Relationship between measurable sets in the Lebesgue sense and sets with the Baire property on the real line

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Huye, 2016

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214003433

A dissertation submitted in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE IN APPLIED MATHEMATICS

In the College of Science and Technology

Supervisor: Prof. Vitalij Tjatyrko

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Huye, 2016

Declaration

I declare that this Dissertation contains my own work except where specifically acknowledged. And in addition, it has not been previously submitted for any comparable academic award.

Student's names: Emmanuel TWAGIZIMANA

Signed..........................

Date...........................

Dedication

To my parents, brothers and sisters this work is dedicated too.

Acknowledgements

I wish to sincerely thank all those who have contributed in one way or another to this work. Words can only inadequately express my deep gratitude to my supervisor, Prof. Vitalij TJATYRKO, for his encouragement, fruitful comments and insightful suggestions have been a crucial formative influence on the present study.
I am also grateful to Venuste NYAGAHAKWA for being the second reader and he has supported me in every possible way since the beginning of my thesis. His critical and careful reading of my writing has saved me from a lot of errors.

Grateful acknowledgements are also due to Dr. Isidore MAHARA for his expert guidance, valuable suggestions and kind assistance for class observation. I am also indebted to the valuable support of ISP through EAUMP-UR node. I will forever be thankful to Dr. Froduald MINANI, the coordinator of master program in Applied Mathematics. And to all teaching staff in master' s program of Applied Mathematics. I would also like to thank my colleagues of the class, students and Professors that I have been meeting in the summer schools organized by EAUMP for their unforgotten intellectual discussions and support.

Finally, I extend my thanks to Claudine UZAYISENGA, Solange MUKESHIMANA, Jean Bosco NSHIMIYIMANA, John HARERIMANA, and Modeste KAYONGA, your kindness and advice have been to me a special help, I am so thankful.

Abstract

This thesis is based on some concepts from real analysis and topology on the real line $\mathbb{R}$. The goal is to clarify the relationship between the family $\mathcal{L}(\mathbb{R})$ of Lebesgue measurable sets and the family $\mathcal{B}_p(\mathbb{R})$ of sets possessing the Baire property there. In the text we observe some common properties of these families as well as common properties of their complements in the power set $\mathcal{P}(\mathbb{R})$ of all subsets of the real line. The thesis ends by a theorem which shows that despite of all similarities the family $\mathcal{L}(\mathbb{R})$ of Lebesgue measurable sets and the family $\mathcal{B}_p(\mathbb{R})$ of sets possessing the Baire property are completely different.

Key words:

Lebesgue outer measure, Lebesgue measure, Baire property, meager set, second category set, null set, equivalence relation, invariance of Lebesgue measure, Baire Category Theorem, Axiom of Choice, Bernstein set, Borel set, Vitali set and σ-algebra.

ACRONYMS

UR: University of Rwanda

ISP: International Science Programme, Uppsala university, Sweden

EAUMP: Eastern Africa Universities Mathematics Program

Contents

Chapter 1

Introduction

The concept of Lebesgue measurable sets on the real line $\mathbb{R}$ is an important part of Lebesgue integration theory. The family $\mathcal{L}(\mathbb{R})$ of the Lebesgue measurable sets includes all open sets of $\mathbb{R}$ as well as the family $\mathcal{N}(\mathbb{R})$ of null sets of $\mathbb{R}$. Recall that any open set of the real line is a union of open intervals of $\mathbb{R}$ and $\mathcal{N}(\mathbb{R})$ consists of those subsets of $\mathbb{R}$ which can be covered by a sequence of intervals of arbitrarily small total length. The family $\mathcal{L}(\mathbb{R})$ possesses different set-theoretical properties. It is a σ-algebra of sets what means that countable unions, countable intersections and complements (in $\mathbb{R}$) of elements of $\mathcal{L}(\mathbb{R})$ are again elements of $\mathcal{L}(\mathbb{R})$. Moreover, the family $\mathcal{L}(\mathbb{R})$ is invariant under translations of $\mathbb{R}$. Another family of sets on the real line $\mathbb{R}$ which has similar set-theoretical properties is the family $\mathcal{B}_p(\mathbb{R})$ of sets possessing the Baire property. The family $\mathcal{B}_p(\mathbb{R})$ is known in topology for years. It is also a σ-algebra of sets which is invariant under translations of $\mathbb{R}$. The family $\mathcal{B}_p(\mathbb{R})$ includes all open sets of $\mathbb{R}$ as well as the family $\mathcal{M}(\mathbb{R})$ of meager sets of $\mathbb{R}$ i.e. sets which are countable unions of nowhere dense subsets of $\mathbb{R}$. Let us note that both families $\mathcal{N}(\mathbb{R})$ and $\mathcal{M}(\mathbb{R})$ are σ-ideal of sets what means that they are closed under operations of taking countable unions and subsets of its elements. It is interesting that even the complements $\mathcal{L}(\mathbb{R})^c$ and $\mathcal{B}_p(\mathbb{R})^c$ of the families $\mathcal{L}(\mathbb{R})$ and $\mathcal{B}_p(\mathbb{R})$ in the power set $\mathcal{P}(\mathbb{R})$ of all subsets of $\mathbb{R}$ contain common sets. For example, Vitali sets and Bernstein sets of $\mathbb{R}$ are elements of both families $\mathcal{L}(\mathbb{R})^c$ and $\mathcal{B}_p(\mathbb{R})^c$. Despite of the mentioned above similarities the family $\mathcal{L}(\mathbb{R})$ of Lebesgue measurable sets and the family $\mathcal{B}_p(\mathbb{R})$ of sets possessing the Baire property are very different. This thesis will illuminate this fact.

1.1 Structure of the work

This work consists of seven chapters which are structured as follows.

The first chapter is a short introduction to the subject of the thesis (with some motivation).

The second chapter introduces basic concepts of set theory and general topology of the real line.

The third chapter discusses the Lebesgue measurable sets on the real line and their properties.

The fourth chapter considers the family of sets possessing the Baire property on real line.

The fifth chapter summarizes common properties of the considered earlier families of Lebesgue measurable sets and sets with the Baire property.

The sixth chapter recalls known examples of sets (as Vitali sets and Bernstein sets) which are non Lebesgue measurable and without the Baire property.

The seventh chapter shows that on the real line there are sets which are non-Lebesgue measurable but with the Baire property, and sets which are Lebesgue measurable but without the Baire property.

Chapter 2

Preliminaries and basic concepts

2.1 Elements of Set Theory

A set may be viewed as any well defined collection of **objects** called **elements**. It is usually denoted by capital letters like A,B,C,.... Two ways to specify sets are **tabular form** where all elements are listed, and **set-builder form** where the common property of all elements are mentioned. If p is an element of A, we write $p \in A$ and if p is not an element of A, we write $p \notin A$. The set which does not contain any element is called the **empty set** and it is denoted by $\emptyset$. If every element of A is also an element of B then A is called a **subset** of B and it is written as $A \subset B$. With this terminology, the possibility that $A = B$ exists and a set A will be called a **proper subset** of B if $A \subset B$ and $A \neq B$. Some sets have special symbols like, $\mathbb{N}$ for the set of non negative integers, $\mathbb{Z}$ for the set of integers, $\mathbb{Q}$ for set of rational numbers and $\mathbb{R}$ for the set of real numbers. A collection of sets will be denoted by capital script letters like $\mathcal{A}$, $\mathcal{B}$,... An **index set** is a set whose elements label (or index) elements of another set. For instance, if the elements of a set A may be indexed by means of a set B, then B is an index set.

Let A and B be two non-empty sets. The **union** of A and B denoted by $A \cup B$ is defined as $\{x : x \in A \text{ or } x \in B\}$ and the **intersection** of A and B denoted by $A \cap B$ is defined as $\{x : x \in A \text{ and } x \in B\}$. If $A \cap B = \emptyset$, then A and B are said to be disjoint.
The **simple difference** $A \setminus B$ of A and B is the set $\{x : x \in A \text{ but } x \notin B\}$ and the **symmetric difference** $A \triangle B$ of A and B is the set $A \triangle B = (A \cup B) \setminus (A \cap B)$. If $A \subseteq X$ then $X \setminus A = A^c$ is called **the complement of** A **with respect to** X and it is clear that

$(A \triangle B)^c = A^c \triangle B$. If $\{A_\alpha\}_{\alpha \in I}$ is a collection of sets then $\bigcap_{\alpha \in I} A_\alpha = \{x : x \in A_\alpha \text{ for all } \alpha \in I\}$ and $\bigcup_{\alpha \in I} A_\alpha = \{x : x \in A_\alpha \text{ for some } \alpha \in I\}$ and they are linked by De-Morgan's laws that are $(\bigcup_\alpha A_\alpha)^c = \bigcap_\alpha A_\alpha^c$ and $(\bigcap_\alpha A_\alpha)^c = \bigcup_\alpha A_\alpha^c$.

A family of sets $(A_\alpha)_{\alpha \in H}$ is pairwise disjoint if $A_\alpha \cap A_\beta = \emptyset$ whenever $\alpha \neq \beta$. **The Cartesian product** $A \times B$ of sets A and B is the set of ordered pairs: $A \times B = \{(a, b) : a \in A, b \in B\}$.

2.2 Countable and uncountable set

The **cardinality** of a set A denoted by $|A|$ is the number of elements of A. More precisely, one says that two sets A and B have the same cardinality if there is a bijection $f : A \to B$ of A onto B. A set is said to be **countable** if it is finite or if it has the same cardinality as $\mathbb{N}$. A set is **uncountable** if it is not countable. Let us note that any subset of a countable set is countable and if A and B are countable then the $A \times B$ is countable.

Example 2.1. *The set rational numbers $\mathbb{Q}$ is countable and the set of real numbers $\mathbb{R}$ is uncountable.*

2.3 Relations

Given a non-empty set X, we call a subset $\mathcal{R}$ of $X \times X$ a **relation** on X and write $x\mathcal{R}y$ provided that $(x, y) \in \mathcal{R}$. The relation $\mathcal{R}$ is said to be reflexive provided $x\mathcal{R}x$ for all $x \in X$; the relation $\mathcal{R}$ is said to be symmetric provided $x\mathcal{R}y$ if $y\mathcal{R}x$; and the relation $\mathcal{R}$ is said to be transitive provided whenever $x\mathcal{R}y$ and $y\mathcal{R}z$, then $x\mathcal{R}z$ for every $x, y, z \in X$.

An important type of relation which will be used is an equivalence relation defined as follows:

Definition 2.1. *A relation $\mathcal{R}$ on X is called an* ***equivalence relation*** *provided it is reflexive, symmetric and transitive.*

Example 2.2. *Let $\mathcal{R}$ be the relation on $\mathbb{R}$ defined by saying that" $x\mathcal{R}y$ if and only if $x - y \in \mathbb{Q}$". It is an equivalence relation on $\mathbb{R}$;*

(i) *It is clear that $x\mathcal{R}x$ for all $x \in X$ since $x - x = 0 \in \mathbb{Q}$ and hence it is reflexive.*

(ii) *Assume that* $x, y \in \mathbb{R}$ *and* $x\mathcal{R}y$ *i.e.* $x - y \in \mathbb{Q}$. *Since* $x - y = -(y - x) \in \mathbb{Q}$ *then* $y\mathcal{R}x$ *i.e.* $\mathcal{R}$ *is symmetric.*

(iii) *Suppose* $x, y, z \in \mathbb{R}$, $x\mathcal{R}y$ *and* $x\mathcal{R}z$. *Then* $x - y \in \mathbb{Q}$ *and* $y - z \in \mathbb{Q}$. *Note that* $(x - y) + (y - z) = x - z \in \mathbb{Q}$ *which shows that* $x\mathcal{R}z$ *i.e.* $\mathcal{R}$ *is transitive.*

Definition 2.2. *A* ***partition*** *of a set* X *is a collection of mutually disjoint subsets of* X *whose union is* X.

Let $x \in X$ and $\mathcal{R}$ be an equivalence relation on X. The set $E_x = \{y \in X : x\mathcal{R}y\}$ is called the **equivalence class** of x under $\mathcal{R}$. It is easy to see that $E_x \cap E_y \neq \emptyset$ if and only if $E_x = E_y$. The collection of all equivalence classes under $\mathcal{R}$ is a partition of X.

2.4 σ- Algebra

Definition 2.3. *A σ-algebra $\mathcal{A}$ is a family of subsets of a non-empty set X such that:*

(i) $\emptyset, X \in \mathcal{A}$.

(ii) *If $A \in \mathcal{A}$ then $A^c \in \mathcal{A}$.*

(iii) *If $A_1, A_2, \ldots \in \mathcal{A}$ then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$.*

Example 2.3. *Let X be a non-empty set and let $\mathcal{P}(X)$ be the family of all subsets of X. Then, the collection $\mathcal{P}(X)$ is a $\sigma - algebra$ on X.*

The intersection of σ-algebras containing a collection of sets $\mathcal{A}$ is called the **σ-algebra generated by $\mathcal{A}$**.

2.5 Axiom of Choice

The Axiom of Choice is stated as follows [7]:
Suppose that $\mathcal{H} = \{H_\alpha : \alpha \in I\}$ is a non-empty collection, indexed by some set I, of non-empty disjoint subsets of X. Then there exists a set $E \subset X$ which contains precisely one element from each of the sets H_α.

2.6 Element of General Topology

2.6.1 Topological spaces

A topology τ on a non-empty set X is a collection of subsets of X such that:

(i) $\emptyset, X \in \tau$.

(ii) If $O_i \in \tau$ for $i = 1, 2, ..., n$ then $\bigcap_{i=1}^{n} O_i \in \tau$.

(iii) If $O_i \in \tau$ for $i \in I$ then $\bigcup_{i \in I} O_i \in \tau$.

Thus a collection of subsets of a set X is a topology in X if it includes the empty set and X, and if finite intersections and arbitrary unions of sets in the collection are also in the collection.

The couple (X, τ) is called a topological space.

Example 2.4. *Let X be a non empty set, $\tau = \{\emptyset, X\}$ is the minimal topology that can be defined on X. If τ is the collection of all subsets of Xthen τ is called the discrete topology on X. It is the largest topology which can be defined on X.*

The basic concepts of topology are the notions of *open* and *closed* sets.

Definition 2.4. *Let (X, τ) be a topological space on X. A subset A of X is said to be **open** in (X, τ) if $A \in \tau$ and A is said to be **closed** in (X, τ) if $X \setminus A \in \tau$.*

Definition 2.5. *Let (X, τ) be a topological space and let $A \subseteq X$. The **interior**of A denoted by Int(A) is the union of all open sets contained in A and the **closure** of A denoted by Cl(A) is the intersection of all closed sets containing A.The boundary of A, denoted ∂A is the set $\partial A =$ Cl$(A) \setminus$ Int(A).*

Let (X, τ) be a topological space and let A, B be subsets of X [3]:

(i) If O is an open set in (X, τ) and $O \subset A$, then $O \subset \text{Int}(A)$.

(ii) If D is a closed set in (X, τ) and $A \subset D$ then $\text{Cl}(A) \subset D$.

(iii) If $A \subset B$ then $\text{Int}(A) \subset \text{Int}(B)$.

(iv) If $A \subset B$ then $\text{Cl}(A) \subset \text{Cl}(B)$.

(v) A is open if and only if $\text{Int}(A) = A$.

(vi) A is closed if and only if $\text{Cl}(A) = A$.

(vii) $\text{Int}(X \setminus A) = X \setminus \text{Cl}(A)$.

(viii) $\text{Cl}(X \setminus A) = X \setminus \text{Int}(A)$.

2.6.2 Topology of the real line

The set of real numbers plays an important role in Real Analysis since many concepts in topology are abstractions of properties of $\mathbb{R}$ [6]. The interval $(a, b) = \{x : a < x < b\}$ is called **open** and the interval $[a, b] = \{x : a \leq x \leq b\}$ is called **closed** but the intervals $[a, b) = \{a \leq x < b\}$ and $(a, b] = \{x : a < x <\}$ are neither open nor closed. The collection of all open intervals defines a topology τ_E on $\mathbb{R}$ as follows.

Definition 2.6. *A set $G \subset \mathbb{R}$ is **open** in $(\mathbb{R}, \tau_E)$ if for every $x \in G$ there exist a $\delta > 0$ such that $(x-\delta, x+\delta) \subset G$ and G is said to be **closed** if $\mathbb{R} \setminus G$ is open in $(\mathbb{R}, \tau_E)$.*

The topological space $(\mathbb{R}, \tau_E)$ is called the real line.

Definition 2.7. *Let $(\mathbb{R}, \tau_E)$ be the real line and let A be a subset of $\mathbb{R}$. The set A is said to be **dense** in $\mathbb{R}$ if $Cl(A) = \mathbb{R}$ and A is said to be **nowhere dense** in $(\mathbb{R}, \tau_E)$ if $Int(Cl(A)) = \emptyset$.*

Example 2.5. *The set of rational numbers $\mathbb{Q}$ is dense in $\mathbb{R}$ because $Cl(\mathbb{Q}) = \mathbb{R}$ while the set $\mathbb{N}$ is nowhere dense since $Int(Cl(\mathbb{N})) = \emptyset$.*

A set A is said to be **meager** or **first category** set in $(\mathbb{R}, \tau_E)$ if it is a countable union of nowhere dense sets. If A is not a meager in $(\mathbb{R}, \tau_E)$ then it is of **second category**. A set A is called **comeager** or **residual** in $\mathbb{R}$ if A^c is meager in $\mathbb{R}$ i.e. $A \subseteq \mathbb{R}$ is comeager in $\mathbb{R}$ if and only if it contains a countable intersection of dense open sets [2].

Example 2.6. *The set $\mathbb{Q}$ of rational numbers is of first category in $\mathbb{R}$ and the set of irrational numbers is of second category in $\mathbb{R}$.*

Proposition 2.1. *Let $(\mathbb{R}, \tau_E)$ be the real line. Then :*

(a) *Any subset of a nowhere dense set is nowhere dense.*

(b) *The union of finitely many nowhere dense sets is nowhere dense.*

(c) *The closure of a nowhere dense set is nowhere dense.*

(d) *Every finite subset of $\mathbb{R}$ is a nowhere dense set.*

(e) *If A is an open subset of $\mathbb{R}$, then $Cl(A) \setminus A$ is of first category.*

2.7 σ- ideal of sets

Let X be a non-empty set and let $\mathcal{P}(X)$ be the family of all subsets of X

Definition 2.8. **(a)** *A family $\mathcal{I} \subset \mathcal{P}(X)$ of sets is called an **ideal of sets** of X if it satisfies the following conditions:*

(i) *If $A \in \mathcal{I}$ and $B \in \mathcal{I}$ then $A \cup B \in \mathcal{I}$.*

(ii) *If $A \in \mathcal{I}$ and $B \subseteq A$,then $B \in \mathcal{I}$.*

(b) *An ideal $\mathcal{I}$ which is closed under countable unions is called a $\sigma-ideal$ of sets on X i.e if $A_1, A_2, \cdots \in \mathcal{I}$ then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$. Note that the collection of all nowhere dense sets is an ideal of sets $\mathbb{R}$ and the collection of all meager sets is a σ-ideal of sets on $\mathbb{R}$.*

Theorem 2.1 ([2])**.** ***(Baire Category Theorem)****: The complement of any set of first category on the real line is dense in $\mathbb{R}$.*

Proof. Let $A = \bigcup A_n$ be a representation of A as a countable union of nowhere dense sets. For any interval I, let I_1 be a closed subinterval of $I \setminus A_1$. Let I_2 be a closed subinterval of $I_1 \setminus A_2$, and so on. Then $\cap I_n$ is a non-empty subset of $I \setminus A$, hence A^c is dense. To specify all the choices in advance, it suffices to arrange the (denumerable) class of closed intervals whith rational endpoints into a sequence, take $I_0 = I$, and for $n > 0$ take I_n to be the first term of the sequence that is contained in $I_{n-1} \setminus A_n$. □

A Baire space is a topological space in which the union of every countable collection of closed sets with empty interior has empty interior.

2.8 Standard Cantor Set

The Cantor set is a set of points lying on a single line segment that has a number of remarkable properties. The Cantor set is an example of subset of $\mathbb{R}$ which does not contains an interval [9].

Let $E_0 = [0, 1]$, by removing the segment $(\frac{1}{3}, \frac{2}{3})$ then $E_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. Remove the middle thirds of these intervals of E_1 and we get $E_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{3}{9}] \cup [\frac{6}{9}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$. Continuing in this way, we obtain a sequence of compact sets E_n such that $E_1 \supset E_2 \supset E_3, \ldots$ and E_n is the union of 2^n intervals, each of length $3^{\frac{1}{n}}$. The set $C = \bigcap_{n=1}^{\infty} E_n$ is called the Cantor set .

2.9 Borel sets on the real line

A Borel set is any subset of $\mathbb{R}$ that can be formed from open sets (or from closed sets) through the operations of countable union, countable intersection and relative complement[3, 7]. A Borel $\sigma - algebra$ is the smallest σ-algebra which includes all open sets.

Definition 2.9. *A set is called a G_δ if it is the intersection of a countable collection of open sets and a set is called an F_σ if it is the union of a countable collection of closed sets.*

It is clear that G_δ and F_σ sets are Borel sets.

Chapter 3

Lebesgue measurable sets on the real line

Definition 3.1. *Let X be a set and $\mathcal{A}$ be a σ-algebra consisting of subsets of X. A* ***measure*** *on $(X, \mathcal{A})$ is a function $\mu : \mathcal{A} \longrightarrow [0, \infty]$ such that:*

(i) $\mu(\emptyset) = 0$*;*

(ii) *If $\{A_i \in \mathcal{A} : i \in \mathbb{N}\}$ is a collection of disjoint sets in $\mathcal{A}$ then $\mu\left(\bigcup_{i=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu\left(A_n\right)$.*

The triple $(X, \mathcal{A}, \mu)$ is called a measure space.

The simplest example of a measure is the counting measure that assigns to each finite set the cardinality of that set and to infinite sets the value of $+\infty$.
Some properties of measure are:

(i) If $A, B \in \mathcal{A}$ and $B \subset A$, then $\mu(B) \leqslant \mu(A)$.

(ii) If $A, B \in \mathcal{A}$ and $B \subset A$ and $\mu(B) < \infty$ then $\mu(A \setminus B) = \mu(A) - \mu(B)$.

(iii) If $A, B \in \mathcal{A}$, $\mu(A \cap B) < \infty$ then $\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$.

(iv) Sub additivity of μ: if $(A_i)_{i \geqslant 1} \subset \mathcal{A}$, such that $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$, then $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu(A_i)$.

Definition 3.2. *A measure μ is a finite measure if $\mu(X) < \infty$ and a measure is σ-finite if there exist sets $E_{i \in I} \in \mathcal{A}$ such that $\mu(E_i) < \infty$ for each i and $X = \bigcup_{i=1}^{\infty} E_i$. If μ is a finite measure, then $(X, \mathcal{A}, \mu)$ is a finite measure space, and similarly, if μ is a σ-finite measure, then $(x, \mathcal{A}, \mu)$ is a σ-finite measure space.*

3.1 Null sets

A null set $N \subset \mathbb{R}$ is a set that can be covered by a countable union of intervals of arbitrarily small total length i.e. the way of defining the length of set is to start with intervals nonetheless. Suppose that I is bounded (interval of any kind; I= $[a, b]$, I=(a,b), I=[a,b) or I=(a,b]). We simply define the length of I as $l(I) = b - a$ in each case and as a particular case we have $l(\{a\})$= $l([a, a]) = 0$, which is normal to say that a one-element set is **null set** in the sense that its length is zero. A finite set is not interval but since a single point has length 0, adding finitely many such lengths together should still get 0 [1, 7].

The concept is that if we decompose a set into a finite number of disjoint intervals, we compute the length of this set by adding the length of the pieces. In general it may not be always possible to decompose set into intervals; therefore we consider systems of intervals that cover a given set.

Theorem 3.1 ([7]). *If $(N_n)_{n\geq 1}$is a sequence of null sets on $\mathbb{R}$, then their union $N = \bigcup_{n=1}^{\infty} N_n$ is also a null set. It is clear that if N is a null set and $M \subseteq N$ then M is also a null set. Furthermore, if $N_1, N_2, ...$ are null sets then $\bigcup_{n=1}^{\infty} N_i$ is also a null set and hence the collection of null sets forms a σ-ideal of sets.*

3.2 Lebesgue outer measure

The simple concept of null sets provides the key to our idea of length since it tells us what we can ignore. A quite general notation of length is now provided by:

Definition 3.3. *The Lebesgue outer measure of any set $A \subseteq \mathbb{R}$ is given by :*

$$\mu^*(A) = \inf \left\{ \sum_{n=1}^{\infty} l(I_n) : I_n \text{ are intervals } \text{ and } A \subseteq \bigcup_{n=1}^{\infty} I_n \right\}.$$

It follows that the Lebesgue outer measure is the infimum of lengths of all possible covers of A.

The concept of null set is consistent with that Lebesgue outer measure.

Theorem 3.2 ([7]). *$A \subseteq \mathbb{R}$ is a null set if and only if $\mu^*(A) = 0$.*

Theorem 3.3. *The outer measure is countably subadditive i.e. for any sequence of sets $(E_i)_{i\geq 1}$, $\mu^*(\bigcup_{i=1}^{\infty} E_i) \leqslant \sum_{i=1}^{\infty} \mu^*(E_i)$.*

Proof. If $\sum_i \mu^*(E_i) = \infty$, then it is satisfied.

Suppose that the sum is finite. Let $\epsilon > 0$ be given . For each i, there is a sequence (I_{ik}) of open intervals such that $E_i \subseteq \bigcup_k I_{ik}$ and $\sum_k l(I_{ik}) < \mu^*(E_i) + \frac{\epsilon}{2^i}$. Clearly, the doubly-indexed sequence (I_{ik}) is a cover for $\bigcup_i E_i$. That is $\bigcup_i E_i \subseteq \bigcup_i \bigcup_k I_{ik}$. We have $\mu^*(\bigcup_i E_i) \leqslant \sum_{i,k} l(I_{ik}) = \sum_i \sum_k l(I_{ik}) \leqslant \sum_i \left(\mu^*(E_i) + \frac{\epsilon}{2^i}\right) = \sum_i \mu^*(E_i) + \epsilon$. Since $\epsilon > 0$ is arbitrary, the above equality gives $\mu^*(\bigcup_i E_i) \leqslant \sum_i \mu^*(E_i)$ [2]. $\square$

There are some basic properties of Lebesgue outer measure [4];

(i) $\mu^*(A) = 0$ if A is at most countable.

(ii) μ^* is monotonic i.e $\mu^*(A) \leqslant \mu^*(B)$ whenever $A \subseteq B$.

(iii) $\mu^*(A) = \sup\{\mu^*(F) : A \supset F, F \text{ compact }\}$.

(iv) $\mu^*(A + x) = \mu^*(A)$ where $A + x = \{a + x : a \in A\}$ for all $x \in \mathbb{R}$ i.e. μ^* is translation invariant.

(v) If $\mu^*(A) = 0$, then $\mu^*(A \cup B) = \mu^*(B)$ and $\mu^*(B \setminus A) = \mu^*(B)$ for all $B \subseteq \mathbb{R}$.

(vi) If $\mu^*(A \triangle B) = 0$ then $\mu^*(A) = \mu^*(B)$.

Theorem 3.4. *For any interval $I \subseteq \mathbb{R}$, $\mu^*(I) = l(I)$.*

Proof. Suppose that I is bounded and its end points are a, b. For any given $\epsilon > 0$, we have $I \subseteq (a, b) \cup (a - \epsilon, a + \epsilon) \cup (b - \epsilon, b + \epsilon)$. Thus, $\mu^*(I) \leqslant l\left((a, b)\right) + l\left((a - \epsilon, a + \epsilon)\right) + l\left((b - \epsilon, b + \epsilon)\right) = b - a + 4\epsilon$. As $\epsilon > 0$ is arbitrary, we conclude that $\mu^*(I) \leqslant b - a = l(I)$. We need to prove that $\mu^*(I) \geqslant l(I)$. Firstly, consider the case when $I = [a, b]$. Let (I_k) be any sequence of open intervals that covers I. Since I is compact, by the Heine-Borel Theorem, there is a finite sub collection $J_i : 1 \leqslant i \leqslant n$ of (I_k) still covers I. By reordering and deleting if necessary, we can assume that $a \in J_1 = (a_1, b_1), b_1 \in J_2 = (a_2, b_2), b_2 \in J_3 = (a_3, b_3), \ldots, b_{n-1} \in J_n = (a_n, b_n)$ and $b \in J_n$. We then have $b - a < b_n - a_1 = \sum_{i=2}^{n}(b_i - b_{i-1}) + (b_1 - a_1 < \sum_{i=1}^{n} l(J_i) \leq \sum_i l(I_i)$. hence, $l(I) \leq \mu^*(I)$. This proves the result when $I = [a, b]$.

If $I = (a, b)$ then, as we have shown above, $l([a, b]) = \mu^*([a, b]) \leq \mu^*\left((a, b)\right) + \mu^*(a) + \mu^*(b) = \mu^*\left((a, b)\right)$. Hence $l(I) = l([a, b]) \leq \mu\left((a, b)\right)$ as we desired [3, 2]. □

3.3 Lebesgue measure

We wish however, to ensure that if sets (E_m) are pairwise disjoint then the inequality in the Theorem 3.3 becomes the equality. If the pairwise disjoint sets (E_n) have union E , then the length of the sets $B_n = E \setminus \left(\bigcup_{k=1}^{\infty} E_k\right)$ may be expected to decrease to 0 as $n \to \infty$. Combining this with finite additivity leads quite naturally to the demand that length should be countably additive i.e. that $\mu^*(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \mu^*(E_n)$ when $E_i \cap E_j = \emptyset$ for $i \neq j$. In order to define the good sets which have this property, it also seems plausible that such a set should apportion the outer measure of every set in $\mathbb{R}$. This remark leads to the following definition:

Definition 3.4. *A set $E \subset \mathbb{R}$ is called **(Lebesgue) measurable** if for every subset A of $\mathbb{R}$; $\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$. If E is a Lebesgue measurable set, then the Lebesgue measure of E is defined to be its outer measure $\mu^*(E)$ and is written as $\mu(E)$.*

Example 3.1. **(1)** *If $A = \{a\}$ then $\mu(A) = 0$.*

(2) *Let $G = \bigcup_{i=1}^{\infty} G_i$ where G_i is a removed open interval in $[0,1]$. The Cantor set $C = [0,1] \setminus G$. $\mu(G) = \frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \ldots = \frac{1}{3}\left(1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \ldots\right) = 1$. The Lebesgue measure of C is $\mu(C) = 0$.*

Theorem 3.5 ([7])**.**

(i) Any null set is Lebesgue measurable.

(ii) Any interval is Lebesgue measurable.

3.4 Basic properties of Lebesgue measure

The family of all Lebesgue measurable subsets of $\mathbb{R}$ will be denoted by $\mathcal{L}(\mathbb{R})$ in our future considerations. The basic properties of Lebesgue measure are the following [1];

(a) Both $\emptyset, \mathbb{R} \in \mathcal{L}(\mathbb{R})$, $\mu(\emptyset) = 0$ and $\mu(\mathbb{R}) = \infty$.

(b) If $E \in (\mathbb{R})$ then so is $E^c \in \mathcal{L}(\mathbb{R})$.

(c) If $\mu^*(E) = 0$, then $E \in \mathcal{L}(\mathbb{R})$.

(d) If $E_1, E_2 \in \mathcal{L}(\mathbb{R})$ then $E_1 \cup E_2 \in \mathcal{L}(\mathbb{R})$ and $E_1 \cap E_2 \in \mathcal{L}(\mathbb{R})$.

(e) If $E \in \mathcal{L}(\mathbb{R})$, then $E + x_0 \in \mathcal{L}(\mathbb{R})$.

(f) Every interval is measurable and $\mu(I) = \mu^*(I) = l(I)$.

(g) If $(E_i)_{1 \leqslant i \leqslant n} \subset \mathcal{L}(\mathbb{R})$, then $\forall A \subset \mathbb{R}$, $\mu^*(\bigcup_{i=1}^{n} A \bigcap E_i) = \mu^*(A \bigcap (\bigcup_{i=1}^{n} E_i)) = \sum_{i=1}^{n} \mu^*(A \cap E_i)$. In particular, if $A = \mathbb{R}$ then $\mu(\bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} \mu(E_i)$.

(h) If $(E_i)_{i\geqslant 1} \in \mathcal{L}(\mathbb{R})$ then $\bigcup_{i=1}^{\infty} E_i \in \mathcal{L}(\mathbb{R})$ and $\bigcap_{i=1}^{\infty} E_i \in \mathcal{L}(\mathbb{R})$.

(i) Every open set and every closed set are measurable.

Chapter 4

Sets with Baire property on the real line

Let $(\mathbb{R}, \tau_E)$ be the real line. A subset of $\mathbb{R}$ is said to have the **Baire property** in $(\mathbb{R}, \tau_E)$ if it can be represented in the form $A = G\triangle P$ where G is open and P of first category [11].

Theorem 4.1. *A set has the Baire property on $\mathbb{R}$ if and only if it can be represented in the form $A = F\triangle Q$, where F is closed and Q is of first category.*

Proof. If $A = G\triangle P$, G open and $P$of first category, then $N = \mathrm{Cl}(G) \setminus G$ is a nowhere dense closed set, and $Q = N\triangle P$ is of first category. Let $F = \mathrm{Cl}(G)$. Then A=$G\triangle P$= $(\mathrm{Cl}(G)\triangle N)\triangle P$ =$\mathrm{Cl}(G)\triangle(N\triangle P)$. Conversely, if $A = F\triangle Q$, where F is closed and Q is of first category. Let G be the interior of F. Then N=$F \setminus G$ is nowhere dense P= $N\triangle Q$ is of first category and A=$F\triangle Q$= $(G\triangle N)\triangle Q$= $G\triangle(N\triangle Q)$=$G\triangle P$ [2]. □

Theorem 4.2. *If A has the Baire property, then so does its complement.*

Proof. For any two sets A and B we have $(A\triangle B)^c = A^c\triangle B$. Hence if $A = G\triangle P$ then $A^c = G^c\triangle P$. □

Theorem 4.3. *The class of sets having the Baire property is a $\sigma - algebra$. It is the $\sigma - algebra$ generated by the open sets together with the sets of first category.*

Proof. Let $A_i = G_i\triangle P_i$ (i=1,2,...) be any sequence of sets having the Baire property. Put $G = \cup G_i$, $P = \cup P_i$ and $A = \cup A_i$. Then G is open, P is of first category and $G \setminus P \subset A \subset G \cup P$. Hence $G\triangle A \subset P$ is of first category, and $A = G\triangle(G\triangle A)$ has the Baire property. The result, together with Theorem 4.2 shows that the class in question is a $\sigma - algebra$. □

Obviously meager sets and open sets have the Baire property. The following proposition shows that meager sets and sets with the Baire property are invariant under translations of $\mathbb{R}$.

Proposition 4.1 ([10]). *Let $x \in \mathbb{R}$. If $M \subseteq \mathbb{R}$ is a meager set then $x+M = \{x+m : m \in M\}$ is a meager set and if $B \subseteq \mathbb{R}$ has the Baire property then $x + B = \{x + b : b \in B\}$ has the Baire property too .*

Theorem 4.4. *A set has the Baire property if and only if it can be represented as the disjoint union of a G_δ set and a set of first category (or as an F_σ set minus a set of first category).*

Proof. Since the closure of any nowhere dense set is a nowhere dense, any set of first category is contained in an F_σ set of first category. If G is an open and P of first category, let Q be an F_σ set of first category that contains P. Then the set $E = G \setminus Q$ is a G_δ, and we have $G \triangle P = [(G \setminus Q) \triangle (G \cap Q)] \triangle (P \cap Q) = E \triangle [(G \triangle P) \cap Q]$. The set $(G \triangle P) \cap Q$ is of first category and disjoint to E. Hence any set having the Baire property can be represented as the disjoint union of a G_δ set and a set of first category.
Conversely, any set that can be so represented belongs to the σ-algebra generated by open sets and the sets of first category; it therefore has the Baire property. The parenthetical statement follows by complementation, with the aid of Theorem 4.2 [2]. □

Theorem 4.5 ([2]). *For any set $A \subset \mathbb{R}$ of second category having the property of Baire and for any measurable set A with $\mu(A) > 0$, there exists a positive number δ such that $(x + A) \cap A \neq \emptyset$ whenever $|x| < \delta$.*

The family of subsets of $\mathbb{R}$ with the Baire property will be denoted by $\mathcal{B}_p(\mathbb{R})$ in our future considerations.

4.1 The common properties of Lebesgue measurable sets and sets with the Baire property

Let us recall some facts about the collections $\mathcal{L}(\mathbb{R})$ and $\mathcal{B}_p(\mathbb{R})$.

Both families

(1) are σ- algebras of sets on $\mathbb{R}$.

(2) contain the family of Borel sets.

(3) are invariant under translations of $\mathbb{R}$ i.e. if $E \in \mathcal{L}(\mathbb{R})$ (resp. $E \in \mathcal{B}_p(\mathbb{R})$) then $E + x_0 \in \mathcal{L}(\mathbb{R})$ (resp. $E + x_0 \in \mathcal{B}_p(\mathbb{R})$) for all $x_0 \in \mathbb{R}$.

If $E \subseteq \mathbb{R}^2$, let $E_x = \{y : (x, y) \in E\}$. We have the following propositions:

Proposition 4.2 ([8]). **(1)** *If E is measurable in $\mathbb{R}^2$ then E_x is measurable for all x except a set of measure zero.*

(2) *If E is a null set, then E_x is a null set for all x except a set of measure Zero.*

(3) *If E is measurable and it has positive measure, then E_x has positive measure for positively many x.*

The category analogue to the above proposition is the following ;

Proposition 4.3 ([8]). **(1)** *If E has Baire property in $\mathbb{R}^2$ then E_x has the Baire property for all x except a set of first category.*

(2) *If E is of first category then E_x is of first category for all x except a set of first category.*

Even the complements $\mathcal{L}(\mathbb{R})^c$ and $\mathcal{B}_p(\mathbb{R})^c$ of the families $\mathcal{L}(\mathbb{R})$ and $\mathcal{B}_p(\mathbb{R})$ in the power set $\mathcal{P}(\mathbb{R})$ of all subsets of $\mathbb{R}$ have common properties as we will see in the following part.

Chapter 5

Examples of sets which are non-measurable and without the Baire property

With the aid of the Axiom of Choice, it is possible to show the existence of subsets of $\mathbb{R}$ which are not Lebesgue measurable, and the existence of sets without the Baire property. We will consider two such constructions due to **Vitali** and **Bernstein**.

5.1 Construction of Vitali sets of the real line

Let $(\mathbb{R}, \tau_E)$ be the real line and let $\mathbb{Q}$ be the set of rational numbers. Define an equivalence relation $\mathcal{R}$ on $\mathbb{R}$ as follows : if $x, y \in \mathbb{R}$ then $x\mathcal{R}y$ if and only if $x - y \in \mathbb{Q}$. Since $\mathcal{R}$ is an equivalence relation, the set $\mathbb{R}$ will be partitioned into disjoint equivalence classes $\{E_\alpha : \alpha \in I\}$ of sets. So $\mathbb{R} = \bigcup_\alpha E_\alpha$. By using the Axiom of Choice, we pick a single element $x_\alpha \in E_\alpha$ in each class E_α. Then we put them together in order to create a new set $V = \{x_\alpha : \alpha \in I\}$ such that $\mid V \cap E_\alpha \mid = 1$. This new set V is called a **Vitali set** .

Proposition 5.1. *Let V be a Vitali set. Then we have:*

(a) *If $r_1, r_2 \in \mathbb{Q}$ and $r_1 \neq r_2$ then $(V + r_1) \cap (V + r_2) = \emptyset$.*

(b) $\mathbb{R} = \bigcup_{r \in \mathbb{Q}} (V + r)$.

(c) *The set V is not a meager.*

Proof. **(a)** Let $x \in (V+r_1)\cap(V+r_2)$ with $r_1, r_2 \in \mathbb{Q}$ and $r_1 \neq r_2$. Then x can be represented in two ways: $x = x_\alpha + r_1 = x_\beta + r_2$ for some $x_\alpha, x_\beta \in V$. But $x_\alpha - x_\beta = r_2 - r_1 \in \mathbb{Q}$ implies that x_α and x_β are in the same equivalence class. Since $|V \cap E_\alpha| = 1$ for all $\alpha \in I$, then $x_\alpha = x_\beta$. This shows that $r_1 = r_2$. Which is a contradiction.

(b) If $x \in \mathbb{R}$, then x belongs to a unique equivalence classE_α. Let u_α be the representative of E_α in V, i.e. $V \cap E_\alpha = \{u_\alpha\}$. So, $x - u_\alpha = r$ for some $r \in \mathbb{Q}$. It follows that $x = u_\alpha + r \in (V + r)$.

(c) If V is a meager in $\mathbb{R}$, then each $V + r, r \in \mathbb{Q}$, is a meager subset of $\mathbb{R}$ and using proposition (b), then $\mathbb{R}$ is a meager. which is is a contradiction with the Baire Category Theorem.

□

Theorem 5.1. *Any Vitali set is non- Lebesgue measurable and it does not possess the Baire property.*

Proof. Assume that V is Lebesgue measurable. Then $V+r$ is also Lebesgue measurable and $\mu(V) = \mu(V+r)$. It follows from Proposition 5.1 (b) that Vitali set V can not be a null set so, $\mu(V) > 0$. Using the Theorem 4.5; if V is measurable, there exists a number $\delta > 0$ such that $(V + x) \cap V \neq \emptyset$ whenever $|x| < \delta$. But if $x = r \in \mathbb{Q}$ and $r \neq 0$, then $(V + r) \cap V = \emptyset$ which contradicts Theorem 4.5. Hence V can not be measurable in the Lebesgue sense. □

Assume that V has the property of Baire. Then $V + r$ also has the Baire property. It follows from Proposition 5.1,(c) that V can not be meager. Using the Theorem 4.5; if V possess the Baire property, there exists a number $\delta > 0$ such that $(V + x) \cap V \neq \emptyset$ whenever $|x| < \delta$. But if $x = r \in \mathbb{Q}$ and $r \neq 0$, then $(V + r) \cap V = \emptyset$ which contradicts Theorem 4.5. Therefore Vitali set V is not Lebesgue measurable and does not possess the Baire property on the real line.

5.2 Bernstein sets

Definition 5.1. *A Bernstein set B is a subset of the real line such that both B and B^c meet every uncountable closed subset P of $\mathbb{R}$ i.e. $(P \cap B) \neq \emptyset$ and $P \cap (\mathbb{R} \setminus B) \neq \emptyset$.*

The Bernstein set can be constructed as follows: let c denote the cardinality of the real line $\mathbb{R}$; we can well order all uncountable closed subsets of $\mathbb{R}$ in a collection indexed by the ordinals less than c say $\mathcal{F} = \{F_\alpha : \alpha < c\}$. Let x_0 and y_0 be the first two elements of F_0. Let x_1 and y_1 be the two first members of F_1 different from x_0 and y_0. If $1 < \alpha < c$ and x_β and y_β have been defined for all $\beta < \alpha$, let x_α and y_α be the first two elements of $F_\alpha \setminus \bigcup_{\beta<\alpha}\{x_\beta, y_\beta\}$. This set is non-empty for each α, and so x_α and y_α can be defined similarly for all $\alpha < c$. Put $B = \{x_\alpha : \alpha < c\}$. Since $x_\alpha \in B \cap F_\alpha$ and $y_\alpha \in B^c \cap F_\alpha$ for each $\alpha < c$, the set B has the property that both it and its complement meet every uncountable closed set. So B is a **Bernstein set**.

Bernstein sets are classical examples of non Lebesgue measurable sets and sets without the Baire property as well.

Theorem 5.2 ([2])**.** *Any uncountable G_δ subset of $\mathbb{R}$ contains a nowhere dense closed set C of measure zero that can be mapped continuously onto* $[0, 1]$.

Theorem 5.3. *Every measurable subset of either B or B^c is a null set, and any subset of B or B^c that has the Baire property is of first category.*

Proof. Let A be any measurable subset of B. Any closed set F contained in A must be countable (since every uncountable closed set meets B^c), hence $\mu(F) = 0$. Therefore $\mu(A) = 0$, by using the definition of Lebesgue outer measure. Similarly, if A is a subset of B having the Baire property, then $A = E \cup P$, where E is G_δ and P is of first category. The set E must be countable since every uncountable G_δ set contains an uncountable closed set, by Theorem 5.2, and therefore meets B^c. Hence A is of first category. The same reasoning applies to B^c. □

Theorem 5.4. *Any set with positive outer measure has a non-measurable subset. Any set of second category has a subset that lacks the Baire property.*

Proof. If A has positive outer measure and B is a Bernstein set, Theorem 5.3 shows that the subset $A \cap B$ and $A \cap B^c$ can not both be Lebesgue measurable. If A is of second category, the two subsets can not both have the Baire property. □

Corollary 5.5. *Any Bernstein set B is non-Lebesgue measurable and lacks the Baire property.*

Proof. Suppose that B is Lebesgue measurable. It follows from Theorem 5.3 that $\mu(B) = 0$ then $\mu(\mathbb{R}) = 0$. Which is a contradiction. Using the Theorem 5.4, B and B^c are subsets of $\mathbb{R}$ can not both have the Baire property. Which is a contradiction. □

Chapter 6

Examples of sets which are non-measurable but with the Baire property and sets which are measurable but without the Baire property on the real line

By analysing the real line we get the following theorem;

Theorem 6.1 ([2]). *The real line $\mathbb{R}$ can be decomposed into two complementary sets A and B such that A is of first category (meager) and B is of measure zero. That is $\mathbb{R} = A \cup B$ with $A \cap B = \emptyset$, $\mu(B) = 0$ and $A = \cup_{i=1}^{\infty} A_i$ where $Int(Cl(A_i)) = \emptyset$.*

Let $\mathcal{M}(\mathbb{R})$ and $\mathcal{N}(\mathbb{R})$ be the collections of all meager sets and all null sets, respectively, on $\mathbb{R}$.

Corollary 6.2. *Every subset of $\mathbb{R}$ can be written as disjoint union of a meager set and a null set of $\mathbb{R}$.*

Proof. Assume that $X \subseteq \mathbb{R}$. Since $\mathbb{R} = A \cup B$ with $A \cap B = \emptyset$, $A \in \mathcal{M}(\mathbb{R})$, and $B \in \mathcal{N}(\mathbb{R})$, then $X = (X \cap A) \cup (X \cap B)$. Since $\mathcal{M}(\mathbb{R})$ (resp. $\mathcal{N}(\mathbb{R})$) is σ-ideal of sets, it follows that $X \cap A \in \mathcal{M}(\mathbb{R})$ and $X \cap B \in \mathcal{N}(\mathbb{R})$, with $(X \cap A) \cup (X \cap B) = \emptyset$. $\square$

By using Theorem 6.1 and the Corollary 6.2, it is possible to prove the following propositions:

(1) $\mathcal{M}(\mathbb{R}) \nsubseteq \mathcal{N}(\mathbb{R})$ and $\mathcal{N}(\mathbb{R}) \nsubseteq \mathcal{M}(\mathbb{R})$.

Proof. By using Theorem 6.1, $\mathbb{R} = A \cup B$ with $A \cap B = \emptyset$, $A \in \mathcal{M}(\mathbb{R})$, $B \in \mathcal{N}(\mathbb{R})$.

Since the families $\mathcal{M}(\mathbb{R})$ and $\mathcal{N}(\mathbb{R})$ are σ-ideals of sets we can observe the following: If $A \in \mathcal{N}(\mathbb{R})$ then $\mathbb{R} \in \mathcal{N}(\mathbb{R})$. But this is impossible. So $A \notin \mathcal{N}(\mathbb{R})$. The same reasoning is valid for B; if $B \in \mathcal{M}(\mathbb{R})$ then $\mathbb{R} \in \mathcal{M}(\mathbb{R})$. This is also impossible. So $B \notin \mathcal{M}(\mathbb{R})$. □

Briefly, we have that $A \in \mathcal{M}(\mathbb{R})$ but $A \notin \mathcal{N}(\mathbb{R})$ and $B \in \mathcal{N}(\mathbb{R})$ but $B \notin \mathcal{M}(\mathbb{R})$ which shows that the collection of meager sets $\mathcal{M}(\mathbb{R})$ and the collection of null sets $\mathcal{N}(\mathbb{R})$ of $\mathbb{R}$ are very different.

(2) $\mathcal{N}(\mathbb{R}) \nsubseteq \mathcal{B}_p(\mathbb{R})$ and $\mathcal{M}(\mathbb{R}) \nsubseteq \mathcal{L}(\mathbb{R})$, and hence even $\mathcal{L}(\mathbb{R}) \nsubseteq \mathcal{B}_p(\mathbb{R})$ and $\mathcal{B}_p(\mathbb{R}) \nsubseteq \mathcal{L}(\mathbb{R})$.

Proof. Let V be a Vitali set subset of $\mathbb{R}$. It is known that $V \notin \mathcal{L}(\mathbb{R})$ and $V \notin \mathcal{B}_p(\mathbb{R})$. By using the corollary 6.1, V can be decomposed as $V = (A \cap V) \cup (B \cap V)$. Where $A \cap V \in \mathcal{M}(\mathbb{R})$ and $B \cap V \in \mathcal{N}(\mathbb{R})$. Since $\mathcal{L}(\mathbb{R})$ and $\mathcal{B}_p(\mathbb{R})$ are σ-algebra, it follows that $A \cap V \notin \mathcal{L}(\mathbb{R})$ and $B \cap V \notin \mathcal{B}_p(\mathbb{R})$. □

The same reasoning can be even done for Bernstein sets. This was the main aim of this thesis as stated in the introduction.

Conclusion

The main goal of this study was to clarify if the family $\mathcal{L}(\mathbb{R})$ of Lebesgue measurable sets and the family $\mathcal{B}_p(\mathbb{R})$ of sets with the Baire property on the real line $\mathbb{R}$ coincide or not. The problem was motivated by the existence of different common properties of both families. (Let us recall that they are σ-algebras of sets invariant under translations of the real line, they contain the same family $\mathcal{B}(\mathbb{R})$ of all Borel sets of $\mathbb{R}$, and even their complements $\mathcal{L}(\mathbb{R})^c$ and $\mathcal{B}_p(\mathbb{R})^c$ in the power set $\mathcal{P}(\mathbb{R})$ have common elements as Vitali sets and Bernstein sets.) By the use of a well known decomposition of $\mathbb{R}$ into a set A of first category and a set B of measure zero, we observed that any subset of the real line could be decomposed into a set of first category and a set of measure zero. In particular, any Vitali set V (or Bernstein set) could be decomposed into a set $A \cap V$ of first category and a set $V \cap B$ of measure zero. This implied the existence of sets which are non-Lebesgue measurable but with the Baire property, and sets which are Lebesgue measurable but without the Baire property. Therefore the family $\mathcal{L}(\mathbb{R})$ of Lebesgue measurable sets and the family $\mathcal{B}_p(\mathbb{R})$ of sets with the Baire property are completely different.

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