Multifield Inflation With Nonminimal Couplings

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Abstract

Theories of cosmic inflation suggest that during its early stage, the universe experienced a sudden burst of expansion due to one or more scalar fields. These inflationary models offer exact forecasts for multiple observable variables, such as precise arrangements of temperature differences in the cosmic microwave background radiation. Realistic models of high-energy physics often feature multiple scalar fields, which are expected to have nonminimal connections to spacetime curvature. These connections occur as renormalization counterterms when scalar fields are quantized in a curved spacetime.

Within this thesis, I present a recent study that concentrates on multifield inflationary models that include nonminimal couplings within a broad category. One characteristic of these models is their ability to exhibit an impressive attractor behavior. This means that, regardless of the specific couplings and initial conditions, the scalar fields primarily evolve along one particular trajectory for the majority of the inflationary period. This behavior results in reliable predictions for measurable quantities over significant areas of phase space and parameter space, ultimately aligning with the most up-to-date observational data.

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1 The investigation of nonminimal couplings in the early universe

1.1Introduction

The coupling of a scalar field to the Ricci spacetime curvature scalar in the action was a pivotal moment for Brans and Dicke. This breakthrough led to the substitution of G, Newton's constant, with a dynamic gravity strength that is capable of fluctuating in both temporal and spatial dimensions. There have been many theoretical incentives that have emerged since the pioneering work of Brans and Dicke for using these nonminimal connections beyond the initial exploration of Mach's principle. These motivations span from dimensional compactification in higher-dimensional theories to examining effective connections in supergravity and other aspects of theoretical physics.

At present, one of the primary reasons for considering nonminimal couplings stems from their commonplace but crucial role as counterterms when quantizing a scalar field that interacts with itself in curved spacetime. Even if the initial coupling is set to zero, quantum corrections will inevitably result in a nonzero coupling. Additionally, during the renormalization-group flow, the nonminimal coupling generally increases as the energy scale increases, lacking an ultraviolet fixed point. Therefore, it becomes reasonable to investigate models that incorporate significant nonminimal couplings at high energies, particularly around or above the Grand Unified Theory (GUT) scale. Therefore, when considering the early universe, it becomes essential to factor in the potential impact of nonminimal couplings.

1.2Nonminimal Couplings and Inflation

According to the dominant theories of cosmic inflation, the observable universe underwent a phase of swift expansion during its early days, which was fueled by the actions of scalar fields. Throughout history, a diverse array of inflationary models has been created, many of which feature nonminimal couplings. Older models, such as "induced-gravity inflation" and "Extended inflation," utilized a straightforward potential and incorporated a Brans-Dicke field to spur on the rapid expansion.

Following this, alternative models have delved into nonminimal couplings that are more broad in scope. These couplings produce an effective gravitational coupling, G_{eff}, that results from a combination of an unadorned coupling constant and contributions from a scalar field that is tethered to the Ricci curvature scalar. One such example is "Higgs inflation," which has gained recognition in recent times. This model involves a scalar field that tends to settle into a minimum of its potential towards the conclusion of inflation, resulting in a gravitational coupling that is nearly constant for the majority of cosmic history. As a result, these models do not contradict the limitations imposed by Solar System observations on scalar-tensor gravity.

One can express the action of the original Brans-Dicke theory through the following relation:

$$S_{BD} = \int d^4x \sqrt{-\tilde{g}} \left[\Phi \tilde{R} - \frac{\omega}{\Phi} \tilde{g}^{\mu\nu} \partial_\mu \Phi \, \partial_\nu \Phi \right], \tag{1.1}$$

The constant ω, which lacks physical units, is present in the equation where $\tilde{g}_{\mu\nu}(x)$ is the metric of spacetime, where the indices μ and ν range from 0 to 3. In a four-dimensional spacetime, the Brans-Dicke field Φ has dimensions of (mass)^2. As scalar fields with mass dimensions in a four-dimensional spacetime are commonly used by high-energy theorists, it is necessary to rescale the Brans-Dicke field to achieve a suitable scaling. By substituting Φ with ϕ^2/(8ω), we can represent the action of Eq. (1.1) using the rescaled field ϕ as follows:

$$S_{BD} = \int \ d^4x\sqrt{-\tilde{g}}\left[f_{BD}(\phi)\tilde{R} - \frac{1}{2}\tilde{g}^{\mu\nu}\,\partial_\mu\phi\,\partial_\nu\phi\right], \qquad (1.2)$$

The nonminimal coupling function can be expressed in the following manner:

$$f_{BD}(\phi) = \frac{1}{2}\xi\phi^2, \qquad (1.3)$$

The coupling constant ξ, which lacks any dimension, has a direct correlation to the initial Brans-Dicke parameter, with $\xi = 1/(8\omega)$. The quadratic component of this equation precisely mirrors the emergence of quantum corrections in relation to scalar fields in the curvature of spacetime. As a result, it is an accurate representation of the proper format for counterterms that consider these corrections.

The initial proposition by Brans and Dicke states that the potency of gravity in a given area, $G_{\text{eff}}(x)$, is dependent on the field ϕ(x) and can be represented as $\phi(x)$: $G_{\text{eff}}\,(x) = 1/(8\pi\xi\phi^2)$. However, it is possible to broaden this association by integrating a constant mass, M_0, into the function f(ϕ).

$$f(\phi) = \frac{1}{2}[M_0^2 + \xi\phi^2], \qquad (1.4)$$

with $(16\pi G_{\text{eff}})^{-1} = f(\phi)$.

Moreover, it is possible to expand this structure to include models containing N scalar fields:

$$f(\phi^I) = \frac{1}{2}[M_0^2 + \sum_{I=1}^{N} \xi_I (\phi^I)^2] \tag{1.5}$$

Our exploration delves into various models that allow for the expression of the action in a particular manner, such as:

$$S = \int d^4x\sqrt{-\tilde{g}}\left[f(\phi^I)\tilde{R} - \frac{1}{2}\delta_{IJ}\tilde{g}^{\mu\nu}\,\partial_\mu\phi^I\,\partial_\nu\phi^J - \tilde{V}(\phi^I)\right] \tag{1.6}$$

In this context, capital letters of the Latin alphabet (I, J = 1, 2, ..., N) are employed as indices for the field-space. The tilde symbolizes quantities in the Jordan frame, where non-minimal couplings, $f(\phi^I)\tilde{R}$, are kept explicit in the action.

To compare the predictions of this set of models with current astrophysical observations - particularly the exact measurements of cosmic microwave background radiation (CMB) - it is beneficial to work in the Einstein frame. Physicists have created a sturdy gauge-invariant methodology for managing gravitational disturbances in this frame.

In order to transform the gravitational aspect of the equation (1.6) into the well-known Einstein-Hilbert form, a conformal transformation is utilized. This method is reminiscent of Dicke's earlier research on Brans-Dicke gravitation. The transformation consists of rescaling the spacetime metric tensor $\tilde{g}_{\mu\nu}(x) \to g_{\mu\nu}(x) = \Omega^2(x)\tilde{g}_{\mu\nu}(x)$. The conformal factor $\Omega^2(x)$ is positive and directly related to the nonminimal coupling function that appears in equation (1.6).

$$\Omega^2(x) = \frac{2}{M_{\rm pl}^2} f(\phi^I(x)), \tag{1.7}$$

The reduced Planck mass, denoted as $M_{\rm pl} \equiv 1/\sqrt{8\pi G} = 2.43 \times 10^{18}\text{GeV}$, is related to Newton's gravitational constant, G. Following the implementation of this conformal transformation, the action detailed in Equation (1.6) undergoes the subsequent transformation:

$$S = \int d^4x\sqrt{-g}\left[\frac{M_{\rm pl}^2}{2}R - \frac{1}{2}\mathcal{G}_{IJ}(\phi^K)g^{\mu\nu}\,\partial_\mu\phi^I\,\partial_\nu\phi^J - V(\phi^I)\right], \tag{1.8}$$

The transformation that conforms to a given set of rules results in the creation of a manifold for field-space, which contains a metric in the Einstein frame that is presented below:

$$\mathcal{G}_{IJ}(\phi^K) = \frac{M_{\rm pl}^2}{2f(\phi^K)}\left[\delta_{IJ} + \frac{3}{f(\phi^K)} f_{,I} f_{,J}\right], \qquad (1.9)$$

where $f_{,I} = \partial f / \partial \phi^I$.

When applying the conformal transformation to models that contain multiple scalar fields, an intriguing aspect arises. In contrast to the extensively examined situation of a single-field model, it is, in general, not feasible to discover a rescaling of the scalar fields ϕ^I that simultaneously transforms the gravitational component of the action into the Einstein-Hilbert form and results in canonical kinetic terms for the scalar fields. For $M_0 \neq 0$ and $N \geq 2$ scalar fields, the conformal transformation generates a field-space manifold featuring a metric, $\mathcal{G}_{IJ}(\phi^K)$, that is not conformal to flat. Therefore, these models, after undergoing the conformal transformation, resemble nonlinear sigma models.

During the process of transformation to the Einstein frame, the conformal factor results in a stretching of the potential. The following observations were made:

$$V(\phi^I) = \frac{M_{\rm pl}^4}{[2f(\phi^I)]^2}\tilde{V}(\phi^I), \qquad (1.10)$$

The discovery made by Dicke regarding the correlation between particle masses and the Brans-Dicke field is further expanded upon in this study. Through the process of conformal transformation, the potential of simple inflationary models is stretched, leading to substantial modifications in the dynamics of inflation. This sets it apart from models that contain minimally coupled fields. Of particular interest is the emergence of a stable single-field attractor behavior, which is discussed in greater detail in Section 1.4.

The study of models that involve multiple scalar fields with nonminimal couplings has been expanded upon through the pioneering work on multifield inflation. This formalism is covariant with respect to both ordinary gauge transformations ($x^\mu \rightarrow x'^\mu$) and reparameterizations of the field-space coordinates ($\phi^I \rightarrow \phi'^I$), enabling us to examine their dynamics. To analyze these models, we consider perturbations around a Friedmann-Lemaître-Robertson-Walker spacetime metric, with the spatial curvature assumed to be flat for convenience. This choice is advantageous as the radius of curvature stretches exponentially fast during the initial stages of inflation, making a spatially flat background an appropriate approximation for subsequent dynamics. Thus, our methodology is as follows:

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu \\ &= -(1+2A)dt^2 + 2a(\partial_i B)dx^i dt + a^2\left[(1-2\psi)\delta_{ij} + 2\,\partial_i\,\partial_j E\right]dx^i dx^j, \end{aligned}$$
(1.11)

The scale factor is represented by a(t) in this context, whereas the scalar degrees of freedom of the metric perturbations are described by $A(x^\mu), B(x^\mu), \psi(x^\mu)$, and $E(x^\mu)$. With the spacetime symmetries in mind, it is important to note that at the background level, the fields may only rely on time.

$$\phi^I(x^\mu) = \varphi^I(t) + \delta\phi^I(x^\mu) \tag{1.12}$$

The expression of the background fields' velocity vector's magnitude is as follows:

$$|\dot{\varphi}^I| \equiv \dot{\sigma} = \sqrt{\mathcal{G}_{IJ}\dot{\varphi}^I\dot{\varphi}^J} \tag{1.13}$$

The equation of motion is satisfied by the background fields, with overdots indicating derivatives with respect to cosmic time, t.

$$\mathcal{D}_t\dot{\varphi}^I + 3H\dot{\varphi}^I + \mathcal{G}^{IJ}V_{,J} = 0 \tag{1.14}$$

In this context, the symbol $H \equiv \dot{a}/a$ denotes the Hubble parameter. Additionally, a directional derivative for vectors A^I on the field-space manifold has been introduced and is presented in a covariant manner.

$$\mathcal{D}_t A^I \equiv \dot{\varphi}^J\mathcal{D}_J A^I = \dot{A}^I + \Gamma^I_{JK}A^J\dot{\varphi}^K \tag{1.15}$$

The creation of the Christoffel symbols Γ^I_{JK} is dependent on the metric of the field-space, $\mathcal{G}_{IJ}$. At the background level, the Friedmann equations are presented as:

$$\begin{aligned} H^2 &= \frac{1}{3M_{\rm pl}^2}\left[\frac{1}{2}\dot{\sigma}^2 + V(\varphi^I)\right], \\ \dot{H} &= -\frac{1}{2M_{\rm pl}^2}\dot{\sigma}^2. \end{aligned} \tag{1.16}$$

The generation of self-consistent inflationary solutions can be achieved through the utilization of Equations (1.14) and (1.16), which are applicable across a wide range of initial conditions and parameter spaces. Such solutions are characterized by the condition that $|\dot{H}| \ll H^2$.

In order to examine the fluctuations' dynamics, an extension of the gauge-invariant Mukhanov-Sasaki variable to a multifield scenario is required. This extension encompasses the creation of a perturbation vector, $Q^I(x^\mu)$, which is a linear amalgamation of the field fluctuations, $\delta\phi^I$, and the metric perturbation, ψ. However, since the field-space manifold is curved, it's crucial to use a representation of the field fluctuations that remains covariant under reparameterizations of the field-space coordinates. This representation is equivalent to Eq. (1.17) at linear order in the field fluctuations, which is enough for our current objectives.

$$Q^I \equiv \delta\phi^I + \frac{\dot{\varphi}^I}{H}\psi \tag{1.17}$$

Equation (1.17) expresses the representation of $Q^I(x^\mu)$ as a linear combination of fluctuations in the field and perturbations in the metric.

The equation of motion for the fluctuations Q^I is obeyed to a linear order.

$$\mathcal{D}_t^2 Q^I + 3H\mathcal{D}_t Q^I + \left[\frac{k^2}{a^2}\delta^I_J + \mathcal{M}^I_J\right] Q^J = 0 \tag{1.18}$$

After conducting a Fourier transformation, in which the equation $\nabla^2 Q^I = -k^2 Q^I$ with comoving wavenumber k is utilized, the equation of motion leads to the mass-squared matrix, which is as follows:

$$\mathcal{M}^I_J \equiv \mathcal{G}^{IK}\left(\mathcal{D}_J \mathcal{D}_K V\right) - \mathcal{R}^I_{LMJ} \dot{\varphi}^L \dot{\varphi}^M - \frac{1}{a^3 M_{\text{pl}}^2} \mathcal{D}_t \left(\frac{a^3}{H} \dot{\varphi}^I \dot{\varphi}_J\right) \tag{1.19}$$

The equation presents three contributions that result in the effective mass of fluctuations Q^I being distinct from one another. The first contribution is derived from the potential's second derivative and is similar to what is found in single-field models. The second contribution is proportional to $\mathcal{R}^I{}_{LMJ}$ and is a result of the field-space manifold's curvature that is constructed from $\mathcal{G}_{IJ}$ and calculated at the background order in the fields, φ^I. Lastly, the third contribution is proportional to $1/M_{\text{pl}}^2$, and it arises from the perturbations in the coupled metric.

1.3 Anticipated Observational Outcomes

Equation (1.18) introduces couplings between fluctuations Q^J with Q^J, and so forth, even when examined in a linear order. The existence of multiple interacting degrees of freedom in multifield models can lead to the emergence of unique observational characteristics not present in simpler, single-field models. Two notable examples that have been extensively studied are the amplification of non-Gaussianities in the primordial power spectrum of curvature perturbations and the amplification of isocurvature perturbations alongside adiabatic modes. In single-field models, non-Gaussianities are usually suppressed, and isocurvature modes are not present when there is only one scalar degree of freedom.

Recent observations of the Cosmic Microwave Background have imposed strict limitations on the existence of primordial non-Gaussianities and isocurvature perturbations. As a result, various types of multifield models may be in contradiction with the current observational data.

In order to conduct a quantitative analysis of these multifaceted features, we will introduce covariant measures that will allow us to explore the spectra of perturbations. Furthermore, we will also introduce a unit vector for this purpose.

$$\hat{\sigma}^I \equiv \frac{\dot{\varphi}^I}{\dot{\sigma}} \tag{1.20}$$

To indicate the direction of the evolution of background fields, a unit vector is introduced and symbolized as $\hat{\sigma}^I$. The directions in field space that are perpendicular to this vector are encompassed by

$$\hat{s}^{IJ} \equiv \mathcal{G}^{IJ} - \hat{\sigma}^I \hat{\sigma}^J \tag{1.21}$$

As a result, the perturbations Q^I can be broken down into two distinct components. The first component is aligned with the direction of motion of the background fields, commonly referred to as the adiabatic direction. The second component is orthogonal to the motion of the background fields, known as the isocurvature directions. This breakdown can be expressed as follows:

$$Q_\sigma \equiv \hat{\sigma}_I Q^I,\ \delta s^I \equiv \hat{s}^I_J Q^J \tag{1.22}$$

The definition for the gauge-invariant curvature perturbation, marked as $\mathcal{R}_c$,is as follows:

$$\mathcal{R}_c \equiv \psi - \frac{H}{(\rho + p)} \delta q \tag{1.23}$$

The symbols ρ and p in the given context represent the energy density and pressure of the background order, respectively. The perturbed fluid's momentum flux is denoted by δq, which is given by $T_i^0 = \partial_i \delta q$. By utilizing the format of the action as presented in Equation (1.8), it can be proved that:

$$\mathcal{R}_c = \frac{H}{\dot{\sigma}} Q_\sigma \tag{1.24}$$

The initial curvatures of space, known as $\mathcal{R}_c(x)$, are responsible for causing temperature differences in the cosmic microwave background (CMB). When photons move from areas of space with a greater-than-average gravitational potential, they must expend more energy to climb out of the potential well, leading to a slight redshift. Conversely, photons originating from areas with less-than-average gravitational potential experience a lesser redshift. Thus, the statistical characteristics of the small temperature variances in the CMB serve as a snapshot of the primary non-uniformities, which in turn, help to restrict models of early-universe expansion.

A critical observation to note is that the connection between Q_σ and δs^I is only present when there are changes in the field space of the background fields. As a result, characteristics like non-Gaussianities and isocurvature perturbations can be amplified in multifield models when the turn-rate, ω^I, is not zero during the latter stages of inflation (usually within the last 60 efolds of inflation). The covariant turn-rate can be precisely defined as:

$$\omega^I \equiv \mathcal{D}_t \hat{\sigma}^I \tag{1.25}$$

In models that incorporate multiple fields, it is not mandatory for the turn-rate ω^I to remain at a small value while inflation is occurring. As a result, there is a possibility for the magnification of certain features that may not be discernible in the cosmic microwave background (CMB).

Starting with an analysis of the limit $\omega^I \to 0$, we observe that the perturbations Q_σ and δs^I remain uncoupled. In this situation, the effective masses of the perturbations can be expressed as follows:

$$\mathcal{M}_{\sigma\sigma} \equiv \hat{\sigma}_I \hat{\sigma}^J \mathcal{M}_J^I,\ \mathcal{M}_{ss} \equiv \hat{s}_I{}^J \mathcal{M}_J^I \tag{1.26}$$

When the values of $|\mathcal{M}_{\sigma\sigma}|, |\mathcal{M}_{ss}| \ll H^2$, , any perturbation will experience changes during the inflation process as a nearly weightless scalar field in a (quasi-) de Sitter space. As a result, we can assume that each perturbation will grow to an amplitude that is roughly equal to:

$$\mathcal{P}_Q \simeq \left(\frac{H}{2\pi}\right)^2 \quad (1.27)$$

The power spectrum, denoted as $\mathcal{P}_Q \equiv (2\pi)^{-2}k^3|Q_\sigma|^2$. We have evaluated this formula for modes that are approximately at the Hubble scale, or $k \simeq aH$. In the same vein, we have also assessed $\mathcal{P}_S$, which is the power spectrum that is linked to the conventionally normalized isocurvature perturbations $S^I \equiv (H/\dot{\sigma})\delta s^I$. This is accomplished by utilizing Eqs. (1.16) and (1.24), as well as the standard definition of the slow-roll parameter. The resulting calculations are as follows:

$$\epsilon \equiv -\frac{\dot{H}}{H^2} \quad (1.28)$$

Thus, it is foreseeable that an abundance of perturbations in curvature will occur throughout the inflationary process. The exact amplitude of these perturbations can be estimated as:

$$\mathcal{P}_R \simeq \frac{1}{2M_{\rm pl}^2\epsilon}\left(\frac{H}{2\pi}\right)^2 \quad (1.29)$$

and similarly, for $\mathcal{P}_S$.

During the inflationary epoch, the fields $\varphi^I(t)$ that form the background undergo gradual changes, which leads to variations in both H(t) and ϵ(t). As a result, when modes of differing comoving wavenumbers k cross outside the Hubble radius, where $k = aH$, they do so with small differences in amplitude, represented by , $\mathcal{P}_R(k)$. By carrying out additional analysis, it is possible to determine the spectral tilt of the curvature perturbations.

$$n_s \equiv 1 + \frac{\partial \ln\, \mathcal{P}_R}{\partial \ln\, k} = 1 - 6\epsilon + 2\eta_{\sigma\sigma} \quad (1.30)$$

Where $\eta_{\sigma\sigma} \equiv M_{\rm pl}^2 \frac{\mathcal{M}_{\sigma\sigma}}{V}$

The ϵ_σ parameter serves as a generalization for slow-roll motion in the adiabatic direction. In the event where $\omega^I \to 0$, the amplitude and spectral tilt of primordial curvature perturbations in multifield models bear resemblance to those in single-field models, with one crucial deviation. If $|\mathcal{M}_{ss}| \ll H^2$, then multifield models may amplify a significant fraction of isocurvature modes, which results in a $\beta_{\rm iso}\,(k) \equiv \mathcal{P}_S(k)/[\mathcal{P}_R(k) + \mathcal{P}_S(k)] \sim \mathcal{O}(1)$ at pertinent wavenumbers k. This amplification may be inconsistent with current observations.

When the fields turn in field space during inflation $\omega^I \neq 0$, significant deviations from the single-field case can occur. This can trigger a power transfer from the isocurvature to the adiabatic modes, which can affect the amplitude and tilt of $\mathcal{P}_R(k)$. The transfer function $T_{RS}(t_*, t)$ relates the power spectrum at a specific time, such as 50 or 60 efolds before the end of inflation (t_*), to its value at a later time (t).

$$\begin{aligned} \mathcal{P}_R(k) &= \mathcal{P}_R(k_*)[1 + T_{RS}^2(t_*, t)], \\ n_s &= n_s(t_*) + \frac{1}{H}\left(\frac{\partial T_{RS}}{\partial t_*}\right)\sin(2\Delta), \end{aligned} \tag{1.31}$$

where $\Delta \equiv \arccos\left(T_{RS}/\sqrt{1 + T_{RS}^2}\right)$.

If the transfer of power between the isocurvature and adiabatic modes is even slightly modest, multifield models could potentially deviate from their agreement with the most recent, high-precision measurements of variables such as n_s. Additionally, due to the scale-dependent nature of T_{RS}, these processes couple modes of different wave numbers, k, and thus magnify non-Gaussianities, resulting in the coefficient of the bispectrum, f_{NL}, being much greater than the standard O(1).

When viewed through the Einstein frame, the anisotropic pressure of perturbations is of little significance at the primary order meaning $\left(\Pi^i_j \propto T^i_j \sim 0 \text{ for } i \neq j\right)$. As a result, the evolution of tensor perturbations h_{ij} is akin to that of single-field models. This also means that the power spectrum around the pivot scale k_* follows the equation $\mathcal{P}_T \simeq 128\left(H^2/M_{\rm pl}^2\right)$. This prediction leads to a calculation of the tensor-to-scalar ratio, r.

$$r \equiv \frac{\mathcal{P}_T}{\mathcal{P}_R} = \frac{16\epsilon}{[1+T_{RS}^2]} \tag{1.32}$$

When there is a significant transfer of power from isocurvature to adiabatic modes, the predictions for r, much like the cases of n_s and f_{NL}, can deviate significantly from the typical predictions for a single field.

2 Conformal Transformations Involving Multiple Scalar Fields

There are numerous models that are intriguing, which include scalar fields that possess nonminimal couplings to the Ricci curvature scalar of spacetime. It is commonly understood that if a solitary scalar field is subjected to nonminimal coupling, a conformal transformation can be implemented to establish a new frame. In this new frame, both the gravitational aspect of the Lagrangian and the kinetic aspect of the rescaled field take on a canonical form. Our investigation centers around determining the conditions under which it is possible to render the Lagrangian's gravitational and kinetic terms as canonical when dealing with more than one scalar field that exhibits nonminimal coupling. Although a specific subset of two-field models permits such a transformation, it is typically not the case for models that involve more than two scalar fields with nonminimal coupling. In the fields of cosmology and particle physics, scalar fields that possess nonminimal couplings to the spacetime Ricci curvature scalar are prevalent. These couplings are widely applicable, appearing in various contexts such as scalar-tensor theories like Jordan-Brans-Dicke gravity and induced-gravity models. They also emerge in low-energy formulations derived from theories with higher dimensions, including supergravity, string theory, and other Kaluza-Klein models, as well as in f(R) models of gravity that result from a conformal transformation. Furthermore, it is well-established that nonminimal couplings are a natural consequence when addressing the renormalization of scalar fields within curved spacetime backgrounds. In fact, in many models, the strength of nonminimal coupling, denoted by ξ, tends to increase unbounded due to the influence of renormalization-group flow.

In order to achieve the well-known Einstein-Hilbert form for the gravitational portion of the Lagrangian, we may utilize a conformal transformation on the spacetime metric, which is represented as $g_{\mu\nu} \to \hat{g}_{\mu\nu}$. Additionally, we have the option to modify the scalar field's scale by transforming it from $\phi \to \hat{\phi}$. This involves adjusting the kinetic term for $\hat{\phi}$ in the altered Lagrangian to attain a canonical form. As a result, the system's dynamics within the transformed frame exhibit behavior similar to that of a scalar field with minimal coupling in conventional (Einsteinian) gravitational theory.

Our inquiry delves into the conditions that allow for a combination of conformal transformation and field rescalings to present both the gravitational and kinetic terms of a Lagrangian in canonical form. This holds true even when dealing with multiple non-minimally coupled scalar fields. As non-minimal couplings in scalar fields are widespread within curved spacetimes and particle physics models (including extensions to the standard model) often involve various scalar fields with potential importance in the early universe, it is crucial to grasp the transformation traits of a diverse range of models.

Upon close examination of the subject, it becomes apparent that only a particular subset of models, consisting of two scalar fields that are not minimally coupled, permits the desired transformation. In contrast, models that involve more than two nonminimally coupled scalar fields usually do not present this possibility. It should be highlighted that a conformal transformation on the metric of spacetime can always be performed to operate within a more manageable framework. Nevertheless, it is usually not feasible to identify a transformed framework in which both the gravitational sector and the kinetic terms of the scalar fields achieve a canonical form.

Within Section 2.1, we take a closer look at the single-field situation, re-examining the traditional process of transformation and clarifying the notation used. We then proceed to Section 2.2, where we broaden our analysis to encompass N scalar fields that possess nonminimal couplings. Within this section, we differentiate between the scenarios where N equals 2 and when N is greater than 2. Finally, we present our concluding observations in Section IV.

2.1. Single Field Case

The Ricci tensor and Ricci curvature scalar are obtained by contracting the Riemann tensor:

$$R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu},\ R = g^{\mu\nu}R_{\mu\nu} \qquad (2.3)$$

Our analysis will be focused on a spacetime that encompasses D dimensions, including one for time, and our metric will be characterized by a signature of (-, +, +, +, +, ...). Our method of adopting the Christoffel symbols is as follows: The equation represented as

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma}\left[\partial_{\mu}g_{\sigma\nu} + \partial_{\nu}g_{\sigma\mu} - \partial_{\sigma}g_{\mu\nu}\right] \ (2.1)$$

is a complex mathematical formula. It displays the relationship between various variables such as g and Γ, with μ, ν, and λ acting as indices. The equation involves the partial derivatives of g with respect to μ and ν, as well as σ and λ. The formula also includes the metric tensor g and the affine connection Γ.

$$R^{\lambda}_{\mu\nu\sigma} = \partial_{\nu}\Gamma^{\lambda}_{\mu\sigma} - \partial_{\sigma}\Gamma^{\lambda}_{\mu\nu} + \Gamma^{\eta}_{\mu\sigma}\Gamma^{\lambda}_{\eta\nu} - \Gamma^{\eta}_{\mu\nu}\Gamma^{\lambda}_{\eta\sigma}. \qquad (2.2)$$

To obtain the Ricci tensor and Ricci curvature scalar, it is necessary to contract the Riemann tensor.

$$R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu},\ R = g^{\mu\nu}R_{\mu\nu} \qquad (2.3)$$

When there is only one field involved, the action can be expressed as follows:

$$S = \int d^D x\sqrt{-g}\left[f(\phi)R - \frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - V(\phi)\right] \tag{2.4}$$

When dealing with a solitary field, the corresponding operation can be phrased as follows: The covariant derivatives can be symbolized by ∇. The assumption is always made that $f(\phi)$ will have a positive value. When discussing minimal coupling, it is standard to use the representation $f(\phi) \rightarrow (16\pi G_D)^{-1}$, where G_D represents the value of the gravitational constant in D dimensions and is comparable to Newton's constant. The frame in which $f(\phi)$ is not a constant, as illustrated in Equation (2.4), is typically referred to as the Jordan frame.

In order to simplify our calculations, we will utilize natural units where c and $\hbar$ both equal 1. The metric tensor $g_{\mu\nu}$ will be considered dimensionless as a result. This means that in D dimensions, lengths and times have dimensions of (mass) $)^{-D}$,, and the covariant volume element in the action integral, $d^D x\sqrt{-g}$, has dimensions of (mass) $)^{-D}$. The Ricci scalar $R(g_{\mu\nu})$ has dimensions of $[(\partial_x g)^2] \sim (\text{ mass })^2$. To preserve the integrand's dimensionlessness, the scalar field's kinetic term mandates that ϕ exhibit dimensions of $[\phi] \sim (\text{mass})^{\frac{D-2}{2}}$.

One possible way to add further parameters involves using a (reduced) Planck mass in D dimensions.

$$M_{(D)}^{D-2} \equiv \frac{\mathbf{1}}{\mathbf{8\pi G_D}} \tag{2.5}$$

When $D = 4$, $M_{(4)} = M_{\text{pl}} = 1/\sqrt{8\pi G_4} = 2.43 \times 10^{18}\text{GeV}$.

There exist a multitude of model lineages that exhibit behavior similar to that of Eq. (2.4). In these models, the scalar field has a conventional kinetic term, while the gravitational sector diverges from the Einstein-Hilbert form. One such example is the nonminimal coupling associated with the renormalization counterterm, which can be represented as:

$$f(\phi) = \frac{1}{2}[M_0^{D-2} + \xi\phi^2] \tag{2.6}$$

The variable ξ in this context represents the degree of the nonminimal coupling, whereas M_0 denotes a certain mass scale. It's worth noting that the mass scale M_0 may not be equivalent to $M_{(D)}$. Assuming the sign conventions of Eq. (2.6), a field that is conformally coupled is represented by $\xi = -\frac{1}{4}(D-2)/(D-1)$. If the potential V(ϕ) of a scalar field includes solutions that break symmetry by having a non-zero vacuum expectation value v, then the strength of observed gravity will be $M_{(D)}^{D-2} = M_0^{D-2} + \xi v^2$. It's also possible to have M_0=0, such as in the case of induced-gravity models.

Jordan-Brans-Dicke gravity is a model that is frequently encountered in the field. Its relevant action is typically expressed through the following statement:

$$S_{\text{JBD}} = \int d^D x\sqrt{-g}\left[\Phi R - \frac{\omega}{\Phi} g^{\mu\nu}\nabla_\mu\Phi\nabla_\nu\Phi\right] \tag{2.7}$$

The action can be restructured to resemble the format of Equation (2.4) by adjusting the field to $\Phi \rightarrow \phi^2/(8\omega)$. This reformulation comprises a canonical kinetic term for ϕ and a nonminimal coupling f(ϕ) that is comparable to Equation (2.6), with $M_0 = 0$ and $\xi = 1/(8\omega)$.

A transformation known as a conformal transformation can be enacted on the metric, which is defined in the following manner:

$$\hat{g}_{\mu\nu} = \Omega^2(x) g_{\mu\nu} \tag{2.8}$$

In our analysis, we regard Ω(x) as a genuine number, thereby signifying that $\Omega^2(x)$ is a positive definite quantity. One must acknowledge that this alteration does not involve any modification of coordinates; the coordinates x^μ stay constant in both frames. Rather, what has been accomplished is the modification of one metric to another, which is contingent on the dimensions of space and time. In order to differentiate between quantities in the transformed frame, we will designate them using a caret symbol (^).

Starting from Eq. (2.8), It is apparent that,

$$\hat{g}^{\mu\nu} = \frac{1}{\Omega^2(x)} g^{\mu\nu},\ \sqrt{-\hat{g}} = \Omega^D(x)\sqrt{-g} \tag{2.9}$$

When Eq. (2.8) is implemented, the Christoffel symbols and Ricci curvature scalar can be computed for the transformed frame, leading to the following outcome:

$$\begin{aligned} \hat{\Gamma}^{\alpha}_{\beta\gamma} = &\ \Gamma^{\alpha}_{\beta\gamma} + \frac{1}{\Omega}\left[\delta^{\alpha}_{\beta}\nabla_{\gamma}\Omega + \delta^{\alpha}_{\gamma}\nabla_{\beta}\Omega - g_{\beta\gamma}\nabla^{\alpha}\Omega\right], \\ \hat{R} = &\ \frac{1}{\Omega^2}\Big[R - \frac{2(D-1)}{\Omega}\Box\,\Omega - (D-1)(D-4) \\ &\times \frac{1}{\Omega^2} g^{\mu\nu}\nabla_{\mu}\Omega\nabla_{\nu}\Omega\Big], \end{aligned} \tag{2.10}$$

where

$$\Box\,\Omega = g^{\mu\nu}\nabla_{\mu}\nabla_{\nu}\Omega = \frac{1}{\sqrt{-g}}\partial_{\mu}\left[\sqrt{-g}g^{\mu\nu}\,\partial_{\nu}\Omega\right] \tag{2.11}$$

It is imperative to clarify whether the derivatives are derived from the initial metric $g_{\mu\nu}$ or the transformed metric $\hat{g}_{\mu\nu}$. This is because the Christoffel symbols, and therefore the covariant derivatives, are affected by transformations based on Ω, following the formula stated in Eq. (2.8).

Through the implementation of equations (2.8) to (2.11), we are enabled to rephrase the initial expression in the action that concerns R in the subsequent manner:

$$\begin{aligned} \int\ d^D x\sqrt{-g}f(\phi)R = &\ \int\ d^D x\frac{\sqrt{-\hat{g}}}{\Omega^D}f(\phi) \\ &\times\Big[\Omega^2\hat{R} + \frac{2(D-1)}{\Omega}\Box\,\Omega \\ &+\frac{(D-1)(D-4)}{\Omega^2}g^{\mu\nu}\nabla_{\mu}\Omega\nabla_{\nu}\Omega\Big] \end{aligned} \tag{2.12}$$

Let us analyze each of these terms individually. Starting with the term on the right-hand side, it undergoes a transformation:

$$\int d^D x\sqrt{-\hat{g}}\left[\left(\frac{f}{\Omega^{D-2}}\right)\hat{R}\right] \tag{2.13}$$

In order to obtain the Einstein-Hilbert gravitational action in the transformed frame, it is possible to acknowledge the following:

$$\Omega^{D-2}(x) = \frac{2}{M_{(D)}^{D-2}} f[\phi(x)] \tag{2.14}$$

When integrating the second term on the right-hand side of Equation (2.12) through parts, one must take note that the $\Box$ operator that is applied to Ω is defined with respect to the initial metric $g_{\mu\nu}$, not the transformed metric. By utilizing Equations (2.9), (2.11), and (2.14), the outcome is:

$$\begin{aligned} &\int d^D x\sqrt{-\hat{g}}\,\frac{2(D-1)}{\Omega^{D+1}} f\,\Box\,\Omega \\ &= -\int d^D x\sqrt{-\hat{g}}(D-1)(D-3)M_{(D)}^{D-2}\frac{1}{\Omega^2}\hat{g}^{\mu\nu}\hat{\nabla}_\mu\Omega\hat{\nabla}_\nu\Omega \end{aligned} \tag{2.15}$$

It is important to note that x^μ remains constant when subjected to the conformal transformation, indicating that $\hat{\partial}_\mu = \partial_\mu$. When observing the covariant derivatives in Eq. (2.15), it is apparent that they only impact scalar functions. As a result, it can be concluded that $\nabla_\mu\Omega = \partial_\mu\Omega$. Hence, $\hat{\nabla}_\mu\Omega = \nabla_\mu\Omega$.

One way to express the term on the right side of equation (2.12) is:

$$\begin{aligned} &\int d^D x\sqrt{-\hat{g}}(D-1)(D-4)\left(\frac{f}{\Omega^{D+2}}\right)g^{\mu\nu}\nabla_\mu\Omega\nabla_\nu\Omega \\ &= \int d^D x\sqrt{-\hat{g}}\,\frac{1}{2}(D-1)(D-4)M_{(D)}^{D-2}\frac{1}{\Omega^2}\hat{g}^{\mu\nu}\hat{\nabla}_\mu\Omega\hat{\nabla}_\nu\Omega \end{aligned} \tag{2.16}$$

Through the utilization of Equations (2.9) and (2.14) again, we can reformulate the statement as it appears in Equation (2.16). By bringing together Equations (2.12), (2.15), and (2.16), and acknowledging a simple algebraic correspondence between the coefficients that come before the $\hat{\nabla}_\mu \Omega$ terms, we ultimately reach the following conclusion:

$$\int d^D x \sqrt{-g} f(\phi) R = \int d^D x \sqrt{-\hat{g}} \frac{M_{(D)}^{D-2}}{2} \times \left[\hat{R} - (D-1)(D-2)\frac{1}{\Omega^2} \times \hat{g}^{\mu\nu} \hat{\nabla}_\mu \Omega \hat{\nabla}_\nu \Omega\right] \tag{2.17}$$

Incorporating the canonical Einstein-Hilbert term into the action's gravitational portion has led to the commonly used Einstein frame, which pertains to the frame aligned with the metric $\hat{g}_{\mu\nu}$.

We will now analyze the alterations that occur in the kinetic and potential components of the scalar field in the action when the metric shifts from $g_{\mu\nu} \to \hat{g}_{\mu\nu}$. This can be stated as follows:

$$\int d^D x \sqrt{-g} \left[-\frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - V(\phi)\right] = \int d^D x \sqrt{-\hat{g}} \left[-\frac{1}{2}\frac{1}{\Omega^{D-2}} \hat{g}^{\mu\nu} \hat{\nabla}_\mu \phi \hat{\nabla}_\nu \phi - \hat{V}\right] \tag{2.18}$$

Within this framework, we present a modified potential, depicted as:

$$\hat{V} \equiv \frac{V}{\Omega^D} \tag{2.19}$$

Equation (2.4) presents the action in its entirety, which can then be articulated as follows:

$$\int d^D x \sqrt{-g} \left[f(\phi) R - \frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - V\right] = \int d^D x \sqrt{-\hat{g}} \left[\frac{M_{(D)}^{D-2}}{2} \hat{R} - \frac{1}{2}(D-1)(D-2) M_{(D)}^{D-2} \frac{1}{\Omega^2} \times \hat{g}^{\mu\nu} \hat{\nabla}_\mu \Omega \hat{\nabla}_\nu \Omega - \frac{1}{2}\frac{1}{\Omega^{D-2}} \hat{g}^{\mu\nu} \hat{\nabla}_\mu \phi \hat{\nabla}_\nu \phi - \hat{V}\right] \tag{2.20}$$

Equation (2.14) allows for the substitution of f in place of Ω. As a result, in the transformed frame, the action can be expressed in the following manner:

$$\int d^D x \sqrt{-\hat{g}} \left[\frac{M_{(D)}^{D-2}}{2} \hat{R} - \frac{1}{2}\frac{(D-1)}{(D-2)} M_{(D)}^{D-2} \frac{1}{f^2} \hat{g}^{\mu\nu} \hat{\nabla}_\mu f \hat{\nabla}_\nu f - \frac{1}{4f} M_{(D)}^{D-2} \hat{g}^{\mu\nu} \hat{\nabla}_\mu \phi \hat{\nabla}_\nu \phi - \hat{V}\right] \tag{2.21}$$

When dealing with a singular field, the approach is to rescale the field as $\phi \rightarrow \hat{\phi}$. This guarantees that the transformed frame's new scalar field retains the canonical kinetic term. The rescaling is defined as:

$$-\frac{1}{2}\hat{g}^{\mu\nu}\hat{\nabla}_\mu\hat{\phi}\hat{\nabla}_\nu\hat{\phi} = -\frac{M_{(D)}^{D-2}}{4f}\hat{g}^{\mu\nu}\left[\hat{\nabla}_\mu\phi\hat{\nabla}_\nu\phi + \frac{2(D-1)}{(D-2)}\frac{1}{f}\hat{\nabla}_\mu f\hat{\nabla}_\nu f\right] \tag{2.22}$$

Assuming a singular field, it is reasonable to presume a direct correlation between $\hat{\phi}$ and ϕ. This can be expressed as $\hat{\phi}$ being equivalent to $\hat{\phi} \rightarrow \hat{\phi}(\phi)$, or:

$$\frac{d\hat{\phi}}{d\phi} = F(\phi) \tag{2.23}$$

In a scenario where there is only one field, we can represent it as $f = f(\phi)$ and denote its expression in relation to an unidentified function F.

$$F(\phi) = \left(\frac{d\hat{\phi}}{d\phi}\right) = \sqrt{\frac{M_{(D)}^{D-2}}{2f^2(\phi)}}\sqrt{f(\phi) + \frac{2(D-1)}{(D-2)}[f'(\phi)]^2} \tag{2.24}$$

The action can be formulated by expressing the primes, which represent derivatives with respect to ϕ, in terms of the rescaled field.

$$\begin{aligned} &\int d^Dx\sqrt{-g}\left[f(\phi)R - \frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - V(\phi)\right] \\ &= \int d^Dx\sqrt{-\hat{g}}\left[\frac{M_{(D)}^{D-2}}{2}\hat{R} - \frac{1}{2}\hat{g}^{\mu\nu}\hat{\nabla}_\mu\hat{\phi}\hat{\nabla}_\nu\hat{\phi} - \hat{V}(\hat{\phi})\right] \end{aligned} \tag{2.25}$$

The canonical forms of both the gravitational component and scalar field's kinetic term are now apparent in the second line's action, resembling the Einstein-Hilbert form.

2.2. MULTIFIELD CASE

In the following discussion, we will explore the scenario where there exist multiple scalar fields, each with its own nonminimal coupling to the Ricci curvature scalar. These fields will be denoted by Latin indices, specifically ϕ^i, where i ranges from 1 to N to indicate the field space directions. In a spacetime with D dimensions, the structure of the action within the Jordan frame is as follows:

$$\int d^D x\sqrt{-g}\left[f(\phi^1,\ldots,\phi^N)R-\frac{1}{2}\delta_{ij}g^{\mu\nu}\nabla_\mu\phi^i\nabla_\nu\phi^j - V(\phi^1,\ldots,\phi^N)\right] \tag{2.26}$$

The process of executing a conformal transformation can be observed as shown in Equation (8). This involves substituting $g_{\mu\nu}\to\hat{g}_{\mu\nu}$ and expressing it using a conformal factor $\Omega^2(x)$. In this procedure, the functions Ω(x) and f(x), which are both dependent on x^μ, are the only factors considered. The relationship between these functions and the scalar field, ϕ(x), is not considered. As a result, the steps for this transformation remain the same in a multifield scenario, leading us to the gravitational component of the action in the new frame.

$$\int d^D x\sqrt{-g}f(\phi^i)R = \int d^D x\sqrt{-\hat{g}}\left[\frac{M_{(D)}^{D-2}}{2}\hat{R}-\frac{1}{2}\times\frac{(D-1)}{(D-2)}M_{(D)}^{D-2}\frac{1}{f^2}\hat{g}^{\mu\nu}\hat{\nabla}_\mu f\hat{\nabla}_\nu f\right] \tag{2.27}$$

By utilizing Equation (2.14) to substitute f with Ω, the kinetic and potential components of the scalar fields undergo a similar transformation pattern as demonstrated in Equation (2.18). As a result, the following is obtained:

$$\int d^D x\sqrt{-g}\left[-\frac{1}{2}\delta_{ij}g^{\mu\nu}\nabla_\mu\phi^i\nabla_\nu\phi^j - V(\phi^i)\right] = \int d^D x\sqrt{-\hat{g}}\left[-\frac{1}{4f}M_{(D)}^{D-2}\delta_{ij}\hat{g}^{\mu\nu}\hat{\nabla}_\mu\phi^i\hat{\nabla}_\nu\phi^j-\hat{V}\right] \tag{2.28}$$

in terms of $\hat{V}$ as defined in Eq. (2.19).

By uniting these phrases, we can infer the movement in the modified perspective.

$$\int d^D x\sqrt{-g}\left[f(\phi^i)R-\frac{1}{2}\delta_{ij}g^{\mu\nu}\nabla_\mu\phi^i\nabla_\nu\phi^j-V\right]$$
$$=\int d^D x\sqrt{-\hat{g}}\left[\frac{M_{(D)}^{D-2}}{2}\hat{R}-\frac{1}{2}\frac{(D-1)}{(D-2)}\frac{M_{(D)}^{D-2}}{f^2}\hat{g}^{\mu\nu}\hat{\nabla}_\mu f\hat{\nabla}_\nu f\right.$$
$$\left.-\frac{M_{(D)}^{D-2}}{4f}\delta_{ij}\hat{g}^{\mu\nu}\hat{\nabla}_\mu\phi^i\hat{\nabla}_\nu\phi^j-\hat{V}\right] \tag{2.29}$$

When dealing with the multifield scenario, where $f=f(\phi^1,\ldots,\phi^N)$, the corresponding expression takes the following form:

$$\hat{\nabla}_\mu f=\left(\hat{\nabla}_\mu\phi^i\right)f_{,i} \tag{2.30}$$

demonstrates that the vector field f is equal to the product of the covariant derivative of the vector field ϕ^i with respect to the index μ, and the partial derivative of f with respect to index i. By utilizing the equation where $f_{,i}=\partial f/\partial\phi^i$, we are able to rephrase the derivative components found in the last line of Equation (2.29) by means of a metric in the field space, $\mathcal{G}_{ij}$.

$$\int d^D x\sqrt{-\hat{g}}\left[\frac{M_{(D)}^{D-2}}{2}\hat{R}-\frac{1}{2}G_{ij}\hat{g}^{\mu\nu}\hat{\nabla}_\mu\phi^i\hat{\nabla}_\nu\phi^j-\hat{V}\right] \tag{2.31}$$

with

$$G_{ij}=\frac{M_{(D)}^{D-2}}{2f}\delta_{ij}+\frac{(D-1)}{(D-2)}\frac{M_{(D)}^{D-2}}{f^2}f_{,i}f_{,j} \tag{2.32}$$

One crucial fact to consider is that scalar fields possess dimensions denoted by $[\phi]\sim(\text{ mass })^{(D-2)/2}$,, resulting in $\left[f_{,i}\right]\sim\left[\partial_\phi f\right]\sim(\text{ mass })^{(D-2)/2}\sim[\phi]$.

To achieve a field-space metric in the form of $G_{ij}\rightarrow\tilde{G}_{ij}=\delta_{ij}$through a conformal transformation, it is essential that the Riemann tensor created from the metric is equal to zero ($\tilde{\mathcal{R}}^i{}_{jkl}=0$). As the number of nontrivial components of the Riemann tensor increases with a growing N, we will investigate the Ricci curvature scalar derived from the field-space metric, which is $\tilde{\mathcal{R}} = \tilde{G}^{ij}\tilde{R}^{k}{}_{ikj}$. Although it is possible for a metric $\tilde{\mathcal{G}}_{ij}$ to exist with $\tilde{\mathcal{R}} = 0$, it doesn't necessarily mean that all components of $\tilde{\mathcal{R}}^{i}{}_{jkl}$ are equal to zero. It's feasible that $\tilde{\mathcal{R}}$ vanishes due to cancellations between various nonzero terms within the Riemann tensor. However, the opposite isn't true: if $\tilde{\mathcal{R}}^{i}{}_{jkl} = 0$, then $\tilde{\mathcal{R}} \neq 0$. In other words, $\tilde{\mathcal{R}} \neq 0$ only if $\tilde{\mathcal{R}}^{i}{}_{jkl} \neq 0$. To demonstrate that a conformal transformation to make $\tilde{G}_{ij} = \delta_{ij}$ is impossible, we only need to prove that $\tilde{\mathcal{R}} \neq 0$, which is our objective.

The Ricci curvature scalar that is associated with the G_{ij} metric is a complex subject that involves a variety of terms, including f, $f_{,i}$, and $f_{,ij} = \partial^2 f / \partial\phi^i \partial\phi^j$. Our primary focus is on the condition of conformational flatness within the targeted field space, which remains unchanged under conformal transformations. To simplify this expression, we can perform a conformal transformation in the field space, modifying the metric by $G_{ij} \to \tilde{G}_{ij} = f G_{ij}$. When N is greater than one, the curvature scalar connected to $\tilde{G}_{ij}$ takes on a specific form:

$$\begin{aligned}\tilde{\mathcal{R}}_{(N)} = \; & \tfrac{2(D-1)}{L(\phi^i)}\Big[(D-2)A^{ijkl}\big(f f_{,ij} f_{,kl} - f_{,i} f_{,j} f_{,kl}\big) \\ & +2(D-1)B^{ijklmn}\big(f_{,i} f_{,j} f_{,kl} f_{,mn}\big)\Big]\end{aligned} \tag{2.33}$$

where

$$L(\phi^i) = \Big[(D-2)f + 2(D-1)\textstyle\sum_i \; f_{,i}^2\Big]^2 \tag{2.34}$$

and

$$\begin{aligned} A^{ijkl} &\equiv [\delta^{ij}\delta^{kl} - \delta^{ik}\delta^{jl}], \\ B^{ijklmn} &\equiv [\delta^{ij}A^{klmn} + 2\delta^{ik}A^{jmln}] \end{aligned} \tag{2.35}$$

In the larger scope, every term that includes f and its derivatives is linked to ϕ^i. To achieve the possibility of a conformal transformation that produces $\tilde{G}_{ij} = \delta_{ij}$, it's crucial for $\tilde{\mathcal{R}}^{i}{}_{jkl}$ (and, accordingly, $\tilde{\mathcal{R}}$) to be null throughout all areas of field space, regardless of the specific ϕ^i values. This realization highlights the overall difficulty in discovering such a conformal transformation. However, it's valuable to examine the properties of $\tilde{\mathcal{R}}_{(N)}$ for varying N values, specifically N=2 and N>2, as they may diverge.

It is crucial to acknowledge that if there is only one nonminimally coupled field among N fields, with the other N-1 fields maintaining minimal coupling, a combination of conformal transformation and field rescaling can always be discovered. This combination will produce a new frame in which both the gravitational and kinetic terms in the action follow canonical format. This conclusion arises from the arrangement of A^{ijkl}, particularly in instances where all indices are identical. In these scenarios, the terms nullify, leaving only those that include derivatives of $f(\phi^i)$ along a minimum of two directions in field space.

If $f(\phi^i)$ relies on a single field, then any term within $\tilde{\mathcal{R}}_{(N)}$ and $\tilde{\mathcal{R}}^i{}_{jkl})$ will disappear. This holds true for the entirety of $R_{(N)}$. As such, multifield models that only feature one nonminimally coupled field function similarly to conventional single-field structures. However, a significant difference arises in the form of new interactions between the scalar potential $V(\phi^i)$ and both the nonminimally and minimally coupled fields. This is due to the scaling of

$$V \to \hat{V} = \Omega^{-D} V = \left[2f(\phi)/M_{(D)}^{D-2}\right]^{-D/(D-2)} V(\phi^i). \qquad (2.36).$$

Our focus now shifts to the scenario where several fields are subjected to nonminimal couplings.

1. $N = 2$

   Upon examination of the scenario where N=2, it is noteworthy that Equation (34) concerning $\tilde{\mathcal{R}}$ undergoes a significant simplification. The portion of the equation correlated with B^{ijklmn} is rendered null, leading to the following outcome:

$$\tilde{\mathcal{R}}_{(2)} = \frac{2(D-1)(D-2)}{L(\phi^i)}\left[2ff_{,11}f_{,22} - f_{,1}^2 f_{,22} - f_{,2}^2 f_{,11} - 2f_{,12}\left(ff_{,12} - f_{,1}f_{,2}\right)\right] \qquad (2.37)$$

   Furthermore, when N is equal to 2, we can infer that $\tilde{\mathcal{R}}^i{}_{jkl} \propto \tilde{\mathcal{R}}_{(2)}$, where there is no summation over repeated indices. To be specific:

$$\tilde{\mathcal{R}}^i{}_{iji} = -\frac{(D-1)}{(D-2)}\frac{f_{,i}f_{,j}}{f}\tilde{\mathcal{R}}_{(2)},$$
$$\tilde{\mathcal{R}}^i{}_{jij} = \left[\frac{1}{2} + \frac{(D-1)}{(D-2)}\frac{f_{,j}^2}{f}\right]\tilde{\mathcal{R}}_{(2)}. \qquad (2.38)$$

In cases where N exceeds 2, it is important to note that the relationship between $\tilde{\mathcal{R}}^i_{jkl} \propto \tilde{\mathcal{R}}$ will not hold.

We will now analyse a frequently used phrase for $f(\phi^i)$ in cases where N equals two. The two fields can be labeled as follows: $\phi^1 = \phi$ and $\phi^2 = \chi$. As a result, the nonminimal connections in the action are frequently depicted in the following manner:

$$f(\phi, \chi) = \frac{1}{2}\left[M_0^{D-2} + \xi_\phi \phi^2 + \xi_\chi \chi^2\right] \tag{2.39}$$

The strengths of coupling in this standard expression for f, ξ_ϕ and ξ_χ, are not required to be identical. In this particular version of f, the derivatives of cross-term become null, with $f_{,12} = f_{,\phi X} = 0$, leaving

$$\begin{aligned} L(\phi, \chi)\tilde{\mathcal{R}}_{(2)} = &\quad 2(D-1)(D-2) \\ &\quad \times \left[2ff_{,\phi\phi}f_{,\chi\chi} - f_{,\phi}^2 f_{,\chi\chi} - f_{,\chi}^2 f_{,\phi\phi}\right] \\ = &\quad 2(D-1)(D-2)\xi_\phi \xi_\chi M_0^{D-2} \end{aligned} \tag{2.40}$$

When examining a model featuring two scalar fields, both of which are nonminimally linked to the curvature of spacetime, it becomes clear that conformal transformations cannot be used to render both the field responsible for gravity and the kinetic terms of the scalar fields into a canonical form at the same time. This means that even within the Einstein frame, the actions of these fields would not mirror those of models that are truly minimally coupled.

If N is equal to 2, a conformal transformation can be discovered that can make both the gravitational and kinetic terms canonical, given that $M_0 = 0$,. It is noteworthy that in the case of N=2 and $M_0 = 0$, the relationship $\tilde{\mathcal{R}}^i{}_{jkm} = 0$still holds, even when $\xi_\phi \neq \xi_\chi$. This property does not apply to models where N is greater than 2. Jordan-Brans-Dicke gravity, which has two scalar fields, and induced-gravity models, where one or both fields have non-zero vacuum expectation values v_i, leading to $M_{(D)}^{D-2} = \sum_i \xi_i v_i^2$ below the scale of symmetry breaking, are noteworthy examples.

2. $N > 2$

When dealing with cases that involve nonminimal coupling and have more than two variables (N>2), the B^{ijklmn} term in equation (34) for $\tilde{\mathcal{R}}_{(N)}$ will not be zero. This particular term introduces components into $\tilde{\mathcal{R}}_{(N)}$ that are influenced by the fields ϕ^i themselves, not just their couplings ξ_i. As a result, models that include more than two nonminimally coupled scalar fields do not allow for a conformal transformation capable of producing $\tilde{G}_{ij} = \delta_{ij}$ in most scenarios.

One example to consider would be the expansion of nonminimal couplings to N>2, beyond the standard application:

$$f(\phi^i) = \frac{1}{2}[M_0^{D-2} + \sum_{i=1}^{N} \xi_i (\phi^i)^2] \tag{2.41}$$

In the case of models falling under this category, where $f_{,i} = \xi_i \phi^i$ (without summation), and $f_{,ij} = \xi_i \delta_{ij}$, the Ricci scalar manifests as follows:

$$\begin{aligned} \tilde{\mathcal{R}}_{(N)} = \; & \frac{(D-1)}{L(\phi^i)}\Big[(D-2)M_0^{D-2}\sum_{ijkl}\left(\delta^{ij}\delta^{kl} - \frac{1}{N^2}\delta^{ik}\right)\xi_i\xi_k \\ & + \sum_i (D-2+4(D-1)\xi_i) \\ & \times \sum_{jklm}\left[\delta^{jm}\delta^{kl} - \frac{1}{N^2}(\delta^{jk} + \delta^{ij} + \delta^{ik} - 2\delta^{ij}\delta^{ik})\right] \\ & \times \xi_i\xi_j\xi_k(\phi^i)^2\Big] \end{aligned} \tag{2.42}$$

When examining the coupling constants ξ_i, there may be a temptation to investigate combinations that could produce precise nullifications, resulting in $\tilde{\mathcal{R}}_{(N)} = 0$. Yet, when N>2, the correlation $\tilde{\mathcal{R}}^i{}_{jkl} \propto \tilde{\mathcal{R}}_{(N)}$ is no longer applicable. The comprehensive Riemann tensor comprises various elements, including:

$$\tilde{\mathcal{R}}^i{}_{ijk} = 2(D-1)^2 \frac{f_{,i}^2 f_{,j} f_{,k}}{fL}\left(f_{,jj} - f_{,kk}\right) \tag{2.43}$$

for $i \neq j \neq k$.

Merely requiring $\tilde{\mathcal{R}}_{(N)} = 0$ is not enough to ensure that these components are indeed zero. The conditions for their vanishing are more stringent. Specifically, within the model family containing $f(\phi^i)$, these terms will only disappear if all the coupling constants are identical, i.e., if $\xi_i = \xi$ for every i. Put differently, this necessitates the presence of an O(N) symmetry among the N nonminimally coupled fields. By satisfying this symmetry condition, the expression for $\tilde{\mathcal{R}}_{(N)}$ can be further simplified.

$$\tilde{\mathcal{R}}_{(N)} = \frac{(D-1)(N-1)}{L(\phi^i)}[(D-2)N\xi^2 M_0^{D-2} + [D-2+4(D-1)\xi](N-2)\xi^3 \sum_i (\phi^i)^2] \tag{2.44}$$

When the value of $N > 2, \tilde{\mathcal{R}}_{(N)} \neq 0$. This means that the Riemann tensor does not disappear, and it is impossible to achieve a conformal transformation that would result in $\tilde{G}_{ij} = \delta_{ij}$.

3 Presentation of findings

The fundamental action for a scalar field in a curved spacetime can be expressed as

$$\boldsymbol{\mathcal{S}} = \boldsymbol{\mathcal{S}}_{\mathbf{EH}} + \boldsymbol{\mathcal{S}}_{\boldsymbol{\phi}} = \int \ \boldsymbol{d^4x}\sqrt{-\boldsymbol{g}}\left[\frac{\boldsymbol{M_P^2}}{\mathbf{2}}\boldsymbol{R} - \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{g^{\mu\nu}}\boldsymbol{\partial_\mu\phi\partial_\nu\phi} - \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{\xi R\phi^2} - \boldsymbol{V}(\boldsymbol{\phi})\right]$$

(3.1)

The symbol V(ϕ) is used here to denote a potential for ϕ that is not dependent on R. R, in this context, refers to the Ricci scalar curvature, and we maintain a 'mostly minus' or 'west-coast' metric convention (-+++) with a metric signature of 1. The value of the parameter ξ determines the level of non-minimal coupling (NMC) that exists between ϕ and gravity via the Ricci scalar R. When ξ is set to 0, it corresponds to a scenario of minimal coupling. What inspires the pursuit of a scalar theory with non-minimal coupling? When it comes to gauge fields and chiral fermions, their coupling to curvature is entirely bound by gauge and chiral symmetry, which is not the case for scalar fields. There is no symmetry that prevents a direct coupling term such as $\boldsymbol{R\phi^2}$, which is comparable to scalar masses. In fact, the Higgs field in the Standard Model is assumed to have this coupling. It is a dimension-4 operator that remains consistent with all of the Standard Model and gravitational symmetries. When considering conformal field theories, it is crucial to acknowledge that the action for a free scalar field in a curved spacetime does not possess conformal invariance unless the parameter ξ is not equal to zero. In a four-dimensional setting, conformal invariance necessitates that ξ equals one-sixth. This differs from the approach used to achieve conformal invariance in free vector and fermionic fields.

4 Discussion and Conclusion

Jordan Framework

The inclusion of a direct link between phi and curvature introduces the possibility of a tensor-scalar theory that can be defined by its action.

$$\mathcal{S}_J = \int d^4x\sqrt{-g}\left[\frac{M_P^2}{2}f(\phi)R - \frac{1}{2}g^{\mu\nu}\,\partial_\mu\phi\,\partial_\nu\phi - V(\phi)\right] \tag{4.1}$$

where we have defined

$$f(\phi) = \left(1 - \frac{\xi\phi^2}{M_P^2}\right) \tag{4.2}$$

The equation presented in (4.1) is often referred to as the defining principle of the Jordan framework, which is believed to be a modified version of a scalar tensor theory of gravity.

Curvature Transformation Rules

It is possible to locate the subsequent transformation regulations within spacetime dimensions that are represented by the variable "d".

$$g_{\mu\nu} \to \Omega^{-2}\tilde{g}_{\mu\nu},\;\; g^{\mu\nu} \to \Omega^2\tilde{g}^{\mu\nu},\; g = \det\left(g_{\mu\nu}\right) \to \Omega^{-2d}\tilde{g} \tag{4.3}$$

The conformal transformation of the Ricci scalar, in addition to being four-dimensional, can be expressed as follows:

$$R = \Omega^2\tilde{R} + 6\Omega\,\tilde{\Box}\,\Omega - 6\tilde{g}^{\mu\nu}\left(\partial_\mu\Omega\right)(\partial_\nu\Omega) = \Omega^2\big[\tilde{R} + 6\,\tilde{\Box}\ln\,\Omega - 6\tilde{g}^{\mu\nu}\left(\partial_\mu\ln\,\Omega\right)(\partial_\nu\ln\,\Omega)\big] \tag{4.4}$$

where we have used $\Omega^{-1}\,\tilde{\Box}\,\Omega = \tilde{\Box}\ln\,\Omega$.

By utilizing the aforementioned transformation rules, the action is transformed into its new representation, which is dependent on the arbitrary parameter Ω.

$$\tilde{S} = \int d^4x\sqrt{-\tilde{g}}\Big[\frac{M_P^2}{2} f(\phi)\Omega^{2-d}\left(\bar{R} + 6\,\Box\ln\,\Omega - 6\tilde{g}^{\mu\nu}(\partial_\mu \ln\,\Omega)(\partial_\nu \ln\,\Omega)\right) - \frac{1}{2}\Omega^{2-d}\tilde{g}^{\mu\nu}\,\partial_\mu\phi\,\partial_\nu\phi - \Omega^{-d}V(\phi)\Big] \qquad (4.5)$$

The Einstein Frame in 4 Dimensions

Within the context of 4-dimensional spacetime, the Einstein frame can be identified by the presence of the character Ω, which serves to convert the coefficient $M_P^2\bar{R}/2$ to 1. The only factor that exerts any dimensional influence is the transformation of g. The establishment of the Einstein frame in d spacetime dimensions is determined by the following condition: $f(\phi)\Omega^{2-d} = 1$. In the case of 4 dimensions, this condition can be expressed as:

$$\Omega^2 = f(\phi) \qquad (4.6)$$

which means $\ln\,\Omega = \frac{1}{2}\ln\,f$ and $\partial_\mu\ln\,\Omega = \frac{1}{2}\partial_\mu\ln\,f = \frac{1}{2f}\partial_\mu f$. In the context of 4 dimensions and the Einstein frame, we arrive at the conclusion that

$$\tilde{S}_E = \int d^4x\sqrt{-\tilde{g}}\left[\frac{M_P^2}{2}\tilde{R} - \frac{3}{4f^2}M_P^2\tilde{g}^{\mu\nu}(\partial_\mu f)(\partial_\nu f) - \frac{1}{2f}\tilde{g}^{\mu\nu}\,\partial_\mu\phi\,\partial_\nu\phi - \frac{1}{f^2}V(\phi)\right]$$
(4.7)

After eliminating a total derivative term of $\# \times \tilde{\Box}\ln\,\Omega$, we are left with an equation that involves only ϕ derivatives.

$$\tilde{S}_E = \int d^4x\sqrt{-\tilde{g}}\left[\frac{M_P^2}{2}\tilde{R} - \frac{1}{2}\mathcal{K}^2(f)\tilde{g}^{\mu\nu}\,\partial_\mu\phi\,\partial_\nu\phi - \frac{1}{f^2}V(\phi)\right] \qquad (4.8)$$

with

$$\mathcal{K}^2(f) = \frac{f+6\xi(1-f)}{f^2} = \frac{1+\xi(6\xi-1)\phi^2/M_P^2}{\left(1-\xi\phi^2/M_P^2\right)^2} \qquad (4.9)$$

By introducing a non-minimal kinetic term for a scalar degree of freedom, we can have Einstein gravity alongside it. This leads to the definition of $\tilde{\phi}$ as follows:

$$\mathcal{K}\,\partial_\mu\phi = \partial_\mu\tilde{\phi} \implies d\tilde{\phi} = \mathcal{K}d\phi \qquad (4.10)$$

Assuming the total differential to be $d\phi = (\partial_\mu \phi) dx^\mu$, we can construct an action that has a canonically normalized kinetic term for $\tilde{\phi}$.

$$\tilde{S}_E = \int \; d^4x\sqrt{-\tilde{g}} \left[\frac{M_P^2}{2}\tilde{R} - \frac{1}{2}\tilde{g}^{\mu\nu}\,\partial_\mu\tilde{\phi}\,\partial_\nu\tilde{\phi} - \tilde{V}(\tilde{\phi})\right] \tag{4.11}$$

where $\tilde{V}(\tilde{\phi})$ is implicitly defined in terms of ϕ as

$$\tilde{V}(\tilde{\phi}) = \frac{V(\phi)}{\left(1-\xi\phi^2/M_P^2\right)^2} \tag{4.12}$$

The standard Klein-Gordon equation is satisfied by the field in the Einstein frame.

$$\tilde{\Box}\,\tilde{\phi} - \frac{\partial\tilde{V}}{\partial\tilde{\phi}} = \frac{1}{\sqrt{-\tilde{g}}}\partial_\mu\left(\sqrt{-\tilde{g}}\tilde{g}^{\mu\nu}\,\partial_\nu\tilde{\phi}\right) - \frac{\partial\tilde{V}}{\partial\tilde{\phi}} = 0 \tag{4.13}$$

It is indeed possible to find a precise analytical solution for $\tilde{\phi}$ by utilizing ϕ (up to an integration constant).

$$\pm\frac{\tilde{\phi}}{M_P} = \sqrt{6}\tanh^{-1}\left[\frac{\xi\phi/M_P}{\sqrt{\frac{1}{6}+\left(\xi-\frac{1}{6}\right)\xi\phi^2/M_P^2}}\right] - \sqrt{\frac{6\xi-1}{\xi}}\sinh^{-1}\left[\sqrt{\frac{6\xi-1}{\xi}}\frac{\xi\phi}{M_P}\right] \tag{4.14}$$

Reversing this formula poses a challenge. The $\tanh^{-1}$ function experiences a divergence as it approaches an argument of 1. This divergence occurs when the ratio of ξ times ϕ squared to the Planck mass squared approaches 1. A few noteworthy limits include:

- $\xi = 0$: In this case, we simply have $\phi = \tilde{\phi}$ as the J and E frames are equivalent.
- $\xi = 1/6$: In this case, we have

$$\phi = \sqrt{6}M_P\tanh\left(\frac{\tilde{\phi}}{\sqrt{6}M_P}\right) \implies \tilde{V}(\tilde{\phi}) = V(\phi(\tilde{\phi}))\cosh^4\left(\frac{\tilde{\phi}}{\sqrt{6}M_P}\right) \tag{4.15}$$

- $\xi\phi^2/M_P^2 \gg 1$: In this case, we get

$$\phi = \frac{M_P}{\sqrt{\xi}} \exp\left(\frac{\tilde{\phi}}{\sqrt{6}M_P}\right) \tag{4.16}$$

If the equation for the scalar potential $V(\phi) = \lambda(\phi^2 - v^2)^2/4$, then it leads to the possibility of Higgs inflation in the EF.

In future research, it would be intriguing to examine this framework. However, due to the challenges associated with inverting this formula as well as the possibility of singular points, it is also noteworthy to explore the theory in the Jordan frame.

Non-minimally Coupled Scalars in the Jordan Frame

In the most comprehensive manner possible, we examine the influence of gravity on a scalar field ϕ that has been coupled to it.

$$\mathcal{S}_{\mathcal{J}} = \mathcal{S}_{\text{EH}} + \mathcal{S}_{\phi} = \int \ d^4x\sqrt{-g}\left[\frac{M_P^2}{2}R - \frac{1}{2}g^{\mu\nu}\,\partial_\mu\phi\,\partial_\nu\phi - \frac{1}{2}\xi R\phi^2 - V(\phi) + \mathcal{L}_{\text{m}}\right]$$
(4.17)

We will be examining a flat Friedmann-Robertson-Walker (FRW) universe, which can be described by a metric that is general in nature but remains flat in shape. This metric is known as the "α-time" metric.

$$ds^2 = -a(\eta)^{2\alpha}d\eta^2 + a(\eta)^2\delta_{ij}dx^i dx^j \tag{4.18}$$

so we have $\sqrt{-g} = a(\eta)^{3+\alpha}$ and the Ricci scalar is

$$R = \frac{6}{a^{2\alpha}}\left[\frac{a''}{a} + (1-\alpha)\left(\frac{a'}{a}\right)^2\right] = \frac{6}{a^{2\alpha}}\left[\frac{a''}{a} + (1-\alpha)\mathcal{H}^2\right] \tag{4.19}$$

In this context, the use of primes denotes differentiation with regard to the α-time variable η. Additionally, the α-time Hubble rate $\mathcal{H} = a'/a$ is represented by H. This is connected to the physical Hubble rate $H = \dot{a}/a$ through the following relationship:

$$\mathcal{H} = H\frac{dt}{d\eta} = Ha^{\alpha} \tag{4.20}$$

where dots indicate derivatives w.r.t to cosmic time t.
Some familiar examples of α-time choices

- $\alpha = 0$: Real or cosmic time
- $\alpha = 1$: Conformal time

In order to delve into this particular theory, it is imperative to possess both the equations of motion for the scalar field and the gravitational equations of motion- commonly referred to as the Friedmann equations.

Equation of Motion for the Non-minimally Coupled Scalar Field

To derive the equation of motion for ϕ, one must find the variation of $\mathcal{S}_{\phi}$ with respect to ϕ. This analysis yields the Euler-Lagrange equation.

$$\begin{aligned} \partial_{\mu}\left(\sqrt{-g}\,\frac{\partial\mathcal{L}_{\phi}}{\partial(\partial_{\mu}\phi)}\right) - \sqrt{-g}\,\frac{\partial\mathcal{L}_{\phi}}{\partial\phi} &= 0, \\ -\frac{1}{\sqrt{-g}}\partial_{\mu}\left(\sqrt{-g}g^{\mu\nu}\,\partial_{\nu}\phi\right) + \xi R\phi + \frac{\partial V}{\partial\phi} &= 0, \\ \Box\,\phi - \xi R\phi - \frac{\partial V}{\partial\phi} &= 0, \end{aligned} \tag{4.21}$$

where $\Box = g^{\mu\nu}\nabla_{\mu}\nabla_{\nu}$, which for scalars is equivalent to $\Box\,\phi = (-g)^{-1/2}\,\partial_{\mu}\left(\sqrt{-g}g^{\mu\nu}\,\partial_{\nu}\phi\right)$.

When we write out the equation of motion for our metric in relation to α-time, we arrive at the following result.

$$\phi'' + (3-\alpha)\frac{a'}{a}\phi' - a^{-2(1-\alpha)}\nabla^2\phi = -a^{2\alpha}\left(\xi R\phi + \frac{\partial V}{\partial\phi}\right) \tag{4.22}$$

Energy-momentum tensor for the NMC scalar

The energy momentum tensor for ϕ is defined as

$$T_{\mu\nu}^{\phi} = -\frac{2}{\sqrt{-g}}\frac{\delta\left(\sqrt{-g}\mathcal{L}_{\phi}\right)}{\delta g^{\mu\nu}} = -2\frac{\delta\mathcal{L}_{\phi}}{\delta g^{\mu\nu}} + g_{\mu\nu}\mathcal{L}_{\phi} \quad (4.27)$$

The result for the energy-momentum tensor is

$$T_{\mu\nu}^{\phi} = \underbrace{\partial_{\mu}\phi\,\partial_{\nu}\phi - g_{\mu\nu}\left(\frac{1}{2}g^{\rho\sigma}\,\partial_{\rho}\phi\,\partial_{\sigma}\phi + V(\phi)\right)}_{\text{minimally coupled}} + \underbrace{\xi\left(G_{\mu\nu} + g_{\mu\nu}\,\Box - \nabla_{\mu}\nabla_{\nu}\right)\phi^{2}}_{\text{non-minimally coupled}}$$

(4.28)

where we have used $\Box = \nabla^{\sigma}\nabla_{\sigma} = g^{\rho\sigma}\nabla_{\rho}\nabla_{\sigma}$. It should be noted that the NMC coupled component includes elements that incorporate curvature terms.

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6 Appendix A

curvature perturbation correlator

Our goal is to determine the correlation that exists between two points of the curvature perturbation. To begin with, we examine the Fourier components of the curvature perturbation, which are complex numbers that represent intricate functions.

$$\alpha_1 \equiv \alpha(l) = \int \alpha(l) e^{il\cdot y} d^3y \tag{6.1}$$

The two-point correlator in momentum space is

$$\begin{aligned}
\langle \alpha_1 \alpha_{1'} \rangle &= \int d^3y d^3y' \langle \alpha(\boldsymbol{y}) \alpha(\boldsymbol{y}') \rangle e^{i(\boldsymbol{l}\cdot \boldsymbol{y} + \boldsymbol{l}'\cdot \boldsymbol{y}')} \\
&= \int d^3y d^3y' \langle \alpha(\boldsymbol{y}) \alpha(\boldsymbol{y}') \rangle e^{i(\boldsymbol{l}+\boldsymbol{l}')\cdot \boldsymbol{y}} e^{i\boldsymbol{l}'\cdot(\boldsymbol{y}'-\boldsymbol{y})} \\
&= (2\pi)^3 \delta^{(3)}(\boldsymbol{l}+\boldsymbol{l}') \int d^3y' \langle \alpha(\boldsymbol{y}) \alpha(\boldsymbol{y}') \rangle e^{i\boldsymbol{l}'\cdot(\boldsymbol{y}'-\boldsymbol{y})}
\end{aligned} \tag{6.2}$$

where $\boldsymbol{l}\cdot\boldsymbol{y} + \boldsymbol{l}'\cdot\boldsymbol{y}' = (\boldsymbol{l}+\boldsymbol{l}')\cdot\boldsymbol{y} + \boldsymbol{l}'\cdot(\boldsymbol{y}'-\boldsymbol{y})$ and the definition of the Dirac delta function in three dimensions is

$$\delta^{(3)}(\boldsymbol{q}) = \frac{1}{(2\pi)^3} \int e^{i\boldsymbol{q}\cdot\boldsymbol{y}} d^3y \tag{6.3}$$

with $\boldsymbol{q} = \boldsymbol{l} + \boldsymbol{l}'$. Because of the invariance of translation, we can make x=0. After that, the following information is provided:

$$\langle \alpha_1 \alpha_{1'} \rangle = (2\pi)^3 \delta^{(3)}(\boldsymbol{l}+\boldsymbol{l}') P_\alpha(l) \tag{6.4}$$

where we have defined

$$P_\alpha(l) = P_\alpha(\boldsymbol{l}) = \int d^3y \langle \alpha(0) \alpha(\boldsymbol{y}) \rangle e^{i\boldsymbol{l}\cdot\boldsymbol{y}} \tag{6.5}$$

We employed the following strategy to demonstrate the invariance of rotation, which led to the definition of k as $l \equiv |\boldsymbol{l}|$. We also eliminated the initials:

$$y' \to y.$$

we can also locate the two-point temporal associate.

$$\alpha(\boldsymbol{y}) = \frac{1}{(2\pi)^3}\int \alpha_l e^{-i\boldsymbol{l}\cdot\boldsymbol{y}} d^3 l. \tag{6.6}$$

$$\begin{aligned}
\langle\alpha(\boldsymbol{y})\alpha(\boldsymbol{y}')\rangle &= \frac{1}{(2\pi)^6}\int d^3l d^3l' \langle\alpha_{\mathbf{l}}\alpha\mathbf{l}\rangle e^{-i(\boldsymbol{l}\cdot\boldsymbol{y}+\boldsymbol{l}'\cdot\boldsymbol{y}')} \\
&= \frac{1}{(2\pi)^3}\int d^3l d^3l' \delta^{(3)}(\boldsymbol{l}+\boldsymbol{l}') P_\alpha(l) e^{-i(\boldsymbol{l}\cdot\boldsymbol{y}+\boldsymbol{l}'\cdot\boldsymbol{y}')} \\
&= \frac{1}{(2\pi)^3}\int d^3l P_\alpha(l) e^{-i\boldsymbol{l}\cdot(\boldsymbol{y}-\boldsymbol{y}')}
\end{aligned} \tag{6.7}$$

where $\boldsymbol{l}\cdot\boldsymbol{y}+\boldsymbol{l}'\cdot\boldsymbol{y}' = (\boldsymbol{l}+\boldsymbol{l}')\cdot\boldsymbol{y} - \boldsymbol{l}'\cdot(\boldsymbol{y}-\boldsymbol{y}')$ and Eq. (7.3) with $\boldsymbol{q} = \boldsymbol{l}+\boldsymbol{l}'$. Setting $y' = y$ we get

$$\langle\alpha^2(\boldsymbol{y})\rangle = \frac{1}{(2\pi)^3}\int P_\alpha(l) d^3 l = \int_0^\infty \mathcal{P}_\alpha(l)\frac{dl}{l} \tag{6.8}$$

and we've characterized the range of the bend in the spectrum as

$$\mathcal{P}_\alpha(l) \equiv \frac{l^3}{2\pi^2} P_\alpha(l) \tag{6.9}$$

Thus,

$$\langle\alpha_{\mathbf{l}}\alpha_{\mathbf{l}'}\rangle = (2\pi)^3\delta^{(3)}(\boldsymbol{l}+\boldsymbol{l}')\frac{2\pi^2}{l^3}\mathcal{P}_\alpha(l) \tag{6.10}$$

By examining equation (7.1), we come across a crucial fact: $\alpha_{\mathbf{l}}^* = \alpha_{-\mathbf{l}}$, which serves as the only link between the various Fourier components of the curvature field. Consequently, the perturbations of the field of curvature at any given point x can be expressed as an infinite sum of uncorrelated random variables. Therefore, statistical analysis shows that the probability of α(y) is distributed around the mean. Ultimately, this is the condition that we encounter.

$$\langle|\alpha_{\mathbf{l}}|^2\rangle = \langle\alpha_{\mathbf{l}}\alpha_{\mathbf{l}}^*\rangle = \langle\alpha_{\mathbf{l}}\alpha_{-\mathbf{l}}\rangle = (2\pi)^3\delta^{(3)}(0)P_\alpha(l) = P_\alpha(l) = \frac{2\pi^2}{l^3}\mathcal{P}_\alpha(l) \qquad (6.11)$$

Under Gaussian statistics, it is a proven fact that all odd-order higher correlators are unequivocally zero. Nevertheless, the idea that the curvature perturbation's statistics may be somewhat non-Gaussian is considered. To investigate the extent of non-Gaussianity, the three-point correlator is examined and expressed as:

$$\left\langle\alpha_{\mathbf{l}_1}\alpha_{\mathbf{l}_2}\alpha_{\mathbf{l}_3}\right\rangle = (2\pi)^3\delta^{(3)}(\boldsymbol{l}_1 + \boldsymbol{l}_2 + \boldsymbol{l}_3)B_\alpha(l_1, l_2, l_3) \qquad (6.12)$$

where $B_\alpha(l_1, l_2, l_3)$ is the bispectrum and is quantified as

$$B_\alpha(l_1, l_2, l_3) = \frac{6}{5}f_{\mathrm{NL}}(l_1, l_2, l_3)[P_\alpha(l_1)P_\alpha(l_2) + P_\alpha(l_2)P_\alpha(l_3) + P_\alpha(l_3)P_\alpha(l_1)]. \qquad (6.13)$$

The non-linearity parameter, denoted as f_{NL}, is a quantity that relies on the arrangement of the $\boldsymbol{l}_1, \boldsymbol{l}_2$, and $\boldsymbol{l}_3$ momentum vectors. The stringent observational limitations placed on f_{NL} validate the highly Gaussian nature of the curvature perturbation. In fact, these constraints are so rigorous that f_{NL} is limited almost to the point of observability.

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