 1.12** A steady flow is a flow with a constant velocity. It means that \( \frac{\partial \vec{V}}{\partial t} = 0 \).

**Definition 1.13** A streamline is a curve drawn in the fluid that maps \( t \mapsto \vec{s}(t) \), in \( \mathbb{R}^3 \) so that its tangent at each point is in the direction of the fluid velocity \( \vec{V} \) at that point, i.e.,

\[
\vec{V} \times d\vec{s} = 0,
\]

where \( d\vec{s} = \vec{i} dx + \vec{j} dy + \vec{k} dz \), stands for a line element in the Cartesian frame and \( \vec{V} = u\vec{i} + v\vec{j} + w\vec{k} \), where \( u, v, w \) are the velocity components.

The expansion of equation (1.1) (under the hypothesis that \( u, v, w \) are non-zero) follows below

\[
\vec{V} \times d\vec{s} = 0
\]

\[
\Leftrightarrow \begin{vmatrix}
\vec{i} & \vec{j} & \vec{k} \\
u & v & w \\
dx & dy & dz
\end{vmatrix} = 0
\]

\[
\Leftrightarrow \begin{cases}
vdz - wdy = 0 \\
u dz - wdx = 0 \\
udy - vdx = 0
\end{cases}
\]
The differential equations (1.3) when solved give the family of streamlines at any instant. When the flow is steady the streamlines will have the same form at all $t$. These equations may also be formulated as $ds = \lambda \vec{V}$, for some non-zero $\lambda = \lambda(t) \in \mathbb{R}$, and this formulation is then valid for a flow in two dimensional space too (in this case the first two equations of (1.3) will be true). Note that streamlines cannot intersect one another, except at points (and times) when $\vec{V}$ is either singular or zero, by standard properties of differential equations.

**Definition 1.14** *Stream tube* is the surface formed instantaneously by all streamlines which pass through a given closed curve in the fluid.

**Definition 1.15** *Path lines* of a fluid particle are the curves that this particle describes during the motion. The path lines equations are described by

$$
\frac{d\vec{s}}{dt} = \vec{V} \tag{1.4}
$$

$$
\begin{align*}
\frac{dx}{dt} &= u \\
\frac{dy}{dt} &= v \\
\frac{dz}{dt} &= w,
\end{align*}
$$

where $\vec{s} = (x(t), y(t), z(t))$, is the spatial position vector for each material particle. Thus path lines do not, in general, coincide with streamlines. They do so, only when the motion is steady. Assume that we have a function $s(x, y, z, t)$ that depends on both position and time. Now evaluate this along a path line of a particle to get $\vec{s}(x(t), y(t), z(t), t)$. Then the total derivative is

$$
\frac{d\vec{s}}{dt} = \frac{\partial \vec{s}}{\partial t} + \frac{\partial \vec{s}}{\partial x} \frac{dx}{dt} + \frac{\partial \vec{s}}{\partial y} \frac{dy}{dt} + \frac{\partial \vec{s}}{\partial z} \frac{dz}{dt}
$$

$$
\iff
$$
\[
\frac{d\mathbf{s}}{dt} = \frac{\partial \mathbf{s}}{\partial t} + u \frac{\partial \mathbf{s}}{\partial x} + v \frac{\partial \mathbf{s}}{\partial y} + w \frac{\partial \mathbf{s}}{\partial z}
\]

\[\Leftrightarrow\]

\[
\frac{d\mathbf{s}}{dt} = \frac{\partial \mathbf{s}}{\partial t} + (u \mathbf{i} + v \mathbf{j} + w \mathbf{k}) \cdot (\mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z}) \mathbf{s}
\]

\[\Leftrightarrow\]

\[
\frac{d\mathbf{s}}{dt} = \frac{\partial \mathbf{s}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{s}.
\]

Where \(\nabla = \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z}\) stands for the differential operator. The differential operator (1.5) is called the total differential operator:

\[
\frac{d}{dt} = \frac{\partial}{\partial t} + (\mathbf{V} \cdot \nabla).
\]

(1.5)

The acceleration can be derived by applying the operator in (1.5) to the velocity. That is if \(\mathbf{V} = \mathbf{V}(x, y, z, t)\),

\[
\ddot{\mathbf{a}} = \frac{d\mathbf{V}}{dt} = \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{V}
\]

\[\Leftrightarrow\]

\[
\ddot{\mathbf{a}} = \frac{\partial}{\partial t} (u \mathbf{i} + v \mathbf{j} + w \mathbf{k}) + (u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z})(u \mathbf{i} + v \mathbf{j} + w \mathbf{k})
\]

\[\Leftrightarrow\]

\[
\ddot{\mathbf{a}} = \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) \mathbf{i} + \left( \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) \mathbf{j} + \left( \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) \mathbf{k},
\]

where the second term on the right hand side of (1.6) is called the convective acceleration.
1.3. FLUID DYNAMICS

1.3.2 The Continuity Equation: Mass Conservation

The law of conservation of mass states that the net flow of mass is equal to the rate at which mass is created within the control volume. If we perform an analysis by considering the three-dimensional flows in a control volume regarded as a rectangular parallelepiped, then in a Cartesian frame of reference, we have:

\[ \frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} = 0, \]  

(1.7)

where \( \rho \) is the fluid density. Equation (1.7) says

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = 0. \]  

(1.8)

The equation (1.8) holds for unsteady or steady, inviscid or viscous, compressible or incompressible fluids in three dimensions. If we assume that the density \( \rho \) is constant the equation (1.8) simplifies to

\[ \nabla \cdot \vec{V} = 0. \]  

(1.9)

From now, we will assume that the fluid density is constant and utilize the equation (1.9) which is called the continuity equation for incompressible fluids.

Definition 1.16 The vorticity \( \vec{w} \) is defined as the curl of the velocity \( \vec{V} \) i.e.

\[ \vec{w} = \text{curl} \vec{V} = \nabla \times \vec{V}. \]  

(1.10)

If we expand equation (1.10) we will get:

\[ \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = \left( \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) \vec{i} - \left( \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} \right) \vec{j} + \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \vec{k}. \]  

(1.11)

From the definition 1.16 we can define two important concepts:

Definition 1.17 A fluid flow is called a rotational flow if

\[ \nabla \times \vec{V} \neq 0, \]  

(1.12)

at every point in the flow.

In this case the fluid elements have a non-zero angular velocity.

Definition 1.18 If

\[ \nabla \times \vec{V} = 0, \]  

(1.13)

at every point in the flow, then the fluid flow is called an irrotational flow.
In this case, the fluid elements have no angular velocity. In two dimensions, equation (1.13) becomes:

\[ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0. \] (1.14)

We will frequently use equation (1.14) in the next sections.

**Definition 1.19 The Bernoulli equation** for an incompressible inviscid fluid is given below: \( p + \frac{1}{2} \rho \vec{V}^2 = \text{const} \), where \( p \) is the pressure, \( \rho \) the density, \( \vec{V} \) the fluid velocity and \( \vec{V}^2 \) the square of the velocity.

It is clear to see that the pressure decreases as the velocity increases and conversely, the velocity decreases as the pressure increases. This is the physical meaning of the Bernoulli equation \([10]\). This equation will be used in section 2.1.8 to explain how the lift force is generated around a spinning cylinder.

### 1.3.3 Some Multivariate Calculus

To define the circulation, let us start by stating the theorems of Green and Stokes.

**Theorem 1.1 (Green’s Theorem):** Let \( C \) be a positively oriented, piecewise, smooth, simple closed curve in the plane and let \( D \) be the region bounded by \( C \). Let \( F(x, y) = P(x, y)\vec{i} + Q(x, y)\vec{j} \), be a vector field of class \( C^1 \). Then

\[ \oint_C P \, dx + Q \, dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA. \] (1.15)

And the vector form of 1.1 is

\[ \oint_C F \cdot ds = \iint_D (\nabla \times F) \cdot k \, dA, \] (1.16)

where \( k \) is the unit outward normal vector.

**Theorem 1.2 (Stokes’ theorem):** Let \( S \) be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve \( C \) with positive orientation. Let \( F \) be a vector field whose components have continuous partial derivatives on an open region in \( \mathbb{R}^3 \) that contains \( S \). Then

\[ \oint_C F \cdot dr = \iint_S (\nabla \times F) \cdot dS \] (1.17)

**Definition 1.20 Circulation:** Let \( C \) be a closed curve in a fluid. Let \( \vec{V} \) and \( ds \) be the velocity and the line segment of \( C \), respectively. The circulation denoted by \( \Gamma \), is defined as the curve integral

\[ \Gamma = \oint_C \vec{V} \cdot ds. \] (1.18)
1.4. STREAM FUNCTION

The circulation is related to the vorticity as follows

\[ \Gamma = \oint_C \vec{V} \cdot ds = \int_S (\nabla \times \vec{V}) \cdot dS, \]

(1.19)

thus the circulation about a curve is equal to the vorticity integrated over any open surface \( S \) bounded by \( C \) [10]. This is an immediate consequence of Theorem 1.2; so we get the following corollary:

**Corollary 1.1** If the flow is irrotational everywhere within the contour of integration over any surface bounded by \( C \) then the circulation \( \Gamma = 0 \).

1.4 Stream Function

Consider a two-dimensional motion of air considered as an incompressible fluid. Let \( A \) be a fixed point in the plane of motion, \( P \) an arbitrary point and \( ABP \) and \( ACP \) two curves in the plane joining \( A \) to \( P \). Assume that there is no air created or destroyed within the region \( R \) bounded by the two curves. And now, the condition of continuity can be expressed in the following form. The rate at which air flows into the region \( R \) from right to left across the curve \( ABP \) is equal to the rate at which it flows out from right to the left across the curve \( ACP \). The flux from right to left across \( ACP \) is equal to the flux from right to the left across any curve joining \( A \) to \( P \). Once the base point \( A \) has been fixed this flux therefore depends solely on the position of \( P \), and the time \( t \). If we denote the flux by \( \psi \), it is a function of \( P \), and the time \( t \). The function \( \psi = \psi(x,y,t) \) is called the stream function (where in Cartesian coordinates \( P = (x,y) \)). The existence of this function is merely a consequence of the assertion of the continuity of incompressible air.

Let \( \psi_1, \psi_2 \) be the values of the stream functions at the points \( P_1 \) and \( P_2 \) respectively. From the same principle, the flux across \( AP_2 \) is equal to the flux across \( AP_1 \) plus that \( P_1P_2 \). Hence the flux across \( P_1P_2 \) equals \( \psi_2 - \psi_1 \). It follows from this that if we take a different base point, \( B \) say, the stream function merely changes by the flux from right to left across \( BA \). Hence the steam function is uniquely determined up to an additive constant. Moreover, if \( P_1 \) and \( P_2 \) are points of the same streamline, there is clearly no flux over the streamline between \( P_1 \) and \( P_2 \), and thus \( \psi_2 - \psi_1 = 0 \). Therefore the stream function is constant along a streamline. The
equations of the streamlines are therefore obtained from $\psi = c$, by giving arbitrary values to the constant $c$. When the motion is steady, the streamline pattern is fixed. When the motion is not steady, the pattern changes from instant to instant with $t$.

1.4.1 Velocity Potential and Stream Function

Consider a two dimensional velocity field in the form

$$\vec{V} = u(x, y, t)\vec{i} + v(x, y, t)\vec{j}. \quad (1.20)$$

If the vector field is irrotational then by (1.14) $\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$, and by (1.15) $\oint_C ud\!x + v\!dy = 0$, along any closed curve $C$. Hence choosing a base point $A$, and letting $P = (x, y)$ be a variable point, the curve integral along a curve $\Gamma$ between $A$ and $P$,

$$\oint_{\Gamma} ud\!x + v\!dy, \quad (1.21)$$

does not depend on which curve $\Gamma$ we use. So we can define

$$\phi(x, y, t) = -\int_A^P ud\!x + v\!dy. \quad (1.22)$$

Note that

$$u = -\frac{\partial \phi}{\partial x}, \quad v = -\frac{\partial \phi}{\partial y}. \quad (1.23)$$

The function $\phi$ is called the velocity potential and we have $\vec{V} = -\nabla \phi$, that is up to a sign change, the gradient of $\phi$.

Now consider the stream function. If $\Gamma$ is curve from $A$ to $P$, it is defined as the flux across $\Gamma$, and this flux (at a point on the curve) is given as the scalar product of $\vec{V}$ and the normal $(-dy, dx) = -dy\vec{i} + dx\vec{j}$ to the curve, and hence as the curve integral

$$\oint_{\Gamma} v\!dx - u\!dy. \quad (1.24)$$
Again, by the preceding section, this does not depend on which curve \( \Gamma \) we choose (if the fluid is incompressible) and so defines a function of \( P, \)

\[ \psi(x, y, t) = \int_A P v \, dx - u \, dy. \]  

(1.25)

Clearly, this function satisfies

\[ v = \frac{\partial \psi}{\partial x}, \quad -u = \frac{\partial \psi}{\partial y}. \]  

(1.26)

Hence the gradient \( \nabla \psi = (-v, u) = -v \vec{i} + u \vec{j} \). The combination of the equations (1.23) and (1.26) yields the Cauchy Riemann type equations

\[ \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \quad \frac{\partial \psi}{\partial x} = -\frac{\partial \phi}{\partial y}. \]  

(1.27)

And taking the partial derivatives with respect to \( x \) and \( y \) of the equations (1.27) we obtain:

\[ \frac{\partial^2 \phi}{\partial x^2} = \frac{\partial^2 \psi}{\partial x \partial y} = \frac{\partial^2 \psi}{\partial y \partial x} = -\frac{\partial^2 \phi}{\partial y^2}. \]

\[ \Rightarrow \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0. \]  

(1.28)

In the same way, it can be shown that

\[ \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0. \]  

(1.29)

**Theorem 1.3** If we have a two dimensional, irrotational, incompressible flow with \( \phi \) equal to its velocity potential and \( \psi \) its stream function then each of \( \phi \) and \( \psi \) satisfies the Laplace equation in (1.28) and/or in (1.29) and the gradient of the two functions are orthogonal:

\[ \nabla \phi \cdot \nabla \psi = 0. \]  

(1.30)

Theorem 1.3 means that the contours of the velocity potential and the stream function cross at right-angles. The proof follows immediately by \( \nabla \phi \cdot \nabla \psi = (-u, -v) \cdot (v, -u) = -uv + uv = 0. \)

### 1.5 Analytic and Harmonic Functions

By convention, we write a complex variable as \( z = x + iy \) with \( x, y \) real numbers which are identified as Cartesian coordinates. The complex number \( z \) can also be written as \( z = re^{i\theta} \), where the modulus of \( z \) is \( r = \sqrt{x^2 + y^2} \), and \( \theta = \tan^{-1}\left(\frac{y}{x}\right) \), if \( x \neq 0 \). And the Euler’s theorem states that \( e^{i\theta} = \cos \theta + i \sin \theta \), and it implies that \( x = r \cos \theta \), and \( y = r \sin \theta \). We can define functions of the complex variable, \( f(z) \) as it is done for functions of a real variable.

For example \( f(z) = z^2 \), \( f(z) = \frac{1}{z} \). Suppose that \( f(z) \) is given, we can substitute \( z = x + iy \), and write \( f(z) = u(x, y) + iv(x, y) \), where \( u(x, y) \) and \( v(x, y) \) are real two-dimensional real valued functions. Let \( f(z) = z^2 \), then \( f(z) = f(x + iy) = (x + iy)^2 = x^2 - y^2 - 2ixy \), and yielding \( u(x, y) = x^2 - y^2 \), and \( v(x, y) = 2xy \).
**1.5.1 Complex Analytic Functions**

Consider the complex valued function \( f(z) = u(x, y) + iv(x, y) \), where \( z = x + iy \). Provided that \( u(x, y) \) and \( v(x, y) \) have continuous partial derivatives, the **Cauchy Riemann Equations** are defined as follows:

\[
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{1.31}
\]

The Cauchy Riemann equations in polar coordinates can be written as:

\[
\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}. \tag{1.32}
\]

where \( x = r \cos \theta, \ y = r \sin \theta, \ r^2 = x^2 + y^2, \) and \( \tan \theta = \frac{y}{x} [3] \). (Here we assume \( r \neq 0, \ x \neq 0 \)).

**Theorem 1.4** The function \( f(z) = u(x, y) + iv(x, y) \), is differentiable at the point \( z = x + iy \), of the region in the complex plane if and only if the partial derivatives \( \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \), are continuous and satisfy the Cauchy Riemann equations at \( z = x + iy \).

**Definition 1.21** A function is said to be **analytic at a point** \( z_0 \) if \( f(z) \) is differentiable in a neighborhood of \( z_0 \).

The function \( f(z) \) is said to be **analytic in a region** \( D \) if it is analytic at every point in the region \([3]\). The points at which is not analytic are called singularities. If \( f(z) \) and \( g(z) \) are analytic functions, then so are

1. \( f(z) \pm g(z) \)
2. \( f(z) \cdot g(z) \)
3. \( f(z)/g(z) \), (if \( g(z) \) is non-zero in \( D \))
4. \( f(g(z)) \)

It follows that polynomial are analytic functions in \( D \). Power series within their circles of convergence are also analytic functions. The real and imaginary parts of an analytic function are called **conjugate functions**. Note that they related to each other by the Cauchy-Riemann equations.

**Theorem 1.5** We can construct a function

\[
\Theta(z) = \phi(x, y) + iv(x, y), \tag{1.33}
\]

called the complex velocity potential, using the velocity potential \( \phi(x, y) \) and the stream function \( \phi(x, y) \). It is an analytic function.
1.5. ANALYTIC AND HARMONIC FUNCTIONS

Proof:
Since $\phi$ and $\psi$ are continuous functions of their argument $z = x + iy$, so is $\Theta$. By (1.27)
$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \frac{\partial \psi}{\partial x} = -\frac{\partial \phi}{\partial y}.$$ That means that $\Theta$ satisfies the Cauchy-Riemann Equations in (1.31) and so the theorem is a consequence of the characterization of analytic functions in theorem 1.4.

Note that $\phi, \psi$ satisfy Laplace’s equation (1.28). This is always true of the real or imaginary part of an analytic function, and such functions are called harmonic.

Proposition 1.1 For the functions $\phi$ and $\psi$ satisfying the Cauchy Riemann equations, the level curves $\phi = \text{constant}$, and $\psi = \text{constant}$, meet at right angles to each other.

Proof:
The gradient $\left( \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y} \right)$, at $P = (x, y)$ is orthogonal to the level curve $\phi$ through $P$. Similarly the level curve of $\psi$ through $P$ is orthogonal to
$$\left( \frac{\partial \psi}{\partial x}, \frac{\partial \psi}{\partial y} \right).$$

But, by the Cauchy Riemann equations, these gradients are orthogonal:
$$\left( \frac{\partial \psi}{\partial x}, \frac{\partial \psi}{\partial y} \right) = \left( -\frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial x} \right), \text{ and so } \left( \frac{\partial \psi}{\partial x}, \frac{\partial \psi}{\partial y} \right) \cdot \left( -\frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial x} \right) = -\frac{\partial \phi}{\partial y} \cdot \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial x} \cdot \frac{\partial \phi}{\partial y}. $$ And then the level curves are also orthogonal.

1.5.2 Singularities

Definition 1.22 A point where the complex function $f(z)$ fails to be analytic is called a singularity.

By Laurent series expansion, the function $f(z)$ can be expressed in powers of $z - z^*$, in the neighborhood of $z^*$.

i.e., $f(z) = \cdots + a_3(z-z^*)^3 + a_2(z-z^*)^2 + a_1(z-z^*) + a_0 + b_1(z-z^*)^{-1} + b_2(z-z^*)^{-2} + \cdots$. If the number of terms with negative exponent in the above Laurent series expansion is finite in number, then $z = z^*$, is called the pole. The coefficient $b_1$ is called the residue of the function at $z^*$.

Theorem 1.6 (Cauchy Residue Theorem). Let $C$ be a closed contour inside and upon which $f(z)$ is analytic, except at finite number of poles $z_1, \ldots z_n$ within $C$. If the residues at the poles are $r_1, r_2, r_3, \cdots r_n$, then
$$\oint f(z)dz = 2\pi i(r_1 + r_2 + r_3 + \cdots + r_n).$$ (1.34)

Definition 1.23 A stagnation point is defined as the point in the fluid flow where the fluid velocity vanishes.
For the velocity potential $\phi$, we know that \( \left( \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y} \right) = -\vec{V} \). Hence $P = (x, y)$ is a stagnation point if and only if $\frac{\partial \phi}{\partial x} = 0$ and $\frac{\partial \phi}{\partial y} = 0$. By the Cauchy Riemann equations, this implies that $\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} = 0$, and also that $\Theta'(z) = 0$, where $\Theta$ is the complex velocity potential. In fact stagnation points $P = (x, y)$ are exactly the points $z = x + iy$ where $\Theta'(z) = 0$.

### 1.5.3 Conformal Transformations

Consider a curve in $C$ in the complex plane. An analytic function $f(z)$, that is defined everywhere in a domain containing $C$, maps $C$ to a new curve $C^* = \{f(z) : z \in C\}$, contained in the complex plane. Suppose that $z_0 \in C$, and that $C$ has a tangent at $z_0$. Then $C$ may be locally parametrized as $\gamma(t) = x(t) + iy(t)$, $t \in [a, b]$, with $\gamma(0) = z_0$, where $x(t)$ and $y(t)$ are functions with continuous derivatives. Also, $\gamma'(t) = x'(t) + iy'(t)$, is non-zero and as a vector, parallel with the tangent of $C$ at $\gamma(0)$. $C^*$ may be locally around $w_0 = f(z_0)$, parametrized as $\delta(t) = f(\delta(t))$, $t \in [a, b]$, $\delta(0) = w_0$. Also $\delta'(t) = f'(\gamma(t)) \cdot \gamma'(t)$, (by the chain rule and the analyticity of $f$) and so $\delta'(0) = f'(z_0) \cdot \gamma'(0)$, and

$$
\arg \delta'(0) = \arg f'(z_0) + \arg \gamma'(0).
$$

(1.35)

Now $\arg \delta'(0)$, is the angle of the tangent at $w_0$ to $C^*$ with the real axis and $\arg \gamma'(0)$, the angle of the tangent at $z_0$ to $C$. So independently of the which curve $C$ we have, this angle is changed by $\arg f'(z_0)$, when we consider the image of $C$ under $f$.

If we have two curves $C_1$ and $C_2$ that meet at $z_0$, with a certain angle $\theta$ between their tangents at $z_0$, we may consider $C_1^* = f(C_1)$, and $C_2^* = f(C_2)$, that meet at $w_0 = f(z_0)$. If the angle (and orientation) between the tangents to $C_1^*$ and $C_2^*$ is equal to angle $\theta$, for all curves $C_1, C_2$, then we say that $f$ is a conformal map at $z_0$. But we saw that an analytic function rotated the tangents by the $\arg f'(z_0)$, so the following has been proved.

**Theorem 1.7** Assume that $f$ is analytic in a domain $D$ in the complex and that $z_0 \in D$, is such that $f'(z_0) \neq 0$. Then $f$ is a conformal map at $z_0$.

### 1.6 Problem Statement

Let us say we want to construct an airplane. Then we want to study the flow around airplane wings to see the optimal shape it can have so that the airplane lifts maximally with minimal energy. The flow around the air plane wings can be treated into two branches; the first branch models the fluid around the airfoil as a viscous fluid, where the phenomena of mass diffusion, viscosity and thermal conductivity are taken into account and this makes this branch more accurate, but also involves much complicated mathematics and physics that are
1.6. PROBLEM STATEMENT

Figure 1.11: A conformal transformation [3]

beyond the scope of this thesis.

Instead, we consider the second branch, that models the fluid flow around the airfoil as an ideal fluid flow. *Ideal fluid flows* refer to fluid motions that are steady, inviscid, incompressible, and irrotational [10]. However such flows do not exist in reality, but there are many practical aerodynamic flows where the influence of the transport phenomena is small, and it can be modeled as being inviscid. Considering the fluid flow around the airfoil as inviscid and incompressible still allows for an accurate model provided certain conditions are met [1]. One of these conditions is that the airplane wing must be moving through the fluid at subsonic speeds; this is very crucial because at speeds approaching the speed of sound, shock waves occur in which the fluid flow no longer becomes continuous, and the perfect fluid idealization breaks down. In particular, we will build our models for airfoils moving through flows regions where the compressibility effects in the fluid flow can be negligible (Mach number between 0.0 and 0.4) [13]. Another assumption is that the flow around the airplane wing satisfies the Kutta Condition (see figure 1.12). The Kutta condition is that the fluid flowing over the upper and lower surfaces of the airfoil meets smoothly at the trailing edge of the airfoil and explains how an inviscid fluid can generate lift. Thus, the Kutta condition accounts for the friction at the boundary of an airfoil that is essential for lift to

Figure 1.12: Inviscid flow over an airfoil with the Kutta condition satisfied. The flow meets smoothly at the trailing edge. [4]
be generated under other additional constraints on the flow around an airfoil \[1\]. With reference to subsection 1.3.2 a fluid flow is said to be an incompressible flow if its density \(\rho\) is constant, in contrast, it is called compressible when \(\rho\) varies. The fluid flow where the density is precisely constant does not exist in nature, but analogous to our discussions of nonviscous flow, there are many aerodynamic problems which can be modeled as incompressible flows without loss of accuracy \[10\]. In this case the equation of continuity (1.7) reduces to (1.8). Furthermore, the absence of frictional shear forces acting on elements of an inviscid fluid causes the motion of the fluid to be purely translational, allowing the flow over an airfoil to be modeled as irrotational \[1\]. Again this irrotationality property of the fluid motion results in equation (1.13) due to the fact that the curl of the velocity vanishes. Since the motion is irrotational, its velocity field \(\vec{V}\) can be expressed as the gradient of a scalar function \(\phi\) in the sense that \(\vec{V} = \nabla \phi\). We call \(\phi\) the velocity potential, and the flows that result from a velocity potential are known as potential flows. We saw in (1.28) that \(\phi\) satisfies the Laplace equation

\[
\nabla^2 \phi = 0,
\]

and solutions to this equation are referred to as harmonic functions. Since Laplace’s equation is a linear homogeneous second order partial differential equation, the sum of particular solutions to the differential equation is also a solution \[10\]. This means that we can study complicated stream functions that one built up from simpler ones, a technique which the rest of the thesis will give an example of.

Due to the fact the air plane wings have complicated geometries; it is difficult to directly solve for the fluid flow around them using Laplace equation and potential flow theory. To do this in a more efficient way; we define a complex potential function in the \(z\)– plane as defined in theorem 1.5. When the complex potential function is transformed using conformal mapping techniques, the stream function and potential remained unchanged. In this thesis we will first solve the flow around a cylinder in the \(z\) plane, and then transform this solution to an airfoil in the \(w\) plane using a specific conformal mapping function. We will also implement the Joukowsky transformation to compute the fluid flow and lift around the Joukowsky airfoils using some computer algebra systems.
1.7 Objectives

We will first go through the theories behind modeling of a two-dimensional fluid flow and summarize some theories of complex analysis and that of conformal mappings. These will comprise the introductory part of the thesis. We will then construct the fluid flow in a complicated domain from known flows in a simpler domain, using conformal mappings.

**The Specific Objectives are:**

1. To study and understand the basic ideas and concepts of conformal mapping.
2. Describe the mathematical model used to model the inviscid incompressible fluid flow around the airplane wing.
3. Apply the theories of conformal mapping using Joukowsky transformation to link the solution for flow around a cylinder to the solution for flow around the airplane wing.
4. Attempt to implement this conformal mapping transformation to compute the fluid flow and lift around some airplane wings.

1.7.1 Scope of the Study

The scope of the study will be limited to the specific objectives listed above; in particular to exhibit the most basic and simple examples of the use of conformal mapping to compute the fluid flow.

1.7.2 Significance of the Study

To introduce in a readable way to mathematics and physics students one of the more fascinating applications of complex function theory to engineering.

1.8 Outline

This thesis is organized as follows: The first part of the thesis consists of the introduction. In this part, a lot of definitions of the important terms were provided. It also includes the description of several key concepts that play an important role in this dissertation including the concepts of fluids dynamics, the concept of circulation, the concept of stream function and velocity potential, the concepts of complex analysis and some concepts of conformal transformation. These concepts explain how the conformal mapping technique is a viable option to transform the potential flows.

The second part consists of the literature about the mathematical model used to model the lifting fluid flow around the circular cylinder and its implementation in MATLAB.
The third part will be about the use of a conformal mapping technique to transform the circular cylinder into another shape such as an ellipse or an airfoil. In this part the complex potential will be described to show how the lift and circulation remained unchanged when the complex potential is subjected to conformal transformation.

After discussing this, the streamlines around the circular cylinder and around the corresponding airfoil will be computed using MATLAB software. Again, the lift force will be calculated using the Kutta-Joukowsky formula. Finally, the lift coefficient as function of the angle of attack for the Joukowsky airfoil 12% and the NACA 0012 airfoil will be compared and the results will be summarized and some ideas for further studies will be suggested.
2.1 Modeling the Fluid around an Airplane Wing

In this section, we will review some elementary incompressible flows which will be superimposed to get more complex incompressible flows to be able to model the fluid flow around an airfoil by simply solving the lifting flow around a cylinder in the plane.

2.1.1 Uniform Flow

We consider the uniform flow as our first elementary incompressible flow. Consider a flow with a free-stream velocity $V_\infty$ that is oriented in the positive $x$ direction as illustrated in figure 2.1 since the uniform flow is irrotational, its velocity $\vec{V}$ can be expressed as $\vec{V} = \nabla \phi$, with the potential function $\phi$. It then follows that

$$\vec{V} = \nabla \phi \iff \begin{cases} \frac{\partial \phi}{\partial x} = u = V_\infty \\ \frac{\partial \phi}{\partial y} = v = 0. \end{cases} \quad (2.1)$$

The potential function $\phi$ is found by integration with respect to $x$ first and then to $y$ and comparing the results to get $\phi = V_\infty x + const$. The constant can be removed from $\phi$ because the velocity if obtained by differentiation, thus $\phi = V_\infty x$. Considering also the stream function we have

$$\begin{cases} \frac{\partial \psi}{\partial y} = u = V_\infty \\ \frac{\partial \psi}{\partial x} = -v = 0. \end{cases} \quad (2.2)$$

Similarly, by integration with respect to $x$ first and then to $y$ and comparing the results gives $\psi = V_\infty y$ which is the stream function of the uniform flow oriented in the positive $x$ direction.

In polar coordinates the potential function and the stream functions are written as follows: $\phi = V_\infty r \cos \phi$, where $x = r \cos \theta$ and $\psi = V_\infty r \sin \theta$. From the equations (2.1) and (2.2), it can be shown that the uniform flow satisfies the Laplace equation and the stream lines (for
\( y = \text{const} \) and the equipotential curves (for \( x = \text{const} \)) are perpendicular. More details about the uniform flow can be found in [5, 23].

![Uniform Flow](image1)

Figure 2.1: Uniform Flow (on the left) [10] and Computed uniform flow in MATLAB with streamlines (blue) and equipotential curves (red) on the right

### 2.1.2 Source and Sink Flow

Consider an incompressible flow in two dimensions where all the streamlines are the straight lines either converging or diverging from the center \( O \) as shown in figure 2.3. If all the streamlines are converging to the center, the flow is called the sink flow. In contrast, when the streamlines are emerging from the central point, the flow is referred to as a source flow. The coordinate system for the figure 2.3 is a cylindrical coordinate system with the \( z \)-axis perpendicular to the page. It is not difficult to show that the source flow is an incompressible flow (\( \nabla \cdot \mathbf{V} = 0 \)), at every point except the origin, where (\( \nabla \cdot \mathbf{V} = \infty \)), and as well as irrotational flow at every point. In the sink flow, the streamlines are directed towards the origin and are still radial lines from the common origin along which the flow velocity is assumed to vary inversely proportional to the distance from the central point. It means that the radial component of the resultant \( V_r = \frac{c}{r} \), where \( c \) is a constant and the tangential

![Source Flow](image2)

Figure 2.2: Source flow [16]
Figure 2.3: Sink flow [4]

Figure 2.4: Source Flow with streamlines (blue) and velocity potential in red (on left) and Sink Flow streamlines (blue) and equipotential curves in yellow (on right) both computed in MATLAB.

Figure 2.5: Volume flow rate from a line source [10]

Component is \( V_0 = 0 \). Considering the mass flow across the surface of the cylinder of \( r \) radius and height \( l \) as illustrated in figure 2.5, it is not difficult to see that the elemental mass flow across the surface element \( dS \) is equal to \( \rho V \cdot dS = \rho V_r (rd\theta)(l) \). Since \( V_r \) is the same at any location for the fixed radius \( r \), the total mass flow across the surface of the cylinder is

\[
\dot{m} = \int_0^{2\pi} \rho V_r (rd\theta)l = \rho rlV_r \int_0^{2\pi} d\theta = 2\pi rl\rho V_r. \tag{2.3}
\]
Because $\rho$ is defined as the mass per unit volume and $\dot{m}$ the mass per second, then we will define the volume flow per second $\dot{v}$ as $\dot{v} = \frac{\dot{m}}{\rho}$. And then, from the equation (2.3), we get

$$\dot{v} = \frac{\dot{m}}{\rho} = 2\pi rlV_r. \quad (2.4)$$

Let $\Lambda$ be the rate of change of the volume (volume flow from the source) per unit length ($\Lambda = \dot{v}/l$) called the source strength. Using the above boundary conditions it can be shown that the potential and the stream functions are $\phi = \frac{\Lambda}{2\pi} \ln r$, and $\psi = \frac{\Lambda}{2\pi} \theta$, respectively with $r$ the distance from the origin [10, 24]. If $\Lambda$ has a positive value, the flow is a source flow and it is a sink flow whenever $\Lambda$ has a negative value.

### 2.1.3 A Combination of a Uniform Flow with a Source and Sink Flow

Consider a uniform stream with the free stream velocity $V_\infty$ oriented from left to right. Superimpose it to a source flow of strength $\Lambda$ located at the origin in polar coordinates as shown in figure 2.6. The stream function for the resulting flow is simply found by addition

$$\psi = V_\infty r \sin \theta + \frac{\Lambda}{2\pi} \theta.$$

![Figure 2.6: A combination (superposition) of a uniform flow and a source flow generates a flow over a semi-infinite body [10]](image)

Figure 2.7: A combination (superposition) of a uniform flow and a source flow generates a flow over a semi-infinite body with streamlines (blue) and equipotential curves (red).
of the stream functions of the two flows. It means

$$\psi = V_\infty r \sin \theta + \frac{\Lambda}{2\pi} \theta$$ \[5\].

This equation satisfies the Laplace equation as it is the sum of the functions where each function satisfies the Laplace equation alone. From this, it is reasonable to say that the equation (2.5) describes an irrotational, incompressible flow. The streamlines of the superimposed flow are found by setting

$$\psi = V_\infty r \sin \theta + \frac{\Lambda}{2\pi} \theta = \text{const.}$$

The velocity field can be found by differentiating equation 2.5 as follows:

$$V_r = \frac{1}{r} \frac{\partial \psi}{\partial \theta} = V_\infty \cos \theta + \frac{\Lambda}{2\pi r}.$$ \[6\]

and

$$V_\theta = -\frac{\partial \psi}{\partial r} = -V_\infty \sin \theta.$$ \[7\]

The stagnation points in the flow can be found by equating the equations 2.6 and 2.7 equal to zero: This is

$$V_\infty \cos \theta + \frac{\Lambda}{2\pi r} = 0,$$

and

$$-V_\infty \sin \theta = 0.$$ \[2.6\]

The coordinates of the stagnation points are obtained by solving for $r$, and $\theta$ thus $(r, \theta) = \left( \frac{\Lambda}{2\pi V_\infty}, \pi \right)$, and is labeled as point $B$ in figure 2.6. The equation of the stream function that goes through the stagnation point is obtained by substituting its coordinates into this equation

$$\psi = V_\infty r \sin \theta + \frac{\Lambda}{2\pi} \theta = \text{const},$$

to get: $\psi = \psi = \frac{\Lambda}{2} = \text{const}$, hence the curve $ABC$ in the schematic 2.6 is the streamline described by $\psi = \frac{\Lambda}{2}$. \[10\].

### 2.1.4 Doublet flow

A doublet flow is a particular, degenerate case a source -sink combination that leads to a singularity. This kind of flow is often used in the theory of incompressible flow. The stream function for a doublet flow is

$$\psi = -\frac{\kappa}{2\pi} \sin \theta.$$ \[2.8\]

Similarly, it can be shown that velocity potential for a doublet flow is given by

$$\phi = \frac{\kappa}{2\pi} \cos \theta.$$ \[2.9\]

We find the streamlines of a doublet flow by setting $\psi = -\frac{\kappa}{2\pi} \sin \theta = \text{const} = c$. And using the fact that $r = d \sin \theta$, it can be shown that the streamlines are a family of circles with diameter $\frac{\kappa}{2\pi c}$, as shown in figure 2.8.
2.1.5 Superposition of a Uniform Flow and a Doublet Flow

The superposition of the uniform flow and the doublet flow yields the flow over a circular cylinder. A circular cylinder is among the most elementary geometrical shapes available and the study of the flow around it has been classically an interesting problem in aerodynamics. Consider a superposition of a uniform flow with a freestream velocity $V_\infty$ and a doublet of strength $\kappa$ as illustrated in figure 2.9. The stream function of the superimposed flow is given as:

$$\psi = V_\infty \sin \theta - \frac{\kappa \theta}{\rho}.$$

Figure 2.9: The synthesis non-lifting flow over a cylinder. [10]
by

\[ \psi = V_\infty r \sin \theta - \frac{\kappa \sin \theta}{2\pi r}, \tag{2.10} \]

\[ \psi = V_\infty r \sin \theta \left(1 - \frac{\kappa}{2\pi V_\infty r^2}\right), \]

\[ \psi = (V_\infty r \sin \theta) \left(1 - \frac{R^2}{r^2}\right), \quad \text{where } R^2 \equiv \frac{\kappa}{2\pi V_\infty}. \tag{2.11} \]

The equation (2.11) shows the stream function of the flow over a circular cylinder of radius

\[ R = \sqrt{\frac{\kappa}{2\pi V_\infty}}. \tag{2.12} \]

The velocity field is:

\[ V_r = \frac{1}{r} \frac{\partial \psi}{\partial \theta} = \frac{1}{r} (V_\infty \cos \theta) \left(1 - \frac{R^2}{r^2}\right) = \frac{1}{r} \left(1 - \frac{R^2}{r^2}\right) V_\infty \cos \theta. \tag{2.13} \]

and

\[ V_\theta = -\frac{\partial \psi}{\partial r} = -\left[(V_\infty \sin \theta) \frac{2R^2}{r^2} + \left(1 - \frac{R^2}{r^2}\right) V_\infty \sin \theta\right] = -\left(1 + \frac{R^2}{r^2}\right) V_\infty \sin \theta. \tag{2.14} \]

From the equations (2.13) and (2.14) , we obtain the stagnation points by solving simultaneously the following equations

\[ -\left(1 + \frac{R^2}{r^2}\right) V_\infty \sin \theta = 0; \]

\[ \left(1 - \frac{R^2}{r^2}\right) V_\infty \cos \theta = 0; \]

and solving for \( r \) and \( \theta \) we get the following stagnation points: \((r, \theta) = (R, 0)\) or \((r, \theta) = (R, \pi)\), denoted \(A\) and \(B\) in figure 2.9. The flow around a circular cylinder is symmetrical about both the horizontal axis and vertical axis through the center of the cylinder as clearly illustrated by the streamline sketched in figure 2.9. Therefore, the pressure gradient is also symmetrical about both axes meaning that the pressure distribution over the bottom of the cylinder is exactly balanced by the pressure distribution over the top of the cylinder and consequently the net lift cannot be generated over the cylinder. Similarly, there is no net drag, because the pressure distribution over the front of the cylinder is also exactly balanced with that over the back of the cylinder. That is why the vortex flow is needed to model the lifting over the circular cylinder \[1\].
2.1.6 Vortex Flow

The last but not the least flow is the elementary flow is called a vortex flow and is a flow where all streamlines are concentric circles about some point \( O \) as shown in the figure 2.11. For a vortex flow the tangential component of the velocity \( V_\theta \) is inversely proportional to the distance from the center. It means that \( V_\theta = \frac{c}{r} \), \( c = \text{const} \). In contrast, the radial component of the velocity is zero (i.e. \( V_r = 0 \)). It is not difficult to show that the vortex flow is both incompressible and irrotational flow except at the origin where the vorticity is infinity. The circulation around a given streamline of radius is given by:

\[
\Gamma = -\oint_C V \cdot ds = -V_\theta(2\pi r)
\]

\[
\Leftrightarrow V_\theta = -\frac{\Gamma}{2\pi r} = \frac{c}{r}
\]

\[
\Leftrightarrow c = -\frac{\Gamma}{2\pi}
\]

\[
\Leftrightarrow \Gamma = -2\pi c. \quad (2.15)
\]

The equation (2.15) the circulation is conventionally called the strength of a vertex flow. The velocity potential \( \phi \) for a vortex flow can be found by the following relations:

\[
\frac{\partial \phi}{\partial r} = V_r = 0.
\]

\[
\frac{1}{r} \frac{\partial \phi}{\partial \theta} = V_\theta = -\frac{\Gamma}{2\pi r}.
\]

Therefore, the potential function \( \phi \) is given by:

\[
\phi = -\frac{\Gamma}{2\pi} \theta, \quad (2.16)
\]
and is found by integration of the above equations. Again, we find the stream function $\psi$ in the same fashion, and it is

$$\psi = \frac{\Gamma}{2\pi} \ln r. \quad (2.17)$$

### 2.1.7 Lifting Flow over a Cylinder

The lifting flow over a cylinder is a combination of a non-lifting flow discussed in section 2.1.5 and a vortex flow of strength $\Gamma$ as shown in the figure 2.12. The stream function and the potential function of the lifting flow over a cylinder are obtained by addition of the stream function and the potential function of the non-lifting flow and those of the vortex flow and are written below:

$$\psi = V_\infty r \sin \theta - \frac{\kappa}{2\pi} \sin \theta r + \frac{\Gamma}{2\pi} \ln r,$$

$$\Leftrightarrow \psi = V_\infty r \sin \theta \left( 1 - \frac{R^2}{r^2} \right) + \frac{\Gamma}{2\pi} \ln r. \quad (2.18)$$

$$\phi = V_\infty r \cos \theta - \frac{\Gamma}{2\pi} \theta + \frac{\kappa}{2\pi} \cos \theta r. \quad (2.19)$$

Note that the streamlines for this flow are no longer symmetric about the horizontal axis through the central point $O$, however, the streamlines are symmetrical about the vertical axis through $O$ and hence they will be no drag. Also note that the circulation is finite and is equal to $\Gamma$ because the vortex of strength $\Gamma$ has been added to the flow. The velocity components are found by differentiating $\psi$. It means that

$$V_r = \left( 1 - \frac{R^2}{r^2} \right) V_\infty \cos \theta. \quad (2.20)$$

$$V_\theta = \left( 1 - \frac{R^2}{r^2} \right) V_\infty \sin \theta - \frac{\Gamma}{2\pi r} \quad [5]. \quad (2.21)$$
And the stagnation points are obtained by setting $V_r = 0$, and $V_\theta = 0$, and then solve for $(r, \theta)$ simultaneously in the equations (2.20) and (2.21) to get $(r, \theta) = \left( R, \arcsin \left( -\frac{\Gamma}{4\pi V_\infty R} \right) \right)$. Because $\Gamma$ is positive, the angle $\theta$ must be 3rd and forth quadrants. This means that there can be two stagnation points on the bottom half of the circular cylinder as shown by the points 1 and 2 in the figure below. If $\Gamma = 4\pi V_\infty R$, there is one stagnation point on the surface of the cylinder with coordinates $\left( R, \frac{\pi}{2} \right)$, labeled as point 3. If $\Gamma > 4\pi V_\infty R$, there are two stagnation points, one inside and the other outside the cylinder, but both are on the vertical axis and are labeled 5 and 4, respectively. From figure 2.14, the stagnation point changes the location as the circulation changes. Hence for the incompressible flow over a circular cylinder, there are an infinite number of possible potential flow solutions, that corresponds to the infinite choices for values of $\Gamma$ and this statements becomes the general statement which holds for the incompressible potential flows over all smooth two dimensional bodies. From the symmetry, or lack of it as illustrated in the previous figures, we have concluded that the lift (normal force) must exist on the body however, the drag is zero. A quantification of these statements by calculating the drag and lift coefficients is done in [10]. These coefficients
are defined as dimensionless quantities(numbers) that areodynamicists use to model all the complex dependencies of shape, inclination and some flow conditions on either airplane lift or drag and as a result the lift coefficient for inviscid, incompressible fluid is

\[ c_l = \frac{\Gamma}{RV_\infty}. \]  

(2.22)

and it relates the lift generated by an airfoil, the dynamic pressure on the fluid flow around the airfoil and the area of the airfoil. The drag coefficient is

\[ c_d = 0, \]  

(2.23)

regardless on whether or not the flow has the circulation about the cylinder. The proof is left for the leader and can be found in [10], section 3.15.

2.1.8 Lift and Drag around two Dimensional Shapes

No matter what the cross section shape of any cylindrical body, one can relate the ideal Lift and Drag force to the complex potential. In practice we will concentrate on cylinders transformed into an airfoil shape. Any bluff shape would have awake of finite thickness, and this would invalidate the theory. The entire flow must be an ideal flow defined in section 1.6 for this theory to apply. Boundary layers and wakes must be vanishingly thin. Consider an incompressible flow over an airfoil as shown in figure 2.15 and let \( A \) be any curve in the fluid flow enclosing the airfoil, then the circulation if given by \( \Gamma \equiv \oint_A V \cdot ds \), where \( V \) is the velocity field around the airfoil and the airfoil is generating a lift. It will turn out that the drag force is always zero; \( F_d = 0 \), and that the lift force is directly proportional to the circulation constant \( \Gamma \). The exact relation for the lift force is

\[ \dot{L} = \rho_\infty V_\infty \Gamma. \]  

(2.24)

Where \( \rho_\infty \) is the fluid density and \( V_\infty \) is the fluid velocity far upstream of the airfoil and \( \Gamma \) is the circulation defined in figure 2.15. This is the Kutta-Joukowsky theorem. For the
proof see [10] page 236 – 238. In fact this theorem states that the lift per unit span \( \dot{L} \) is directly proportional to the circulation. This is a fundamental theorem of aerodynamics as it relates the lift per unit span on an airplane wing (airfoil) to the speed \( V_\infty \) of the airfoil through the fluid, the density of the fluid and the circulation \( \Gamma \). This relation shows that the lift force increases directly as the circulation increases. The Kutta Joukowski formula shows that the circulation is the most important property of the flow in determining the lift.

Let us investigate the reason why a spinning cylinder generates lift. In fact the friction between the fluid and the surface of the cylinder tends to drag the fluid near the surface in the same direction as the rotational motion. Superimposed on the top of the usual nonspinning flow, this extra velocity contribution creates a higher than usual velocity on the top of the cylinder and lower than usual velocity on the bottom as drawn below. The velocities are assumed to be just outside the viscous boundary layer of the surface. From Bernoulli equation, it is known that the pressure decreases, as the velocity increases, as a result the pressure on the top of the cylinder is lower than the pressure on the bottom of the cylinder and consequently this pressure imbalance creates a net upwards force, that is a finite lift. Therefore the prediction embodied in this equation (2.24) that the flow over a circular cylinder can produce a finite lift is verified by experimental observation.

Figure 2.15: Circulation around a lifting airfoil. [10]

Figure 2.16: Creation of lift on spinning cylinder [10]
Chapter 3

Methodology

3.1 The Conformal Mapping Technique

As described in subsection 1.2.1 the airplane wings have complicated geometry, it is then difficult to directly solve for the fluid flow around the airfoils using Laplace equation and potential flow theory. To simplify the problem, the conformal mapping technique is used to extend the application of potential flow theory to practical aerodynamics [1]. The main advantage of conformal mapping in solving the fluid flow problems is that solutions of the Laplace equation for $\phi(x,y)$ and $\psi(x,y)$, remain solutions when subjected to conformal transformation. Let us consider a conformal mapping from $z = x + iy$ plane to a $w = f(z) = u + iv$ plane as defined in subsection 1.5.3. And in section 1.5.1, we have proved that the $z$–plane is conformally mapped to the $w$–plane, in which the Laplace equation is still satisfied for each $\phi(x,y)$ and $\psi(x,y)$. We can now solve for them in the transformed $w = f(z)$ plane, and then compute the solution by the inverse transformation to the original $z$–plane because the transformed boundaries in the $u, v$–plane take a simpler form. In summary, the solution of a flow problem found by conformal mapping consists of the following steps:

1. Map the given flow domain $D \in \mathbb{C}$ onto a geometrically simpler flow domain $D' \in \mathbb{C}$ using conformal transformation, here boundary conditions $\phi$ and $\psi$ remain the same for the transformed boundaries.

2. Solve the Laplace equation of either $\phi$ and or $\psi$ in the transformed domain.

3. Use the inverse transformation to derive $\phi = \phi(x,y)$, $\psi = \psi(x,y)$. Sometimes a sequence of conformal mappings is needed before a solution for $\phi$ and or $\psi$ can be found [12].

For instance, let us consider the $z = x + iy$ plane of a uniform fluid flow horizontal streamlines given by $\phi = iy$, and equipotential curves given by $\psi = x$. In figure 3.1, a conformal mapping given by $f(z) = \sqrt{z}$, is used to transform this complex plane $z$ into a new complex
Figure 3.1: A conformal mapping of a uniform flow with the conformal function $w(z) = \sqrt{z}$. The vertical lines in (a) represent equipotential curves, the horizontal lines represent streamlines. In (b), the conformal map maintains the right angle relationship between the streamlines and equipotential curves [1]

plane $w$. It is clear that this transformation has changed the relative shape of the streamlines and equipotential curves, but the set of curves remains perpendicular. This angle preserving feature is the essential component of conformal mapping. Since the relative orientation of the streamlines and potential curves remains unchanged, any harmonic function in the $z$ plane, is also harmonic in the transformed $w$ plane [12]. In the next section, we will show how the flow around an airfoil shape can be constructed by first solving for the flow around a cylinder in the $z$ plane, and then transforming this solution to an airfoil in the $w$ plane using a specific conformal mapping function.

3.2 Mapping from a Circular Cylinder to an Airfoil with the Joukowsky Transformation

In section 2.1.7 we have seen that the two dimensional incompressible, inviscid, irrotational flow around a cylinder is generated by the superposition of three elementary flows, namely a uniform flow, source-sink flow and a vortex flow. The stream function $\psi$ of such a flow is given by the equation (2.18). The flow around a circular cylinder can be mapped into the flow about another body by using conformal mapping with the complex potential preserved. The most known conformal transformation is the Joukowsky transformation given by:

$$w = f(z) = z + \frac{\lambda^2}{z},$$  \hspace{1cm} (3.1)

where $w$ is the function in the transformed plane and $\lambda$ is the parameter of the transformation that determines the resulting shape of the transformed function. Let $z = x + iy = r(\cos \theta + i \sin \theta) = re^{i\theta}$, this is a circle of radius $r$ and the center at the origin in the $z-$ plane [21].
And this implies that

\[ w = f(z) = z + \frac{\lambda^2}{z} = re^{i\theta} + \frac{\lambda^2}{r}e^{-i\theta} \]  

\[ \iff \]

\[ w = r \cos \theta + ir \sin \theta + \frac{\lambda^2}{r}(\cos \theta - i \sin \theta) \]

\[ \iff \]

\[ w = r \cos \theta + ir \sin \theta + \frac{\lambda^2}{r}(\cos \theta - i \sin \theta) \]

\[ \iff \]

\[ w = a \cos \theta + ib \sin \theta, \]  

(3.3)

where \( a = r + \frac{\lambda^2}{r} \), \( b = a - \frac{\lambda^2}{r} \), and also \( w = u + iv \),

\[ \Rightarrow \begin{cases} 
  u = a \cos \theta \\
  v = b \sin \theta, 
\end{cases} \]

\[ \Rightarrow \begin{cases} 
  \cos \theta = \frac{u}{a} \\
  \sin \theta = \frac{v}{b}. 
\end{cases} \]

But \( \cos^2 \theta + \sin^2 \theta = 1 \), \( \Rightarrow \left( \frac{u}{a} \right)^2 + \left( \frac{v}{b} \right)^2 = 1 \iff \frac{1}{a^2}u^2 + \frac{1}{b^2}v^2 = 1 \). If we set the parameter \( \lambda = r \), then \( a = r + \frac{r^2}{r} \), and \( b = r - \frac{r^2}{r} \), and this implies that \( a = 2r, b = 0 \), and from this it can be seen that the circle in the \( z \)-plane is transformed into a flat plate of length \( 4r \) in the \( w \)-plane. It means that the points of the circle in the \( z \)-plane occupy the strip \( -2r \leq u \leq 2r \), in the \( w \)-plane. For \( \lambda \geq r \), the circle in the \( z \)-plane is transformed into an ellipse in the \( w \)-plane; this is

\[ \left( \frac{u}{r + \frac{\lambda^2}{r}} \right)^2 + \left( \frac{v}{r - \frac{\lambda^2}{r}} \right)^2 = 1, \]  

(3.4)

\[ \Rightarrow \frac{u^2}{r^2(1 + \lambda^2/r^3)^2} + \frac{v^2}{r^2(1 - \lambda^2/r^3)^2} = 1, \]  

(3.5)

which is an equation of ellipse centered at the origin with the semi major axis \( a = r(1+\lambda^2/r^3) \), and the minor axis \( b = r(1 - \lambda^2/r^3) \). Now the implementation of this in MATLAB is shown below:
Figure 3.2: A circular cylinder centered at the origin in the $z$--plane with radius $r = 1$, and $\lambda = 1$ (1). The corresponding flat plate of length 4 in the $w$--plane transformed using Joukowsky transformation (2). This is implemented in MATLAB.

Figure 3.3: (3) A circular cylinder centered at the origin in the $z$--plane with radius $r = 1$, and $\lambda > 1$, and (4) The flow around the corresponding ellipse in the $w$--plane transformed using Joukowsky transformation.

Nevertheless, none of the shapes 3.2 and 3.3 look like an airfoil. The airfoil shape is found by considering a circle in the $z$--plane which is not centered at the origin. The circle in the $z$--plane is slightly offset using the following parameter transformation: $\lambda = r - |t|$, with $t$ the coordinate of the center of the circle in the $z$--plane. In this case the image in the $w$--plane look like the shape of an uncambered airfoil which is symmetric about the $x$--axis, as illustrated in figure 3.4. The $x$ coordinate of the circle consequently determines the thickness distribution of the transformed airfoil. When the center of the circle in the $z$--plane is again offset on the $y$ axis, we get an unsymmetrical, cambered airfoil as shown in figure 3.5. And this shows that the $y$ coordinate of the center of circle determines the curvature of the transformed airfoil in the $w$--plane. The Joukowsky transformation creates the airfoils
3.2. **MAPPING FROM A CIRCULAR CYLINDER TO AN AIRFOIL WITH THE JOUKOWSKY TRANSFORMATION**

shapes known as Joukowsky airfoils. The leading and trailing edges of the transformed airfoil
in the $w$ plane correspond to the $x$ intercepts of the circle in the $z-$ plane [15].

![Figure 3.4](image1.png)

**Figure 3.4:** (a) A circular cylinder centered at $(-0.105, 0)$, (the center is offset on the $x$ axis) in the $z-$plane with radius $r = 1.205$, and $\lambda = 1 - |t|$. (b) The corresponding symmetric uncambered airfoil in $w-$plane obtained using Joukowsky transformation of the cylinder.

![Figure 3.5](image2.png)

**Figure 3.5:** (c) A circular cylinder centered at $(0.3, 0.12)$, (the center is offset on the $x$ and $y$ axes) in the $z-$plane with radius $r = 1.202$, and $\lambda = 1 - |t|$. (d) The corresponding (non symmetric) cambered airfoil in $w-$plane obtained using Joukowsky transformation of the cylinder.
Figure 3.5 (b) also shows that the Joukowsky airfoil has a cusp at the trailing edge, which is a mathematical property that is not present in real airfoil shapes [16]. Besides to above transformation, the flow around the circular cylinder can be transformed by the conformal mapping functions. This is done by expressing the velocity potential \( \phi \) and stream function \( \psi \) as a complex function specifically called the complex potential given by:

\[
\Theta(z) = \phi + i\psi,
\]  

(3.6)
as defined in subsection 1.5.1

**Theorem 3.1** Consider the lifting flow past a cylinder, then its complex potential defined in (3.6) can be written as follows:

\[
\Theta(z) = V_\infty \left( z + \frac{R^2}{z} \right) + i\frac{\Gamma}{2\pi} \ln(z).
\]  

(3.7)

**Proof**

For the uniform flow, the complex potential is given by:

\[
\Theta(z) = \phi + i\psi = V_\infty r \cos \theta + iV_\infty r \sin \theta = V_\infty x + iV_\infty y = V_\infty z \quad [24];
\]  

(3.8)

for the doublet flow, the complex potential is given by:

\[
\Theta(z) = \phi + i\psi = \frac{\kappa}{2\rho r} (\cos \theta - i \sin \theta) = \frac{\kappa}{2\rho r} e^{i\theta} = \frac{\kappa}{2\rho z};
\]  

(3.9)

and for the vortex flow, the complex potential is given by:

\[
\Theta(z) = \phi + i\psi = -\frac{\Gamma}{2\pi} \frac{\theta}{r} + i\frac{\Gamma}{2\pi} \ln(r) = i\frac{\Gamma}{2\pi} \ln(r e^{i\theta}) = i\frac{\Gamma}{2\pi} \ln(z).
\]  

(3.10)

The superposition of the equations (3.8), (3.9) and (3.10) yields:

\[
\Theta(z) = V_\infty z + \frac{\kappa}{2\rho z} + i\frac{\Gamma}{2\pi} \ln(z)
\]

\[
\Leftrightarrow
\]

\[
= V_\infty z + \frac{R^2 V_\infty}{z} + i\frac{\Gamma}{2\pi} \ln(z),
\]  

(3.12)

because \( R = \sqrt{\frac{\kappa}{2\pi V_\infty}} \).

\[
\Leftrightarrow
\]

\[
\Theta(z) = V_\infty \left( z + \frac{R^2}{z} \right) + i\frac{\Gamma}{2\pi} \ln(z).
\]  

(3.13)

**Corollary 3.1** When the complex potential shown in equation (3.13) is transformed by a conformal mapping function, the circulation \( \Gamma \) and the lift \( \hat{L} \) for the circular cylinder in the \( z \) plane are the same as the circulation \( \Gamma \) and the lift \( \hat{L} \) in the \( w \)-plane and this is the result of the angle and orientation preservation property of the conformal mapping technique.
Chapter 4

Results and Analysis

This chapter focuses on computational aspects involved in solving our problem. MATLAB software is used to visualize how the lifting flow around a circular cylinder can be transformed into the flow around the Joukowsky airfoils and then calculate lift.

4.1 Computing the Streamlines around an Airfoil and Lift Calculation

When a Joukowsky transformation is applied to an offset circular cylinder, one can get the Joukowsky airfoils by the use of instance of matlab program. The contour plot of the imaginary part of the complex potential of the equation (3.13) gives the flow around the airfoil. The lift force is calculated using the formula in equation (2.24)

\[ \hat{L} = V_\infty \rho_\infty \Gamma, \]

where

\[ \Gamma = \frac{2r V_\infty \sin \alpha}{2\pi}. \] 

(4.1)

and the angle \( \alpha \) that we have in the equation (4.1) was measured in radians and when converted into degrees amounts to \( \frac{\alpha \pi}{180} \). At zero angle of attack, there is no lift generated on the cylinder and on the airfoil because the fluid flow is symmetric. This is due to the fact that there is a symmetric distribution of the streamlines about the \( x \) axis on both the circular cylinder in the \( z \) plane and the airfoil in the \( w \) plane. Again, looking on the airfoil, it is clear that the streamlines meet at the trailing edge, and therefore the Kutta condition is satisfied. In this experience, we are going to compute the streamlines around the circular cylinder and the corresponding airfoil at different values of angle of attack. In figure 4.2, the calculated lift force is 6.5552 \( N/m \) at \( \alpha = 4 \), in figure 4.3, the lift force found is 13.0785 \( N/m \) at \( \alpha = 8 \), in figure 4.4, we have got the lift force equal to 19.5381 \( N/m \) at \( \alpha = 12 \), and finally we have got 24.322 \( N/m \) as the lift force at \( \alpha = 15 \).
CHAPTER 4. RESULTS AND ANALYSIS

Figure 4.1: The streamlines around circular cylinder plot computed in the $z$ plane and the corresponding symmetric Joukowsky airfoil. The plot was generated with $V_\infty = 200 \text{ m/s}$, $\alpha = 0$, and $\rho = 1.225 \text{ kg/m}^3$. The cylinder parameters used: $x = -0.105 \text{ m}$, $y = 0 \text{ m}$, $r = 1.205 \text{ m}$.

Figure 4.2: The streamlines around circular cylinder plot computed in the $z$ plane and the corresponding symmetric Joukowsky airfoil. The plot was generated with $V_\infty = 200 \text{ m/s}$, $\alpha = 4$, and $\rho = 1.225 \text{ kg/m}^3$. The cylinder parameters used: $x = -0.105 \text{ m}$, $y = 0 \text{ m}$, $r = 1.205 \text{ m}$.

In figure 4.6, we generated a plot of the lift force versus the angle of attack, and this figure shows that there is a strong positive linear relationship between the lift force and the angle of attack. In fact the lift force increases as the angle of attack increases. In figure 4.7, the lift coefficients for Joukowsky 12% airfoil and that for the National Advisory Committee for Aeronautics airfoil denoted as NACA 0012 airfoil, were compared and it is shown that
4.1. COMPUTING THE STREAMLINES AROUND AN AIRFOIL AND LIFT CALCULATION

Figure 4.3: The streamlines around circular cylinder plot computed in the $z$ plane and the corresponding symmetric Joukowsky airfoil. The plot was generated with $V_\infty = 200 \text{ m/s}$, $\alpha = 8$, and $\rho = 1.225 \text{ kg/m}^3$. The cylinder parameters used: $x = -0.105 \text{ m}$, $y = 0 \text{ m}$, $r = 1.205 \text{ m}$.

Figure 4.4: The streamlines around circular cylinder plot computed in the $z$ plane and the corresponding symmetric Joukowsky airfoil. The plot was generated with $V_\infty = 200 \text{ m/s}$, $\alpha = 12$, and $\rho = 1.225 \text{ kg/m}^3$. The cylinder parameters used: $x = -0.015 \text{ m}$, $y = 0 \text{ m}$, $r = 1.205 \text{ m}$.

The lift curve from the Joukowsky airfoil almost matches the lift curve predicted using the NACA 0012 airfoil very well. We have also plotted the absolute error in lift coefficient versus the angle of attack. The figure 4.8 shows that the two lift curves are almost equivalent as the maximum absolute error is $0.0279 \equiv 2.8\%$. The data for the lift coefficient were taken from [20].
Figure 4.5: The streamlines around circular cylinder plot computed in the $z$ plane and the corresponding symmetric Joukowsky airfoil. The plot was generated with $V_{\infty} = 200 \text{ m/s}$, $\alpha = 15$, and $\rho = 1.225 \text{ kg/m}^3$. The cylinder parameters used: $x = -0.015 \text{ m}$, $y = 0 \text{ m}$, $r = 1.205 \text{ m}$.

<table>
<thead>
<tr>
<th>Angle of attack $\alpha$</th>
<th>0</th>
<th>4</th>
<th>8</th>
<th>12</th>
<th>15</th>
</tr>
</thead>
</table>

Table 4.1: Summary of results for lift force calculation.

Figure 4.6: Lift force vs Angle of attack
4.1. COMPUTING THE STREAMLINES AROUND AN AIRFOIL AND LIFT CALCULATION

Table 4.2: Summary of results for lift coefficient calculation.

<table>
<thead>
<tr>
<th>Angle of attack $\alpha$</th>
<th>0.250</th>
<th>1.250</th>
<th>2.250</th>
<th>3.250</th>
<th>4.250</th>
<th>5.250</th>
<th>6.250</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_L(NACA0012)$</td>
<td>0.0281</td>
<td>0.1379</td>
<td>0.2349</td>
<td>0.3545</td>
<td>0.4610</td>
<td>0.5689</td>
<td>0.6875</td>
</tr>
<tr>
<td>$C_L(JOUKOWSKY12%)$</td>
<td>0.0288</td>
<td>0.1438</td>
<td>0.2459</td>
<td>3.724</td>
<td>0.4853</td>
<td>0.5968</td>
<td>0.7051</td>
</tr>
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</table>

Table 4.3: Summary of results for lift coefficient calculation.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_L(NACA0012)$</td>
<td>0.8162</td>
<td>0.9453</td>
<td>1.0365</td>
<td>1.1288</td>
<td>1.2227</td>
<td>1.3144</td>
</tr>
<tr>
<td>$C_L(JOUKOWSKY12%)$</td>
<td>0.8145</td>
<td>0.9228</td>
<td>1.0290</td>
<td>1.1318</td>
<td>1.2301</td>
<td>1.3221</td>
</tr>
</tbody>
</table>

Figure 4.7: Comparison between Joukowsky 12% airfoil and the NACA 0012 airfoil, here we compared the linear dependence of lift coefficient on the angle of attack for both airfoils.
Figure 4.8: Absolute Error in lift coefficients for Joukowsky Airfoil and actual NACA 0012 Airfoil.
Chapter 5

Conclusion

The main goal of our thesis was to apply the conformal mapping technique to model the two dimensional fluid flow around an airplane wing. We reviewed the mathematical model used to describe the two dimensional ideal fluid flow around a circular cylinder obtained by the superposition of simple elementary flows. This model was implemented in MATLAB in order to visualize the streamlines and the equipotential curves for each elementary flow needed in the process of modeling the flow around a circular cylinder. We have used the Joukowsky transformation to map the circular cylinder to a flat plate, an ellipse and to the desired Joukowsky airfoil. We have established the complex potential of the lifting flow around the circular cylinder in theorem 3.1 and it was found that the circulation and the lift force in the $z$ plane remained unchanged in the $w$ plane when the complex potential is transformed by a conformal mapping function. MATLAB software was used to compute and visualize the streamlines around the circular cylinder and the corresponding Joukowsky airfoil. The lift force was calculated at different angle of attack and it was found that the lift force is strongly dependent to the angle of attack in a linear proportion as illustrated in figure 4.6. Moreover we have compared the linear dependence of the lift coefficients on the angle of attack for Joukowsky and NACA 0012 airfoil. And the results showed that they are almost equivalent in the range of 0 to 12.25 degrees of the angle of attack. The plot of the absolute error in $C_L$ was generated to show graphically the maximum absolute error in lift coefficients for Joukowsky Airfoil and actual NACA 0012 Airfoil which is approximately 2.8%.

For further studies one can emphasize on the Karman-Trefftz transformation as an improved transformations to avoid the two drawbacks of the Joukowsky airfoils. These drawbacks include a knife-edge cusp at the trailing edge which presents manufacturing and structural integrity difficulties and the fact that the profiles of the Joukowsky airfoil have the maximum thickness very close to the leading edge, which results in adverse pressure gradient over most of the upper surface leading to earlier boundary layer transition and higher skin friction drag.
Appendix A

MATLAB Codes

```matlab
% JOUKOWSKI TRANSFORMATION
%This codes transforms the circular cylinder in the z-plane into an ellipse
%in the w-plane

clear all
close all
clc
disp('------------------------------------------')
disp(' Joukowski Transformation Input Manager')
disp('------------------------------------------')
disp('------------------------------------------')
s_x = input(' x coordinate of the center of the Circle, X_0 in [m]: ');
s_y = input(' y coordinate of the center of the Circle, Y_0 in [m]: ');
s = s_x + i*s_y;
x = input(' Radius [m]: ');
disp('------------------------------------------')
disp('If Solution visualization is uncorrect try to modify Tolerance TOLL')
disp('------------------------------------------')

lambda = 1.5;
Tollcrancoc

toll = +5e-2;
% Mesh Generation
x = meshgrid(-5:0.02:5);
y = x';
% Complex Plane
z = x + i*y;
% Joukowsky Transformation,
J = Z+lambda^2/z;
% Circle and Joukowsky Airfoil
angle = 0:.01:2*pi;
z_circle = r*(cos(angle)+i*sin(angle))+s;
z_airfoil=z_circle +(1./z_circle );
```
% Plotting Solution
figure[1]
hold on
grid on
fill(real(z_circle), imag(z_circle), 'y')
axis equal
axis([-5 5 -5 5])
xlabel(strcat('x'))
ylabel(strcat('y'))
title(strcat('3 A unity circular cylinder in the z plane centered at the origin'))

figure[2]
hold on
grid on
fill(real(z_airfoil), imag(z_airfoil), 'y')
axis equal
axis([-5 5 -5 5])
xlabel(strcat('u'))
ylabel(strcat('v'))
title(strcat('4 The corresponding ellipse in the w plane'))

% ---------------------------------------------------------------
% JOUKOWSKI TRANSFORMATION
% This codes transforms the circular cylinder in the z-plane into an Airfoil
% in the w-plane

clear all
close all
clc
disp('---------------------------------------------------------------')
disp(' Joukowski Transformation Input Manager')
disp('---------------------------------------------------------------')
disp('---------------------------------------------------------------')
s_x = input(' x coordinate of the center of the Circle, X_0 in [m]: ');
s_y = input(' y coordinate of the center of the Circle, Y_0 in [m]: ');
s = s_x + i*s_y;
x = input(' Radius [m]: ');
disp('---------------------------------------------------------------')
disp('If Solution visualization is uncorrect try to modify Tolerance TOLL')
disp('---------------------------------------------------------------')
% Transformation Parameter.
lambda = r-s;
% Tolerance
 toll = +5e-2;
% Mesh Generation
x = memgrid(-3:0.02:3);
y = x';
% Complex Plane
z = x + i*y;
% Joukowski Transformation,
J = z+lambda^2./z;
% Circle and Joukowsky Airfoil
angle = 0:.01:2*pi;
z_circle = r*(cos(angle)+i*sin(angle)) + s;
z_airfoil = z_circle + (1./z_circle );
% Plotting Solution
figure(1)
hold on
grid on
fill(real(z_circle), imag(z_circle), 'y')
axis equal
axis([-3 3 -3 3])
xlabel(restore('x'))
ylabel(restore('y'))
title(restore('a) A circular cylinder in the z plane for lambda=1-|t|'))

figure(2)
hold on
grid on
fill(real(z_airfoil), imag(z_airfoil), 'y')
axis equal
axis([-3 3 -3 3])
xlabel(restore('u'))
ylabel(restore('v'))
title(restore('b) The Corresponding Symmetric Airfoil in the w plane with lambda=1-|t|'))
APPENDIX A. MATLAB CODES

% JOUKOWSKI TRANSFORMATION
% This code transforms the circular cylinder in the z-plane into an Airfoil
% in the w-plane. It also computes the streamlines around the circular
% cylinder in z-plane and the corresponding airfoil in the w-plane. And it
% finally computes the lift force around the two shapes
%
clear all
close all
clc
disp('-----------------------------------')
disp(' Joukowski Transformation Input Manager ')
disp('-----------------------------------')
v_inf = input(' Asymptotic Speed Modulus [m/s]: ');
v = v_inf/v_inf;
theta = input(' Asymptotic Speed Angle (Angle of attack)[deg]: ');
theta = theta*pi/180;
disp('-----------------------------------')
s_x = input(' x coordinate of the center of the Circle, X_0 in [m]: ');
s_y = input(' y coordinate of the center of the Circle, Y_0 in [m]: ');
s = s_x + i*s_y;
r = input(' Radius [m]: ');
disp('-----------------------------------')
disp(' If Solution visualization is incorrect try modify Tolerance TOLL')
disp('-----------------------------------')

% Fluid parameter
rho = 1.225;
% Transformation Parameter.
lambda = r-s;
% Circulation
beta = (theta);
K = 2*pi*v*sin(beta);
Gamma = K/(2*pi); % Circulation
% Complex Asymptotic speed
w = v * exp(1i*theta);
% Tolerance
toll = +5e-2;
% Mesh Generation
x = meshgrid(-5:.1:5);
y = x';
% Complex Plane
z = x + 1i*y;
% Inside-circle points are Excluded!
for a = 1:length(x)
    for b = 1:length(y)
        if abs(z(a,b)-s) <= r - toll
            z(a,b) = NaN;
        end
    end
end
% Aerodynamic potential
f = w*(z) + (v*exp(-1i*theta)*r^2)./(z-s) + 1i*k*log(z);
% Joukowski Transformation,
J = z+lambda^2./z;
% Circle and Joukowski Airfoil
angle = 0:.01:*pi;
z_circle = r*(cos(angle)+1i*sin(angle)) + s;
z_airfoil = z_circle+lambda^2./z_circle;
% KUTTA JOUKOWSKI THEOREM
L = v_inf*rho*Gamma;
L_str = num2str(L);

% Plotting Solution
figure(1)
hold on
grid on
contour(real(z),imag(z),imag(f),[-5:2:5])
fill(real(z_circle),imag(z_circle),'y')
axis equal
axis([-5 5 -5 5])
xlabel(strcat('x'))
ylabel(strcat('y'))
title(strcat('Flow Around a Circular Cylinder for \alpha =0. Lift Force: ',L_str,' [N/m]'))

figure(2)
hold on
grid on
contour(real(J),imag(J),imag(C),[-5:2:5])
fill(real(z_airfoil),imag(z_airfoil),'y')
axis equal
axis([-5 5 -5 5])
xlabel(strcat('u'))
ylabel(strcat('v'))
title(strcat('Flow Around the Corresponding Airfoil for \alpha =0. Lift Force: ',L_str,' [N/m]'))
Bibliography


