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Semigroups of sets generated by non-Lebesgue measurable subsets of the real line

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A Dissertation Submitted to the Directorate of Research and Graduate Training in Partial Fulfilment of the Requirements for the Award of the Degree of Master of Science in Mathematics

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Dedication

This work is dedicated to both the promotion of my master's degree and my family of origin, which includes my parents, brothers, sisters, and family friends, for their unwavering support and care throughout my life and all of my aspirations for success. This project is also dedicated to my supervisor, whom I want to thank for his encouragement and assistance during my academic career. May the Almighty reward you all. Finally, may the Almighty bless my professors, who played an important role in my accomplishment, and I dedicate this thesis to them.

Declaration

I, Joseline MUNYANEZA, hereby declare that, to the best of my knowledge, this thesis is my own original work. It has not been presented elsewhere for an academic award. The references used are mentioned as recommended.

Some of the results presented in this thesis are also available in the joint paper available online via the link: `https://arxiv.org/abs/2105.11810`. We have submitted this paper for a possible publication.

This work was done under the supervision of Dr. Venuste NYAGAHAKWA.

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Abstract

Let $(\mathbb{R}, +)$ be the additive group of real numbers. The collection $\mathcal{P}(\mathbb{R})$ of all subsets of $\mathbb{R}$ can be decomposed into two disjoint subfamilies, namely, the family $\mathcal{L}(\mathbb{R})$ of all Lebesgue measurable subsets of $\mathbb{R}$ and the family $\mathcal{L}^c(\mathbb{R})$ of all non-Lebesgue measurable subsets of $\mathbb{R}$. The algebraic structure, from the set-theoretical point of view, of the family $\mathcal{L}(\mathbb{R})$ is well known. On the other hand, the family $\mathcal{L}^c(\mathbb{R})$ does not have a well-defined structure from the set-theoretic point of view. In this thesis, we construct subfamilies of the collection $\mathcal{L}^c(\mathbb{R})$, having an algebraic structure of being semigroups of sets. These semigroups are constructed by using the two classical examples of sets that are not measurable in the Lebesgue sense: Vitali selectors of $\mathbb{R}$ and Bernstein subsets of $\mathbb{R}$. In particular, we show that the family $(\mathcal{S}(\mathcal{B}) \vee \mathcal{S}(\mathcal{V})) * \mathcal{N}_0 := \{((U_1 \cup U_2) \setminus N) \cup M : U_1 \in \mathcal{S}(\mathcal{B}), U_2 \in \mathcal{S}(\mathcal{V}), N, M \in \mathcal{N}_0\}$ is a semigroup of sets, which is invariant under translations, and consists of sets which are not measurable in the Lebesgue sense. Here, $\mathcal{S}(\mathcal{B})$ is the collection of all finite unions of some type of Bernstein subsets of $\mathbb{R}$; $\mathcal{S}(\mathcal{V})$ is the collection of all finite unions of Vitali selectors of $\mathbb{R}$; and $\mathcal{N}_0$ is the σ-ideal of all subsets of $\mathbb{R}$ having the Lebesgue measure.

Certification

This is to certify that the thesis on **Semigroups of sets generated by non-Lebesgue measurable subsets of the real line** is a record of the original work done by **Joseline MUNYANEZA** (Reg. Number: 221005842) in partial fulfilment of the requirements for the award of a Master's degree in Applied Mathematics at the University of Rwanda, College of Science and Technology, during the academic year 2023-2024.

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Date: . . . .

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Chapter 1

General introduction

This chapter discusses the thesis's introduction, problem statement, research objectives, motivation and research methodology as well as the structure of this thesis.

1.1 Introduction

In mathematical analysis, a measure on a set is a systematic way to assign a number to each suitable subset of that set, intuitively interpreted as its size. In this sense, a measure is a generalization of the concepts of length, area, and volume.

One of the most important measures that are frequently used in mathematics is the Lebesgue measure. The Lebesgue measure, named after the French mathematician Henri Lebesgue, is the standard way of assigning a measure to subsets of n-dimensional Euclidean space $\mathbb{R}^n, n \geq 1$. For $n = 1, 2$, or 3 the Lebesgue measure coincides with the standard measure of length, area, or volume. In general, it is also called n-dimensional volume, n-volume, or simply volume [5]. The Lebesgue measure is used throughout real analysis, in particular, to define Lebesgue integration. The Lebesgue measure is important for measure theory and the theory of integrals. The sets that can be assigned a Lebesgue measure are called Lebesgue-measurable sets.

Let $\mathcal{P}(\mathbb{R})$ be the family of all subsets of $\mathbb{R}$. An element A of $\mathcal{P}(\mathbb{R})$ is said to be Lebesgue measurable if for any $E \subseteq \mathbb{R}$, the following equality holds [5], [4]:

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \setminus A) \text{ where,}$$

$\mu^*(A) = \inf \left\{ \sum_{n=1}^{\infty} \ell(I_n) : \ell(I) \text{ is the length of the interval }, I_n \text{ are open intervals}, A \subseteq \bigcup_{n=1}^{\infty} I_n \right\}$

is the Lebesgue outer measure of A. Standard subsets of $\mathbb{R}$ have a well-defined Lebesgue measure. This implies that the family $\mathcal{L}(\mathbb{R})$ of all Lebesgue measurable subsets of $\mathbb{R}$ is rich enough. In particular, this family is closed under all basic set operations, it is a $\sigma-$ algebra of sets, and it is invariant under translations. The latter means that if $A \in \mathcal{L}(\mathbb{R})$ and $t \in \mathbb{R}$ then the set $A + t := \{a + t : a \in A\}$ is also an element of $\mathcal{L}(\mathbb{R})$ and $\mu(A) = \mu(A + t)$. Let us point out that the standard sets that we meet in our everyday life are all Lebesgue measurable.

1.2 Problem statement

Even if the family $\mathcal{L}(\mathbb{R})$ of all Lebesgue measurable subsets of $\mathbb{R}$ is rich enough, it doesn't contain every element of $\mathcal{P}(\mathbb{R})$. This means that there exist subsets of $\mathbb{R}$ that are not measurable in the Lebesgue sense [8]. Two important examples of subsets of $\mathbb{R}$ that are not measurable in the Lebesgue sense are Vitali selectors and Bernstein sets [6], [10]. Accordingly, the family $\mathcal{P}(\mathbb{R})$ can be written as a disjoint union of two non-empty families $\mathcal{L}(\mathbb{R})$ and its complement $\mathcal{L}^c(\mathbb{R})$. That is, $\mathcal{P}(\mathbb{R}) = \mathcal{L}(\mathbb{R}) \cup \mathcal{L}^c(\mathbb{R})$, where $\mathcal{L}^c(\mathbb{R}$ is the family of all subsets of $\mathbb{R}$ that are not measurable in the Lebesgue sense.

The algebraic structure of the family $\mathcal{L}(\mathbb{R})$, from the set-theoretical point of view, is well known. Indeed, $\mathcal{L}(\mathbb{R})$ is a σ-algebra of sets. This means that both $\emptyset$ and $\mathbb{R}$ are elements of $\mathcal{L}(\mathbb{R})$; if $A \in \mathcal{L}(\mathbb{R})$ then the complement $\mathbb{R} \setminus A$ of A in $\mathbb{R}$ is also in $\mathcal{L}(\mathbb{R})$; and if $A_1, A_2, \cdots$ is a collection of elements in $\mathcal{L}(\mathbb{R})$ then the union $\bigcup_{i=1}^{\infty} A_i$ is also in $\mathcal{L}(\mathbb{R})$. Hence, family $\mathcal{L}(\mathbb{R})$ is closed under the operation of taking the complement, and it is closed under countable unions of sets [4] and [5]. As a consequence, the family $\mathcal{L}(\mathbb{R})$ is closed under all basic set operations, and in particular, finite union (resp. intersection, difference, symmetric difference) of elements in $\mathcal{L}(\mathbb{R})$ produces again elements of $\mathcal{L}(\mathbb{R})$. Apart from being closed under all basic set operation, the family $\mathcal{L}(\mathbb{R})$ is invariant under translations of $\mathbb{R}$. This means that if $A \in \mathcal{L}(\mathbb{R})$ and $t \in \mathbb{R}$ then $A + t$ is an element of $\mathcal{L}(\mathbb{R})$ and the sets A and $A + t$ have the same Lebesgue measure.

Unlike $\mathcal{L}(\mathbb{R})$, the family $\mathcal{L}^c(\mathbb{R})$ does not have a good structure from the set-theoretic point of view. Indeed, if A and B are elements of $\mathcal{L}^c(\mathbb{R})$ then the union $A \cup B$ can be inside or outside of $\mathcal{L}^c(\mathbb{R})$. In a similar way, if A and B are two elements of $\mathcal{L}^c(\mathbb{R})$ then the intersection $A \cap B$, the difference $A \setminus B$, and the symmetric difference $A \Delta B$ can be inside or outside of the family $\mathcal{L}^c(\mathbb{R})$. This implies that the family $\mathcal{L}^c(\mathbb{R})$ is not closed under all basic set operations, and hence the family cannot be an algebra of sets. However, like the family $\mathcal{L}(\mathbb{R})$, the family $\mathcal{L}^c(\mathbb{R})$ is invariant under translation; that is, if A is an element of $\mathcal{L}^c(\mathbb{R})$ then the set $A + t$ is also an element of $\mathcal{L}^c(\mathbb{R})$ for any given real number t.

In this thesis, we are mainly interested in the following question:

Question 1.2.1: Could we find in the family $\mathcal{L}^c(\mathbb{R})$ subfamilies rich enough and which have some algebraic structures from the set-theoretic point of view?

Question 1.2.1 is related to the questions treated in [13] and [15], where the authors were looking for subfamilies of $\mathcal{L}^c(\mathbb{R})$ containing the collection of all Vitali selections of $\mathbb{R}$ and having some algebraic structures from the set-theoretic point of view.

To answer this question, we consider the family of all Vitali selectors of $\mathbb{R}$ and the family of some type of Bernstein sets of $\mathbb{R}$, and then we construct subfamilies of $\mathcal{L}^c(\mathbb{R})$ having an algebraic structure of semigroups of sets. In addition, the constructed families are invariant under the group $\Phi(\mathbb{R})$ of all translations of $\mathbb{R}$ into itself.

1.3 Objectives

The main objective of this thesis is to construct new families of sets on $\mathbb{R}$ having an algebraic structure from the set-theoretic point of view and for which each element is not measurable in the Lebesgue sense.

The specific objectives are the following:

1. To construct semigroups of sets on $\mathbb{R}$ for which each element is not measurable in the Lebesgue sense, by using Vitali selectors of $\mathbb{R}$.
2. To construct semigroups of sets on $\mathbb{R}$ for which each element is not measurable in the Lebesgue sense, by using Bernstein subsets of $\mathbb{R}$.
3. To construct semigroups of sets on $\mathbb{R}$ for which each element is not measurable in the Lebesgue sense, by using both Vitali selectors and Bernstein subsets of $\mathbb{R}$.

1.4 Motivation

This thesis explores the existence of certain sets on the real line that challenge our intuitive ideas about what can be measured. The two sets that are considered are

Vitali selectors and Bernstein sets of the real line. The Vitali theorem states the existence of Vitali selectors proved in 1905 by Giuseppe Vitali by using the Axiom of Choice, while Bernstein sets were constructed by Felix Bernstein in 1908 by using the method of transfinite induction. Many years have passed since these remarkable sets were constructed but they remain of living interest in Mathematics. They have stimulated further development of many branches of mathematics such as the paradoxical decomposition of simple sets in Euclidean spaces, the theory of large cardinals, the theory of invariant extensions of invariant measures, etc. Therefore, it is very important to study the algebraic structures of these two mathematical concepts that have contradicted the human natural intuition of measurability.

1.5 Research methodology

To achieve the objectives of this thesis, we followed a structured approach. We began by defining the problem under investigation and conducting a comprehensive review of relevant literature, including research articles and books on similar topics. We were focusing on non-Lebesgue-measurable sets on the real line; these are sets that cannot be measured using the conventional Lebesgue measure, as referenced in [6]. These sets predominantly arise under the assumption of the Axiom of Choice; a foundational principle in set theory that allows for the selection of elements from an infinite collection of sets, leading to the construction of sets with properties that defy standard measurability.

The existence of such non-measurable sets underscores the inherent challenges and limitations within measure theory, highlighting the importance of understanding these constraints for our research. To fully appreciate the scope and limitations of measure theory, we delved into specific non-measurable sets, such as Vitali selections and Bernstein sets, as cited in [8, 9].

Starting from the notion of non-Lebesgue measurable sets, as described in [1], the thesis investigates different algebraic structures that can be built from non-Lebesgue-measurable subsets of the real line. As mentioned in the literature [15], the study concentrates on the semigroups of Vitali sets. It then develops a theory of semigroups and ideals of sets, bringing in new ideas that were not touched upon before. We then apply this established theory to Bernstein sets of the real numbers and to Vitali selectors. The final step involves constructing semigroups of sets by combining Vitali sets with Bernstein sets, ensuring that the resulting families are non-measurable and possess desirable algebraic properties, such as being closed under union and invariant under translations.

1.6 Thesis structure

The first chapter begins with an overview of the Lebesgue measure. Then the chapter develops the problem under consideration and the objectives of the thesis, together with the structure of the thesis.

The second chapter is devoted to the concepts of algebraic structures that are used in set theory. We consider mostly the concept of a semigroup, rings, and algebra of sets. We present different ways of showing how ideals of sets can be used in extending given semigroups of sets.

The third chapter deals with the theory related to the Lebesgue measure on the real line. The construction of the Lebesgue measure by using the Lebesgue outer measure, and the properties of the Lebesgue measure are mostly discussed.

The fourth chapter is concerned with non-Lebesgue measurable subsets of the real line. Two classical examples of such sets are constructed, namely, the Vitali selectors and Bernstein sets. Vitali selectors are constructed under the assumption of the Axiom of Choice, while Bernstein sets are constructed by using the transfinite recursion.

The fifth chapter, which is the core part of the thesis, discusses different examples of families for which each element is not measurable in the Lebesgue sense. Those families are constructed by using Vitali selectors and Bernstein subsets of the real line together with the σ-ideal of all Lebesgue measurable subsets of the real line having Lebesgue measure zero.

The thesis ends with a conclusion and a list of recommendations that can be taken into consideration for further studies in the same direction.

Chapter 2

Basic algebraic structures in set theory

The purpose of this chapter is to introduce basic algebraic structures; from the set theoretic point of view, which will be used in the sequel. We mainly consider the concept of rings of sets, algebra of sets, semigroups of sets, and ideal of sets. We further study the behavior of semigroups of sets under different binary operations between them.

2.1 Rings and algebra of sets

This section will provide a basic introduction to the families of sets having algebraic properties. A family of sets is any set whose constituents are also sets. Set families are denoted by capital script letters, such as $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and so forth. For more information, we suggest to read these references [2] and [5].

For a non-empty set X, let $\mathcal{P}(X)$ be the family of all subsets of X.

Definition 2.1.1: A non-empty family $\mathcal{R} \subseteq \mathcal{P}(X)$ of sets is called *a ring of sets* on X if $A\Delta B \in \mathcal{R}$ and $A \cap B \in \mathcal{R}$ whenever $A \in \mathcal{R}$ and $B \in \mathcal{R}$.

Since $A \cup B = (A\Delta B)\Delta(A \cap B)$ and $A \setminus B = A\Delta(A \cap B)$, we have also $A \cup B \in \mathcal{R}$ and $A \setminus B \in \mathcal{R}$ whenever $A \in \mathcal{R}$ and $B \in \mathcal{R}$. Thus a ring of sets is a family of sets that is closed under the operations of taking symmetries, unions, intersections, and differences. In addition a ring of sets must contain the empty set $\emptyset$, since $A \setminus A = \emptyset$.

Definition 2.1.2: Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be ring of sets on X. If $X \in \mathcal{A}$, the family $\mathcal{A}$ is called an *algebra of sets* on X.

According to this definition, a ring of sets is an algebra if and only if it is closed under complement.

Example 2.1.3: (i) The family of all finite subsets of X is a ring on X but not an algebra on X unless X is finite.

(ii) Let $\mathbb{R}$ be the set of real numbers. The family of all bounded subsets of $\mathbb{R}$ is a ring on $\mathbb{R}$ but not an algebra.

Definition 2.1.4: (a) A ring $\mathcal{R} \subseteq \mathcal{P}(X)$ is called a *σ-ring of sets* on X if it is closed under countable unions, i.e. it contains the union $S = \bigcup_{n=1}^{\infty} A_n$ whenever it contains the sets $A_1, A_2, \cdots$

(b) A σ-ring $\mathcal{A} \subseteq \mathcal{P}(X)$ is called a σ-algebra of sets on X if $X \in \mathcal{A}$. From the De Morgan formula $\bigcap_{n=1}^{\infty} A_n = X \setminus \bigcup_{n=1}^{\infty}(X \setminus A_n)$, it follows that each σ-algebra is also closed under countable intersection of sets. Note that a σ-algebra can be defined as an algebra closed under countable unions.

Example 2.1.5: For an arbitrary set Z, the family of all countable subsets of Z is a σ-ring. It will be a σ-algebra if Z is countable.

Definition 2.1.6: Let $\mathcal{A} \subseteq \mathcal{P}(X)$. The smallest σ-algebra of sets on X containing $\mathcal{A}$ is called the *σ-algebra generated* by the family $\mathcal{A}$.

2.1.1 Semigroups and ideal of sets

Families of sets, like algebras or rings of sets, are of fundamental importance in mathematics. In this section, we are mostly interested in other mathematical objects called semigroups and ideals of sets.

Definition 2.1.7: A non-empty set $\mathcal{S}$ is called *a semigroup* if there is a binary operation $\star : \mathcal{S} \times \mathcal{S} \to \mathcal{S}$ for which the associativity law is satisfied, i.e. the equality $(x \star y) \star z = x \star (y \star z)$ holds for all $x, y, z \in \mathcal{S}$. The semigroup $\mathcal{S}$ is called *abelian* if $x \star y = y \star x$ for all $x, y \in \mathcal{S}$.

Consider a family of sets $\mathcal{S} \subseteq \mathcal{P}(X)$ such that for each pair of elements $A, B \in \mathcal{S}$ we have $A \cup B \in \mathcal{S}$. Because the union of sets is both commutative and associative, such a family of sets will be an abelian semigroup in terms of the union operation. This observation leads to the definition below.

Definition 2.1.8: A non-empty family of sets $\mathcal{S} \subseteq \mathcal{P}(X)$ is called *a semigroup of sets* on X if it is closed under finite unions; that is, if $A \in \mathcal{S}$ and $B \in \mathcal{S}$ then $A \cup B \in \mathcal{S}$. If $\mathcal{S}$ is closed under countable unions then it is said to be a σ-semigroup of sets on X.

It is evident that if $\mathcal{S}$ is a semigroup of sets on X with respect to the operation of union of sets then the collection $\{X \setminus S : S \in \mathcal{S}\}$ of all complements of elements of $\mathcal{S}$ in X, is closed under finite intersection of sets, and thus, it is a semigroup of sets with respect to the set-theoretic operation of intersection of sets on X.

Definition 2.1.9: (a) A family $\mathcal{I} \subseteq \mathcal{P}(X)$ of sets is called an *ideal of sets* on X, if it satisfies the following two conditions:

(i) If $D \in \mathcal{I}$ and $E \in \mathcal{I}$, then $D \cup E \in \mathcal{I}$.

(ii) If $D \in \mathcal{I}$ and $E \subseteq D$, then $E \in \mathcal{I}$.

(b) If an ideal of sets $\mathcal{I}$ is closed under countable unions, then it is called a *σ-ideal* of sets on X.

Example 2.1.10: Consider a subset D of a set X. The collection $\mathcal{I}(D) = \{E : E \subseteq D\}$ of all subset of D is a σ-ideal of sets on X. The family $\mathcal{I}_c$ of all countable subsets of X form a σ-ideal of sets on X. The family $\mathcal{I}_f$ of all finite subsets of X forms an ideal of sets on X, but not σ-ideal of sets, whenever X is infinite.

Example 2.1.11 ([13]): If $\mathcal{A} \subseteq \mathcal{P}(X)$ is a non-empty family of sets, consider the collection $\mathcal{S}(\mathcal{A}) = \{\bigcup_{i=1}^{n} A_i : A_i \in \mathcal{A}, n \in \mathbb{N}\}$ of all finite unions of elements of $\mathcal{A}$ and consider the collection $\mathcal{I}(\mathcal{A}) = \{B \in \mathcal{P}(X)$ such that there is $A \in \mathcal{S}(\mathcal{A}) : B \in A\}$. It is clear that family $\mathcal{S}(\mathcal{A})$ is a semigroup of sets, while the collection $\mathcal{I}(\mathcal{A})$ is an ideal of sets on X. The family $\mathcal{S}(\mathcal{A})$ is called *the semigroup of sets generated by* $\mathcal{A}$ while $\mathcal{I}(\mathcal{A})$ is called *the ideal of sets generated by* $\mathcal{A}$. Evidently the inclusions $\mathcal{S}(\mathcal{A}) \subseteq \mathcal{I}(\mathcal{A})$ and $\mathcal{A} \subseteq \mathcal{S}(\mathcal{A})$ hold. If $\mathcal{A}$ is a semigroup of sets then we get the equality $\mathcal{S}(\mathcal{A}) = \mathcal{A}$.

Remark 2.1.1: Using the definition of a semigroup of sets, we can redefine the concept of an ideal of sets in the following manner: A non-empty family $\mathcal{I} \subseteq \mathcal{P}(X)$ is *an ideal* of sets on X if and only if it is a semigroup of sets on X, and if $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$.

Proposition 2.1.12 ([19]): The family $\mathcal{S}(\mathcal{A})$ is a semigroup of sets on X and the family $\mathcal{I}(\mathcal{A})$ is an ideal of sets on X.

For further development on semigroups, we consider the following two binary operations between families of sets, as they were introduced in [11]. Namely, if $\mathcal{A}$ and $\mathcal{B}$ are subfamilies of $\mathcal{P}(X)$, we consider: $\mathcal{A} \vee \mathcal{B} = \{A \cup B : A \in \mathcal{A}, B \in \mathcal{B}\}$ and $\mathcal{A} * \mathcal{B} = \{(A \setminus B_1) \cup B_2 : A \in \mathcal{A}; B_1, B_2 \in \mathcal{B}\}$, where $*$, $\vee$, $\cup$ and $\setminus$ are the usual set operations of junction, conjunction, union and difference of sets, respectively.

For the defined operations, we observe the following: Since the union is a commutative operation, it follows that $\mathcal{A} \vee \mathcal{B} = \mathcal{B} \vee \mathcal{A}$. From the fact that $A \cup B = (A \setminus B) \cup B = (B \setminus A) \cup A$, it follows that $\mathcal{A} \vee \mathcal{B} \subseteq \mathcal{A} * \mathcal{B}$ and $\mathcal{A} \vee \mathcal{B} \subseteq \mathcal{B} * \mathcal{A}$.

Remark 2.1.2: One can observe that the binary operation $\vee$ is both commutative and associative. However, the operation $*$ is not associative nor commutative, as it is illustrated in the following example.

Example 2.1.13: Let F and G be two non-empty subsets of X such that $F \subseteq G$. Consider the families of sets $\mathcal{F} = \{\emptyset, F\}, \mathcal{G} = \{G\}$, and $\mathcal{C} = \{\emptyset, X\}$. Then we have $\mathcal{F} * \mathcal{G} = \{G\}$ while $\mathcal{G} * \mathcal{F} = \{G, G \setminus F\}$. It is clear also that $(\mathcal{F} * \mathcal{G}) * \mathcal{C} = \{\emptyset, G, X\}$ while $\mathcal{F} * (\mathcal{G} * \mathcal{C}) = \{\emptyset, F, G, X\}$.

It is clear that if $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and $\mathcal{D}$ are families of sets on X such that $\mathcal{A} \subseteq \mathcal{B}, \mathcal{C} \subseteq \mathcal{D}$ then $\mathcal{A} \vee \mathcal{C} \subseteq \mathcal{B} \vee \mathcal{D}$. For any family of sets $\mathcal{A}$ on X, the inclusion $\mathcal{A} \subseteq \mathcal{A} \vee \mathcal{A}$ always hold. If $\mathcal{A}$ is a semigroup of sets on X, then $\mathcal{A} \vee \mathcal{A} = \mathcal{A}$. The inclusion $\mathcal{A} \subseteq \mathcal{A} \vee \mathcal{B}$ and $\mathcal{B} \subseteq \mathcal{A} \vee \mathcal{B}$ do not need to hold for any families $\mathcal{A}$ and $\mathcal{B}$ with or without the assumption of being semigroups. If $\mathcal{S}_1$ and $\mathcal{S}_2$ are semigroups of sets; then the union $\mathcal{S}_1 \cup \mathcal{S}_2$ does not need to be a semigroups of sets, however the following lemma holds.

Lemma 2.1.14: If $\mathcal{S}_1$ and $\mathcal{S}_2$ are semigroups of sets on X, then the family $\mathcal{S}_1 \vee \mathcal{S}_2$ is also a semigroup of sets on X.

Proof. Assume that $U_1 \in \mathcal{S}_1 \vee \mathcal{S}_2$ and $U_2 \in \mathcal{S}_1 \vee \mathcal{S}_2$. Then $U_1 = A_1 \cup A_2$ and $U_2 = B_1 \cup B_2$ for some $A_1, B_1 \in \mathcal{S}_1$ and $A_2, B_2 \in \mathcal{S}_2$, Note that $A_1 \cup B_1 \in \mathcal{S}_1$ and $A_2 \cup B_2 \in \mathcal{S}_2$ as $\mathcal{S}_1$ and $\mathcal{S}_2$ are semigroups of sets. It follows that $U_1 \cup U_2 = (A_1 \cup A_2) \cup (B_1 \cup B_2) = (A_1 \cup B_1) \cup (A_2 \cup B_2)$ is an element of $\mathcal{S}_1 \vee \mathcal{S}_2$ □

If $\mathcal{A}$ and $\mathcal{B}$ are families of sets on X then, the equality $\mathcal{S}(\mathcal{A}) \cup \mathcal{S}(\mathcal{B}) = \mathcal{S}(\mathcal{A} \cup \mathcal{B})$ does not need to hold. This means that, the union of the semigroups generated by a given families of sets is not always equal to the semigroups generated by the union of a given family of sets.

Example 2.1.15: On the set $X = \{a, b, c, d\}$, let $A = \{a, b\}, B = \{b, c\}$ and $D = \{c, d\}$, and consider the families $\mathcal{A} = \{A\}$ and $\mathcal{B} = \{B, D\}$. Note that $\mathcal{A} \cup \mathcal{B} = \{A, B, D\}$, $\mathcal{S}(\mathcal{A}) = \mathcal{A}$ and $\mathcal{S}(\mathcal{B}) = \{B, D, B \cup D\}$. It is clear that $\mathcal{S}(\mathcal{A}) \cup \mathcal{S}(\mathcal{B}) = \{A, B, D, B \cup D\}$ while $\mathcal{S}(\mathcal{A} \cup \mathcal{B}) = \{A, B, D, A \cup B, A \cup D, B \cup D, A \cup B \cup D\} = \{A, B, D, A \cup B, B \cup D, X\}$. Since $\mathcal{A} \vee \mathcal{B} = \{A \cup B, A \cup D\} = \{A \cup B, X\}$, we further remark that $\mathcal{S}(\mathcal{A}) \vee \mathcal{S}(\mathcal{B}) = \mathcal{S}(\mathcal{A} \vee \mathcal{B}) = \{A \cup B, A \cup D, A \cup B \cup D\} = \{A \cup B, X\}$.

Lemma 2.1.16: If $\mathcal{A}$ and $\mathcal{B}$ are non-empty families of sets on X then the equality $\mathcal{S}(\mathcal{A} \vee \mathcal{B}) = \mathcal{S}(\mathcal{A}) \vee \mathcal{S}(\mathcal{B})$ always holds.

Proof. Assume that $Y \in \mathcal{S}(\mathcal{A} \vee \mathcal{B})$. Then $Y = \bigcup_{i=1}^{n} Y_i$ where $Y_i \in \mathcal{A} \vee \mathcal{B}$, i.e. $Y_i = A_i \cup B_i$ with $A_i \in \mathcal{A}$, $B_i \in \mathcal{B}$ and $n \in \mathbb{N}$. Hence $Y = \bigcup_{i=1}^{n} (A_i \cup B_i) = (\bigcup_{i=1}^{n} A_i) \cup (\bigcup_{i=1}^{n} B_i)$. Put $A = \bigcup_{i=1}^{n} A_i$ and $B = \bigcup_{i=1}^{n} B_i$. It is clear that $A \in \mathcal{S}(\mathcal{A})$ and $B \in \mathcal{S}(\mathcal{B})$, and hence $Y \in \mathcal{S}(\mathcal{A}) \vee \mathcal{S}(\mathcal{B})$ implying that $\mathcal{S}(\mathcal{A} \vee \mathcal{B}) \subseteq \mathcal{S}(\mathcal{A}) \vee \mathcal{S}(\mathcal{B})$.

Assume that $Y \in \mathcal{S}(\mathcal{A}) \vee \mathcal{S}(\mathcal{B})$. Then $Y = A \cup B$ where $A \in \mathcal{S}(\mathcal{A})$ and $B \in \mathcal{S}(\mathcal{B})$, i.e. that $A = \bigcup_{i=1}^{n} A_i$ and $B = \bigcup_{i=1}^{m} B_i$ where $A_i \in \mathcal{A}$ and $B_i \in \mathcal{B}$ for some n and m in $\mathbb{N}$. We will have two cases to consider:

- If $n = m$ then $Y = \bigcup_{i=1}^{n} (A_i \cup B_i)$ and hence $Y \in \mathcal{S}(\mathcal{A} \vee \mathcal{B})$.
- If $n \neq m$, without loosing generality, assume that $n < m$. We can write $Y = [\bigcup_{i=1}^{n} (A_i \cup B_i)] \cup \left(\bigcup_{i=n+1}^{m} B_i\right)$. For $i = n+1, n+2, \cdots, m$, put $A_i = A_k$, where k is some fixed integer in the set $\{1, 2, \cdots, n\}$. It follows that $Y = [\bigcup_{i=1}^{n} (A_i \cup B_i)] \cup \left[\bigcup_{i=n+1}^{m} (A_i \cup B_i)\right] = \bigcup_{i=1}^{m} (A_i \cup B_i)$. Since $A_i \cup B_i \in \mathcal{A} \vee \mathcal{B}$ for $i = 1, 2, \cdots, m$, it follows that $Y \in \mathcal{S}(\mathcal{A} \vee \mathcal{B})$, and thus $\mathcal{S}(\mathcal{A}) \vee \mathcal{S}(\mathcal{B}) \subseteq \mathcal{S}(\mathcal{A} \vee \mathcal{B})$. □

2.1.2 Operations between semigroups and ideal of sets

In this section, we consider different mathematical operations between semigroups of sets and ideals of sets. In particular, we present a way of extending a given semigroup of sets by using an ideal of sets.

Proposition 2.1.17 ([14]): Let $\mathcal{S}$ be a semigroup of sets on X and let $\mathcal{I}$ be an ideal of sets on X. Then the families $\mathcal{I} * \mathcal{S}$ and $\mathcal{S} * \mathcal{I}$ are semigroups of sets on X such that $\mathcal{S} \subseteq \mathcal{I} * \mathcal{S} \subseteq \mathcal{S} * \mathcal{I}$. Moreover, $\mathcal{I} * (\mathcal{I} * \mathcal{S}) = \mathcal{I} * \mathcal{S}$ and $(\mathcal{S} * \mathcal{I}) * \mathcal{I} = \mathcal{S} * \mathcal{I}$.

The following corollary presents a way of extending a given ideal of sets by using two existing ideals of sets.

Corollary 2.1.18 ([19]): Let $\mathcal{I}_1, \mathcal{I}_2$ be two ideals of sets on the same set X. Then the family $\mathcal{I}_1 \star \mathcal{I}_2$ is an ideal of sets on X such that $\mathcal{I}_i \subseteq \mathcal{I}_1 * \mathcal{I}_2 = \mathcal{I}_2 * \mathcal{I}_1 = \mathcal{I}_1 \cup \mathcal{I}_2$ for $i = 1, 2$.

The following example, shows that if $\mathcal{S}$ is a semigroup of sets and $\mathcal{I}$ is an ideal of sets on the same set X, the family $\mathcal{S} * \mathcal{I}$ does not need to be an ideal of sets.

Example 2.1.19: Let $X = \{3, 2\}, D = \{X\}, E = \{3\}, C = \{2\}, \mathcal{D} = \{D\}, \mathcal{E} = \{E\}$. Note that $\mathcal{S}(\mathcal{D}) = \{D\}, \mathcal{S}(\mathcal{E}) = \{E\}, \mathcal{I}(\mathcal{E}) = \{\emptyset, E\}, \mathcal{S}(\mathcal{D}) * \mathcal{I}(\mathcal{E}) = \{D, C\}$ and $\mathcal{I}(\mathcal{E}) * \mathcal{S}(\mathcal{D}) = \{D\}$. It follows that if $\mathcal{S}$ is a semigroup of sets and an ideal of sets on X then the equality $\mathcal{S} * \mathcal{I} = \mathcal{I} * \mathcal{S}$ and the inclusion $\mathcal{I} \subseteq \mathcal{I} \subseteq \mathcal{S} * \mathcal{I}$ do not need to hold. In addition, the family $\mathcal{S} * \mathcal{I}$ does not need to be an ideal of sets on X.

Lemma 2.1.16 is generalized by the following preposition, for any finite collection of families of sets.

Proposition 2.1.20: Let $\mathcal{A}_i$ be a non-empty family of sets on X, where $i = 1, 2, \cdots, n$, for some $n \in \mathbb{N}$ then we have $\mathcal{S}(\bigvee_{i=1}^{n} \mathcal{A}_i) = \bigvee_{i=1}^{n} \mathcal{S}(\mathcal{A}_i)$.

It follows from Lemma 2.1.17 and Lemma 2.1.14 that if $\mathcal{S}_1$ and $\mathcal{S}_2$ are semigroup of sets then the families $(\mathcal{S}_1 \vee \mathcal{S}_2) * \mathcal{I}$ and $\mathcal{I} * (\mathcal{S}_1 \vee \mathcal{S}_2)$ are semigroups of sets for any ideal of sets $\mathcal{I}$ on X. However, no one of the inclusions $\mathcal{S}_1 * \mathcal{I} \subseteq (\mathcal{S}_1 \vee \mathcal{S}_2) * \mathcal{I}$, $\mathcal{S}_2 * \mathcal{I} \subseteq (\mathcal{S}_1 \vee \mathcal{S}_2) * \mathcal{I}$, $\mathcal{I} * \mathcal{S}_1 \subseteq \mathcal{I} * (\mathcal{S}_1 \vee \mathcal{S}_2)$ and $\mathcal{I} * \mathcal{S}_2 \subseteq \mathcal{I} * (\mathcal{S}_1 \vee \mathcal{S}_2)$ need to hold.

Example 2.1.21: Let X be a non-empty set having at least three elements, i.e. $Card(X) \geq 3$, and let A be a non-empty proper subset of X. Let $B = X \setminus A$. Consider $\mathcal{S}_1 = \{A, X\}$; $\mathcal{S}_2 = \{B, X\}$. Consider the ideals of sets $\mathcal{I} = \mathcal{P}(A)$ and $\mathcal{K} = \mathcal{P}(B)$. It is clear that $\mathcal{S}_1 \vee \mathcal{S}_2 = \{X\}$, and $\emptyset$ and A cannot be elements of $(\mathcal{S}_1 \vee \mathcal{S}_2) * \mathcal{I}$ but $\emptyset$ and A are elements of $\mathcal{S}_1 * \mathcal{I}$. Similarly, the collection $\mathcal{S}_2 * \mathcal{K}$ contains the elements $\emptyset$ and B, but the family $(\mathcal{S}_1 \vee \mathcal{S}_2) * \mathcal{K}$ cannot contain $\emptyset$ and B. Hence $\mathcal{S}_1 * \mathcal{I} \nsubseteq (\mathcal{S}_1 \vee \mathcal{S}_2) * \mathcal{I}$, and $\mathcal{S}_2 * \mathcal{K} \nsubseteq (\mathcal{S}_1 \vee \mathcal{S}_2) * \mathcal{K}$. Further, observe that $\mathcal{I} * (\mathcal{S}_1 \vee \mathcal{S}_2) = \{X\} = \mathcal{K} * (\mathcal{S}_1 \vee \mathcal{S}_2)$. The semigroup $\mathcal{I} * \mathcal{S}_2$ contains the set B and the semigroup $\mathcal{K} * \mathcal{S}_1$ contains the set A. Hence the $\mathcal{I} * \mathcal{S}_2 \nsubseteq (\mathcal{S}_1 \vee \mathcal{S}_2)$ and $\mathcal{K} * \mathcal{S}_1 \nsubseteq \mathcal{K} * (\mathcal{S}_1 \vee \mathcal{S}_2)$.

Proposition 2.1.22: Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be the semigroup of sets on X. If $\mathcal{I}$ is an ideal of sets on X then the following equalities hold:

(i) $(\mathcal{S}_1 \vee \mathcal{S}_2) * \mathcal{I} = (\mathcal{S}_1 * \mathcal{I}) \vee (\mathcal{S}_2 * \mathcal{I})$.

(ii) $\mathcal{I} * (\mathcal{S}_1 \vee \mathcal{S}_2) = (\mathcal{I} * \mathcal{S}_1) \vee (\mathcal{I} * \mathcal{S}_2)$.

Proof. (i) Assume that $A \in (\mathcal{S}_1 \vee \mathcal{S}_2) * \mathcal{I}$. Then $A = [(S_1 \cup S_2) \setminus I] \cup K$ where $S_1 \in \mathcal{S}_1, S_2 \in \mathcal{S}_2$ and $I, K \in \mathcal{I}$. it is clear that $A = (S_1 \setminus I) \cup (S_2 \setminus I) \cup K = [(S_1 \setminus I) \cup K] \cup [(S_2 \setminus I) \cup K] \in (\mathcal{S}_1 * \mathcal{I}) \vee (\mathcal{S}_2 * \mathcal{I})$.

Assume that $A \in (\mathcal{S}_1 * \mathcal{I}) \vee (\mathcal{S}_2 * \mathcal{I})$. Then $A = [((S_1 \setminus N) \cup L) \cup ((S_2 \setminus P) \cup R)]$, where $S_1 \in \mathcal{S}_1, S_2 \in \mathcal{S}_2$ and $N, L, P, R \in \mathcal{I}$. It follow that $A = [(S_1 \cap N^c) \cup (S_2 \cap P^c)]^{cc} \cup I = [(S_1 \cap N^c)^c \cap (S_2 \cap P^c)^c]^c \cup I = [(S_1^c \cup N) \cap (S_2^c \cup P)]^c \cup I$. Furthermore, we get $A = [(S_1^c \cap S_2^c) \cup ((S_1^c \cap P) \cup (S_2^c \cap N) \cup (N \cap P))]^c \cup I$. Let put $J = (S_1^c \cap P) \cup (S_2^c \cap N) \cup (N \cap P) \in \mathcal{I}$ and note that $A = [(S_1^c \cap S_2^c) \cup J]^c \cup I = [(S_1^c \cap S_2^c)^c \cap J^c \cup I] = ((S_1 \cup S_2) \setminus J) \cup I$. Since $S_1 \in \mathcal{S}_1, S_2 \in \mathcal{S}_2$ and $J, I \in \mathcal{I}$ then we have $A \in (\mathcal{S}_1 \vee \mathcal{S}_2) * \mathcal{I}$.

(ii) Assume that $A \in (\mathcal{I} * \mathcal{S}_1) \vee (\mathcal{I} * \mathcal{S}_2)$. Then $A = [(I_1 \setminus U_1) \cup W_1] \cup [(I_2 \setminus U_2) \cup W_2]$ where $I_1, I_2 \in \mathcal{I}, U_1, W_1 \in \mathcal{S}_1$ and $U_2, W_2 \in \mathcal{S}_2$. It is clear that $A = (I_1 \setminus U_1) \cup (I_2 \setminus U_2) \cup (W_1 \cup W_2) = I \cup (W_1 \cup W_2)$, where $I = (I_1 \setminus U_1) \cup (I_2 \setminus U_2)$. Since $I \in \mathcal{I}$ then the set A can be written as $A = [I \setminus (W_1 \cup W_2)] \cup (W_1 \cup W_2) \in \mathcal{I} * (\mathcal{S}_1 \vee \mathcal{S}_2)$.

Assume that $A \in \mathcal{I} * (\mathcal{S}_1 \vee \mathcal{S}_2)$. Then $A = [I \setminus (U_1 \cup U_2)] \cup (W_1 \cup W_2)$ where $I \in \mathcal{I}, U_1, W_1 \in \mathcal{S}_1$ and $U_2, W_2 \in \mathcal{S}_2$. Note that $A = [(I \setminus U_2) \setminus U_1] \cup [(I \setminus U_1) \setminus U_2] \cup (W_1 \cup W_2) = [((I \setminus U_2) \setminus U_1) \cup W_1] \cup [((I \setminus U_1) \setminus U_2) \cup W_2]$. Since the sets $I \setminus U_1$ and $I \setminus U_2$ are elements of I then we have $A \in (\mathcal{I} * \mathcal{S}_1) \vee (\mathcal{I} * \mathcal{S}_2)$. ☐

The following corollary, which may be proven inductively, follows from Proposition 2.1.22.

Corollary 2.1.23: Let $\mathcal{S}_1, \mathcal{S}_2, \cdots, \mathcal{S}_n$ be a finite collection of semigroups of sets on X. If $\mathcal{I}$ is an ideal of sets on X then the following equalities hold:

(i) $(\bigvee_{i=1}^n \mathcal{S}_i) * \mathcal{I} = \bigvee_{i=1}^n (\mathcal{S}_i * \mathcal{I})$.

(ii) $\mathcal{I} * (\bigvee_{i=1}^n \mathcal{S}_i) = \bigvee_{i=1}^n (\mathcal{I} * \mathcal{S}_i)$.

In a similar way, as in Proposition 2.1.22, we can write the following statement.

Proposition 2.1.24: Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be ideals of sets on X and let $\mathcal{S}$ be a semigroup of sets on X. Then the following hold:

(i) $\mathcal{S} * \mathcal{I}_i \subseteq \mathcal{S} * (\mathcal{I}_1 \vee \mathcal{I}_2)$ for $i = 1, 2$ and $(\mathcal{S} * \mathcal{I}_1) \vee (\mathcal{S} * \mathcal{I}_2) \subseteq \mathcal{S} * (\mathcal{I}_1 \vee \mathcal{I}_2)$.

(ii) $\mathcal{I}_i * \mathcal{S} \subseteq (\mathcal{I}_1 \vee \mathcal{I}_2) * \mathcal{S} = (\mathcal{I}_1 * \mathcal{S}) \vee (\mathcal{I}_2 * \mathcal{S})$ for $i = 1, 2$.

Proof. (i). Since $\mathcal{I}_1 \subseteq \mathcal{I}_1 \vee \mathcal{I}_2$ and $\mathcal{I}_2 \subseteq \mathcal{I}_1 \vee \mathcal{I}_2$, then inclusions $\mathcal{S} * \mathcal{I}_1 \subseteq \mathcal{S} * (\mathcal{I}_1 \vee \mathcal{I}_2)$ and $\mathcal{S} * \mathcal{I}_2 \subseteq \mathcal{S} * (\mathcal{I}_1 \vee \mathcal{I}_2)$ follow directly.

Assume that $A \in (\mathcal{S} * \mathcal{I}_1) \vee (\mathcal{S} * \mathcal{I}_2)$. Then $A = ((S_1 \setminus I_1) \cup I_2) \cup ((S_2 \setminus I_3) \cup I_4)$ where $S_1, S_2 \in \mathcal{S}, I_1, I_2 \in \mathcal{I}_1$ and $I_3, I_4 \in \mathcal{I}_2$. We can write that $A = (S_1 \setminus I_1) \cup (S_2 \setminus I_3) \cup (I_2 \cup I_4)$. Note that $(S_1 \setminus I_1) \cup (S_2 \setminus I_3) = [(S_1 \setminus I_1) \cup (S_2 \setminus I_3)]^{cc} = [[(S_1 \setminus I_1) \cup (S_2 \setminus I_3)]^c]^c = [(S_1 \cap I_1^c)^c \cap (S_2 \cap I_3^c)^c]^c = [(S_1^c \cup I_1) \cap (S_2^c \cup I_3)]^c = [(S_1^c \cap S_2^c) \cup (S_1^c \cap I_3) \cup (S_2^c \cap I_1) \cup (I_1 \cap I_3)]^c$. Since $S_2^c \cap I_1 \subseteq I_1, S_1^c \cap I_3 \subseteq I_3$, $I_1 \cap I_3 \subseteq I_1$ and $I_1 \cap I_3 \subseteq I_3$ then $I = (S_1^c \cap I_3) \cup (S_2^c \cap I_1) \cup (I_1 \cap I_3) \in \mathcal{I}_1 \vee \mathcal{I}_2$. It follows that $A = [(S_1^c \cap S_2^c) \cup I]^c \cup (I_2 \cup I_4) = (S_1 \cup S_2) \setminus I) \cup (I_2 \cup I_4)$. Since $\mathcal{S}$ is a semigroup of sets then $S_1 \cup S_2 \in \mathcal{S}$, and thus $A \in \mathcal{S} * (\mathcal{I}_1 \vee \mathcal{I}_2)$.

(ii). Since $\mathcal{I}_1 \subseteq \mathcal{I}_1 \vee \mathcal{I}_2$ and $\mathcal{I}_2 \subseteq \mathcal{I}_1 \vee \mathcal{I}_2$ the inclusions $\mathcal{I}_1 * \mathcal{S} \subseteq (\mathcal{I}_1 \vee \mathcal{I}_2) * \mathcal{S}$ and $\mathcal{I}_2 * \mathcal{S} \subseteq (\mathcal{I}_1 \vee \mathcal{I}_2) * \mathcal{S}$ follow directly.

Assume that $A \in (\mathcal{I}_1 \vee \mathcal{I}_2) * \mathcal{S}$. Then $A = [(I_1 \cup I_2) \setminus S_1] \cup S_2$ for some $S_1, S_2 \in \mathcal{S}, I_1 \in \mathcal{I}_1$ and $I_2 \in \mathcal{I}_2$. We can write $A = [(I_1 \cup I_2) \cap S_1^c] \cup S_2 = (I_1 \cap S_1^c) \cup (I_2 \cap S_1^c) \cup S_2 = (I_1 \setminus S_1) \cup (I_2 \setminus S_1) \cup S_2$. Observe that $I_1 \setminus S_1 \in \mathcal{I}_1$ and $I_2 \setminus S_2 \in \mathcal{I}_2$. Put $K = I_1 \setminus S_1$ and $L = I_2 \setminus S_2$. This implies that $A = (K \cup L) \cup S_2 = (K \cup S_2) \cup (L \cup S_2) = [(K \setminus S_2) \cup S_2] \cup [(L \setminus S_2) \cup S_2]$ and hence $A \in (\mathcal{I}_1 * \mathcal{S}) \vee (\mathcal{I}_1 * \mathcal{S})$.

Assume that $A \in (\mathcal{I}_1 * \mathcal{S}) \vee (\mathcal{I}_1 * \mathcal{S})$. Then $A = [(I_1 \setminus S_1) \cup S_2] \cup [(I_2 \setminus S_3) \cup S_4]$ where $I_1 \in \mathcal{I}_1, S_1, S_2, S_3, S_4 \in \mathcal{S}$ and $I_2 \in \mathcal{I}_2$. Note that $A = [(I_1 \setminus S_1) \cup (I_2 \setminus S_3)] \cup (S_2 \cup S_4)$. Since $(I_1 \setminus S_1 \in \mathcal{I}_1, I_2 \setminus S_3 \in \mathcal{I}_2$, it follows that the set $I = (I_1 \setminus S_1) \cup (I_2 \setminus S_3)$ is an element of $\mathcal{I}_1 \vee \mathcal{I}_2$. Since $\mathcal{S}$ is a semigroup of sets then the set $S = S_2 \cup S_4$ is an element of $\mathcal{S}$. It follows that $A = I \cup S = (I \setminus S) \cup S$ and hence $A \in (\mathcal{I}_1 \vee \mathcal{I}_2) * \mathcal{S}$.
□

Thus the next corollary comes from Proposition 2.1.24 and can also be demonstrated inductively.

Corollary 2.1.25: Let $\mathcal{I}_1, \mathcal{I}_2, \cdots, \mathcal{I}_n$ be a finite collection of ideals of sets on X. If $\mathcal{S}$ is a semigroup of sets on X then the following hold:

(i) $\bigvee_{i=1}^{n}(\mathcal{S} * \mathcal{I}_i) \subseteq \mathcal{S} * (\bigvee_{i=1}^{n} \mathcal{I}_i)$.

(ii) $(\bigvee_{i=1}^{n} \mathcal{I}_i) * \mathcal{S} = \bigvee_{i=1}^{n}(\mathcal{I}_i * \mathcal{S})$.

Chapter 3

Lebesgue measure on the real line

The Lebesgue measure, named after the French mathematician Henri Lebesgue, is the standard way of assigning a measure to subsets of n-dimensional Euclidean space $\mathbb{R}^n, n \geq 1$. For $n = 1, 2$, or 3 the Lebesgue measure corresponds to the standard measure of length, area, or volume. In general, it is also called n-dimensional volume, n-volume, or simply volume. Lebesgue measure is used throughout Real Analysis, in particular to define Lebesgue integration. The Lebesgue measure is important for measure theory and the theory of integrals. The sets that can be assigned a Lebesgue measure are called Lebesgue-measurable sets. Here we will be interested in Lebesgue measure on the real line $\mathbb{R}$.

3.1 General measure theory

Before introducing the Lebesgue measure, we start by giving an overview of the concept of measure in general.

Definition 3.1.1: A *set function* is a function defined on a class of sets; that is, for every set in a given class, a (finite or infinite) function value is defined. More precisely, if $\mathcal{E}$ is a class of sets then a set function $\mu : \mathcal{E} \longrightarrow [-\infty, +\infty]$ associates to every set $A \in \mathcal{E}$ a number $\mu(A)$. The set function is finite, real-valued if it takes real values, i.e. values in $\mathbb{R} = (-\infty, \infty)$.

In the following, we will assume that X is a non-empty set and $\mathcal{A}$ is a σ-algebra of sets on X.

Definition 3.1.2: A (positive) measure μ on X is a mapping $\mu : \mathcal{A} \longrightarrow [0, \infty]$ defined on a σ-algebra $\mathcal{A}$ satisfying the following properties:

(i) $\mu(\emptyset) = 0$.

(ii) For any countable family of pairwise disjoint sets $\{A_i\}_{i=1}^{\infty} \subseteq \mathcal{A}$, we have

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$

If (i) and (ii) both hold, but $\mathcal{A}$ is not a σ-algebra then μ is said to be a *pre-measure*. The property *(ii)* requires implicitly that $\bigcup_{i=1}^{\infty} A_i$ is again in $\mathcal{A}$. This is clearly the case for σ-algebras, but needs special attention if one deals with pre-measures. The elements of $\mathcal{A}$ are called *measurable sets* with respect to μ, while elements belonging to $\mathcal{P}(X) \setminus \mathcal{A}$ are called non-measurable sets with respect to μ.

Definition 3.1.3: Let X be a set and $\mathcal{A}$ be a σ-algebra on X. The pair $(X, \mathcal{A})$ is called a *measurable space*. If μ is a measure on X then the triplet $(X, \mathcal{A}, \mu)$ is called a *measure space*. A *finite measure* is a measure with $\mu(X) < \infty$ and a probability measure is a measure with $\mu(X) = 1$. The corresponding measure spaces are called *finite measure space* (resp. *probability space*). An *exhausting sequence* $\{A_i\}_{i=1}^{\infty} \subseteq \mathcal{A}$ is an increasing sequence of sets if $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$ such that $\bigcup_{i=1}^{\infty} A_i = X$. A measure μ is said to be *σ-finite* and $(X, \mathcal{A}, \mu)$ is called a *σ-finite measure space* if $\mathcal{A}$ contains an exhausting sequence $\{A_i\}_{i=1}^{\infty}$ such that $\mu(A_i) < \infty$ for all $i = 1, 2, \cdots$.

Example 3.1.4 (Counting measure): Define a set function μ on the class $\mathcal{P}(\mathbb{R})$ of all subsets of $\mathbb{R}$ by

$$\mu(A) = \begin{cases} \text{Card}(A) \text{ if } A \text{ is finite} \\ \infty \text{ if } A \text{ is infinite} \end{cases}$$

where Card(A) means the cardinality of the set A i.e. the number of elements of the set A. One can verify that μ is a measure. Hence $(\mathbb{R}, \mathcal{P}(\mathbb{R}), \mu)$ is a measure space. The measure μ is called the **counting** measure.

Example 3.1.5 (Uniform measure): Suppose that X is a non-empty finite or countable set. Consider the set function $\mu : \mathcal{P}(X) \longrightarrow [0, +\infty]$ by

$$\mu(A) = \sum_{x \in A} f(x)$$

for some function $f : X \to [0, +\infty]$. One can show that the set function μ is a measure on X. In particular, for a finite set X with N elements, if $f(x) = \frac{1}{N}$ then μ is a probability measure called the *uniform measure* on X.

3.1.1 Properties of measures

Let us derive some immediate properties of (pre-)measures. One desirable property of a measure is that if $A \subseteq B$ then the measure of B should be greater than or equal to the measure of A. The following are some important properties of measures.

Proposition 3.1.6 (Monotonicity [12]): Let $(X, \mathcal{A}, \mu)$ be a measure space. If $A, B \in \mathcal{A}$ and $A \subseteq B$, then $\mu(A) \leq \mu(B)$. In addition, if $\mu(A)$ is finite then $\mu(B \setminus A) = \mu(B) - \mu(A)$.

Corollary 3.1.7: Let $(X, \mathcal{A}, \mu)$ be a measure space. If $A, B \in \mathcal{A}$ then

$$\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B)$$

Property (ii) of the definition of measure deals with the measure of a countable union of pairwise disjoint measurable sets. The next theorem concerns the countable union of sets which are not necessarily pairwise disjoint.

Theorem 3.1.8 (Countable sub-additivity [12]): Let $(X, \mathcal{A}, \mu)$ be a measure space. If $\{E_j\}$ is a countable collection of sets in $\mathcal{A}$, then

$$\mu\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \sum_{j=1}^{\infty} \mu(E_j)$$

Corollary 3.1.9: Let $(X, \mathcal{A}, \mu)$ be a measure space. If $B, C \in \mathcal{A}$ and $\mu(C) = 0$ then $\mu(B \cup C) = \mu(B)$.

Suppose that $\{A_j\}$ is a countably infinite collection of sets in some measure space with $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots \subseteq A_j \subseteq \cdots$. We know that $A = \bigcup_{j=1}^{\infty} A_j$ will be a measurable set since it is the countable union of measurable sets. It seems reasonable to expect

$$\mu(A) = \lim_{n\to\infty} \mu(A_n)$$

In other words, what we really would like to say is that

$$\mu(\lim_{n\to\infty} A_n) = \lim_{n\to\infty} \mu(A_n)$$

Theorem 3.1.10 (Continuity from below [12]): Let $(X, \mathcal{A}, \mu)$ be a measure space. If $\{A_j\}$ is a countably infinite collection of sets in $\mathcal{A}$ with $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots \subseteq A_j \subseteq \cdots$, then $\bigcup_{j=1}^{\infty} A_j \in \mathcal{A}$ and

$$\mu\left(\bigcup_{j=1}^{\infty} A_j\right) = \lim_{n\to\infty} \mu(A_n)$$

Theorem 3.1.11 (Continuity from above [12]): Let $(X, \mathcal{A}, \mu)$ be a measure space. If $\{E_j\}$ is a countably infinite collection of sets in $\mathcal{A}$ with $E_1 \supseteq E_2 \supseteq E_3 \supseteq \cdots$ and $\mu(E_1) < \infty$ then $E = \bigcap_{n=1}^{\infty} E_n \in \mathcal{A}$ and

$$\mu\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} \mu(E_n)$$

3.1.2 Complete measures

For many measures, sets of measure zero have a special property which is often useful. Let $(X, \mathcal{A}, \mu)$ be a measure space. A subset C of X is a *null* set for the measure μ if $C \in \mathcal{A}$ and $\mu(C) = 0$. In general, an arbitrary subset A of a null set C needs not be measurable, but if A happens to be measurable then Monotonicity property of measure implies that $\mu(A) = 0$.

Definition 3.1.12 (Complete measure): A measure space $(X, \mathcal{A}, \mu)$ is said to be a *complete* measure space if whenever $C \in \mathcal{A}$ with $\mu(C) = 0$ and $A \subseteq C$ then $A \in \mathcal{A}$.

Example 3.1.13: if A is a Lebesgue measurable set with $\mu(A) = 0$ (i.e A is a null set), and $B \subseteq A$, then B is Lebesgue measurable with $\mu(B) = 0$

In other words, a complete measure is one such that every subset A of every null set C is measurable. Complete measures are often more convenient to work with than incomplete measures.

3.1.3 Atomic measures

An "atom" in Measure Theory is a measurable set with positive measure, such that no subset of this set (other than the empty set) has a smaller positive measure. A measure which has no atoms is called non-atomic or atomless.

Definition 3.1.14: Given a measurable space $(X, \mathcal{A})$ and a measure μ on that space, a set $A \subseteq X$ in $\mathcal{A}$ is called an *atom* if $\mu(A) > 0$ and for any measurable subset $B \subset A$ with $\mu(B) < \mu(A)$, the set B has measure zero.

A measure which has no atoms is called *non-atomic* or *diffuse*. In other words, a measure is non-atomic if for any measurable set A with $\mu(A) > 0$, there exists a measurable subset B of A such that $\mu(A) > \mu(B) > 0$.

In Section 3.2, we will see an example of a measure on the real line $\mathbb{R}$, known as the Lebesgue measure, which has no atoms and hence it is a non-atomic measure.

3.2 Lebesgue measure on the real line

We begin by introducing the notions of null sets and Lebesgue outer measure before developing the concept of the Lebesgue measure on the real line. Null sets are defined as follows.

3.2.1 Null sets

A *null set* $A \subseteq \mathbb{R}$ is a set that may be covered by a sequence of intervals of arbitrarily small total length. It means that given any $\varepsilon > 0$ we can find a sequence $\{I_n : n \geq 1\}$ of intervals such that

$$A \subseteq \bigcup_{n=1}^{\infty} I_n \text{ and } \sum_{n=1}^{\infty} \ell(I_n) < \varepsilon.$$

We also say simply that A is a **null** set. Note that the intervals do not need to be disjoint. It follows at once from the definition that the empty set is null.

Theorem 3.2.1 ([12]): If $\{N_n\}_{n=1}^{\infty}$ is a sequence of null sets, then their union

$$N = \bigcup_{n=1}^{\infty} N_n \text{ is also null.}$$

In addition, if B is a null set and $A \subseteq B$ then A is also a null set.

It follows from Theorem 3.2.1 that the sets $\mathbb{N}, \mathbb{Z}$ and $\mathbb{Q}$ of natural numbers, integers and rational numbers, are all null subsets of $\mathbb{R}$.

3.2.2 Lebesgue outer measure

Here we introduce the concept of outer measure that is a basis for the Lebesgue measure.

Definition 3.2.2: By the *Lebesgue outer measure* of a set $A \subseteq \mathbb{R}$, denoted by $\mu^*(A)$, is meant the number

$$\mu^*(A) = \inf \left\{ \sum_{n=1}^{\infty} \ell(I_n) : I_n \text{ is an open interval and } A \subseteq \bigcup_{n=1}^{\infty} I_n \right\} \qquad (3.2.1)$$

where inf is taken over all covering of A by finite or countable systems of intervals I_n.

The outer measure is the infimum of lengths of all possible covers of A. Note that some intervals I_n may be empty; this avoids having to worry whether the sequence $\{I_n\}$ has finitely or infinitely many different members.

Clearly, $\mu^*(A) \geq 0$ for any $A \subseteq \mathbb{R}$. For some sets A, the series $\sum_n \ell(I_n)$ may diverge for a covering of A, so $\mu^*(A)$ may be equal to $+\infty$. It follows from the definition that μ^* is defined on all subsets of $\mathbb{R}$.

In the following, we list different properties of the Lebesgue outer measure.

Theorem 3.2.3 (Monotonicity [3]): If $A \subseteq B$ then $\mu^*(A) \leq \mu^*(B)$. In addition $\mu^*(\emptyset) = 0$.

The theorem follows from the fact that any covering of B by open intervals is also a covering of A so that the latter infimum is taken over a larger collection than the former.

The following theorem shows that the concept of null set is consistent with that of outer measure.

Theorem 3.2.4 ([12]): $A \subseteq \mathbb{R}$ is a null set if and only if $\mu^*(A) = 0$.

The following theorem shows that μ^* is an extension of length to an arbitrary subset of $\mathbb{R}$.

Theorem 3.2.5 (Extention of length [3]): For any interval $I \subseteq \mathbb{R}$ we have $\mu^*(I) = \ell(I)$.

Theorem 3.2.6 (Countable sub-additivity[3]): The Lebesgue outer measure is countably sub-additive, i.e. for any countable collection of sets $\{E_n\}_{n=1}^{\infty}$, we have

$$\mu^*\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \mu^*(E_n)$$

Note that the length of an interval does not change if we shift it along the real line: That is $\ell([a,b]) = \ell([a+t, b+t]) = b-a$. Since the outer measure is defined in terms of the lengths of intervals, it is natural to expect it to share this property.

Theorem 3.2.7 (Translation invariance [3]): The Lebesgue outer measure is translation invariant, i.e. $\mu^*(A) = \mu^*(A+t)$ for each $A \subseteq \mathbb{R}$ and $t \in \mathbb{R}$, where $A + t := \{a + t : a \in A\}$.

The length of interval does not change when the interval is shifted and outer measure is determined by the length of the coverings.

3.2.3 Lebesgue measurable sets

Since the Lebesgue outer measure does not possess the property of σ-additivity, one may try to restrict the outer measure μ^* to a σ-algebra such that the restriction has the property. It turns out that the σ-algebra is also the key property of the abstract concept of measure, and it is uses to provide mathematical foundations for Probability Theory.

Definition 3.2.8: A subset E of $\mathbb{R}$ is said to be *Lebesgue-measurable* if, for every set $A \subseteq \mathbb{R}$, we have

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c) \tag{3.2.2}$$

The set A is referred to as a *test set* for measurability. We denote by $\mathcal{L}(\mathbb{R})$ the *family of all Lebesgue-measurable subsets of the real line.* The criterion in the previous definition is sometimes called the *Carathéodory criterion* of Lebesgue measurability.

Note that for any sets A and E we have $A = (A \cap E) \cup (A \cap E^c)$ and the inequality

$$\mu^*(A) \leq \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

is valid by countably sub-additivity of outer measure as it is stated in Theorem 3.2.6. Hence, a set $E \in \mathcal{L}(\mathbb{R})$ if and only if

$$\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^c) \qquad \text{for all } A \subseteq \mathbb{R} \tag{3.2.3}$$

The set function $\mu : \mathcal{L}(\mathbb{R}) \longrightarrow [0, +\infty]$ defined by $\mu(E) = \mu^*(E)$ for all $E \in \mathcal{L}(\mathbb{R})$ is called the *Lebesgue measure*.

It can be easily observed from the definition that:

(i) $E \in \mathcal{L}(\mathbb{R})$ if and only if $E^c \in \mathcal{L}(\mathbb{R})$.

(ii) $\emptyset \in \mathcal{L}(\mathbb{R})$ by the fact that $\mu^*(A) = \mu^*(A \cap \emptyset) + \mu^*(A \cap \mathbb{R})$ for all $A \subseteq \mathbb{R}$.

(iii) If $\mu^*(E) = 0$ then $E \in \mathcal{L}(\mathbb{R})$ by the fact that

$$\mu^*(A \cap E) + \mu^*(A \cap E^c) = \mu^*(A \cap E^c) \leq \mu^*(A)$$

for all $A \subseteq \mathbb{R}$, since $A \cap E^c \subseteq A$.

The following theorem gives some examples of Lebesgue measurable sets.

Theorem 3.2.9 ([12]): (i) Any null set N is Lebesgue-measurable and $\mu(N) = 0$.

(ii) Any interval $I \subseteq \mathbb{R}$ is Lebesgue-measurable and $\mu(I) = \ell(I)$.

We shall write $\mu(E)$ instead of $\mu^*(E)$ for any E in $\mathcal{L}(\mathbb{R})$ and call $\mu(E)$ the Lebesgue measure of the set E.

3.2.4 Properties of the Lebesgue measure

Since Lebesgue measure is nothing else than the outer measure restricted to a special class of sets, some properties of the outer measure are automatically inherited by Lebesgue measure.

Theorem 3.2.10 (Monotonicity[4]): Suppose that A and B are Lebesgue measurable subsets of $\mathbb{R}$.

1. If $A \subseteq B$, then $\mu(A) \leq \mu(B)$.
2. If $A \subseteq B$ and $\mu(A)$ is finite then $\mu(B \setminus A) = \mu(B) - \mu(A)$.

The proof of the theorem uses the same arguments as for a measure in general.

Theorem 3.2.11 (Countable additivity[3]): The Lebesgue measure μ is countably additive, i.e. for a sequence of pairwise disjoint sets (A_i) we have

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$

Theorem 3.2.12 (Translation invariance[3]): The Lebesgue measure μ is translation-invariant. That is if E is a Lebesgue measurable set, then each set $E + y, y \in \mathbb{R}$ is also Lebesgue measurable and their measures coincide i.e. $\mu(E) = \mu(E + y)$.

Theorem 3.2.13 ([12]): Let α be an element of $\mathbb{R}$ and let A be a Lebesgue measurable subset of $\mathbb{R}$. Then the set αA is Lebesgue measurable and $\mu(\alpha A) = |\alpha|\mu(A)$ where $\alpha A := \{\alpha a : a \in A\}$.

Let us observe that certain set operations on sets in $\mathcal{L}(\mathbb{R})$ again produce sets in $\mathcal{L}(\mathbb{R})$ as it is shown in the following theorem.

Theorem 3.2.14 ([12]): For the class $\mathcal{L}(\mathbb{R})$ of all Lebesgue measurable subsets of $\mathbb{R}$, we have the following:

(i) $\mathbb{R} \in \mathcal{L}(\mathbb{R})$.

(ii) If $E \in \mathcal{L}(\mathbb{R})$ then $E^c \in \mathcal{L}(\mathbb{R})$.

(iii) If $E_n \in \mathcal{L}(\mathbb{R})$ for all $n = 1, 2, \cdots$; then the $\bigcup_{j=1}^{\infty} E_n \in \mathcal{L}(\mathbb{R})$. Moreover, if $E_n \in \mathcal{L}(\mathbb{R})$ for all $n = 1, 2, \cdots$ and $E_j \cap E_k = \emptyset$ for $j \neq k$ then

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n)$$

Let $\mathcal{N}_0$ be the collection of all subsets of $\mathbb{R}$ having the Lebesgue measure zero, then by Theorem 3.2.1, it follow that $\mathcal{N}_0$ is a σ-ideal of sets on $\mathbb{R}$.

The conditions (i)-(iii) mean that $\mathcal{L}(\mathbb{R})$ is a σ-algebra of sets on $\mathbb{R}$. We can therefore summarize the closure properties of the family $\mathcal{L}(\mathbb{R})$ of Lebesgue measurable sets as follows: $\mathcal{L}(\mathbb{R})$ is closed under countable unions, countable intersections, and complements. It contains intervals and all null subsets of $\mathbb{R}$.

Lemma 3.2.15 ([12]): Let A and B be subsets of $\mathbb{R}$. If $A \in \mathcal{L}(\mathbb{R})$ and $\mu(A\Delta B) = 0$ then $B \in \mathcal{L}(\mathbb{R})$ and $\mu(A) = \mu(B)$.

Chapter 4

Non-Lebesgue measurable subsets of the real line

This chapter aims to present examples of subsets of the real line that are not measurable in the Lebesgue sense. Even if many such sets exist, we will be only interested in Vitali selectors and Bernstein subsets of the real line.

4.1 Vitali selectors of the real line

Vitali selectors are the first classical examples of subsets of $\mathbb{R}$ which are not Lebesgue measurable. They were introduced by Giuseppe Vitali [8] in 1905 and and many authors, such as Kharazishivili [1] and V.A Chatyrko [13] developed more general theory about Vitali selectors.

Consider the additive group $(\mathbb{R}, +)$ of the real numbers endowed with the standard topology τ_E, and let Q be a countable dense subgroup of $(\mathbb{R}, +)$. Such a subgroup can be for instance the rational numbers $\mathbb{Q}$, the group $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$, the group $\mathbb{D} = \{a + b\sqrt{2} : a, b \in 2\mathbb{N}\}$ and much more.

Define a relation $\mathcal{R}$ on $\mathbb{R}$ as follows: for $x, y \in \mathbb{R}$, let $x\mathcal{R}y$ if and only if $x - y \in Q$. The relation $\mathcal{R}$ is an equivalence relation on $\mathbb{R}$, and hence it divides $\mathbb{R}$ into equivalence classes. Let $\mathbb{R}/Q = \{E_\alpha(Q) : \alpha \in I\}$ be the set of all equivalence classes, where I is some indexing set. Accordingly, we have the following decomposition of $\mathbb{R}$:

$$\mathbb{R} = \bigcup\{E_\alpha(Q) : \alpha \in I\}. \tag{4.1.1}$$

From Equality (4.1.1), we observe that $\text{Card}(I) = \mathfrak{c}$, where $\mathfrak{c}$ is the continuum. It follows from the definition of the relation $\mathcal{R}$ that the set $\mathbb{R}/Q$ consists of disjoint translated copies of Q by elements of $\mathbb{R}$. This implies that each element of $\mathbb{R}/Q$ is of the form $E_\alpha(Q) = Q + x = \{q + x : q \in Q\}$, for some $x \in \mathbb{R}$. Hence each equivalence class $E_\alpha(Q)$ is dense in $(\mathbb{R}, \tau_E)$.

Definition 4.1.1 ([6, 8]): A *Vitali selector* of $\mathbb{R}$ related to Q is any subset V of $\mathbb{R}$ satisfying the condition: $\text{Card}(V \cap E_\alpha(Q)) = 1$ for each $\alpha \in I$. A Vitali selector is called a *Vitali set* whenever the subgroup Q coincides with the additive group $\mathbb{Q}$ of rational numbers.

If A is a subset of $\mathbb{R}$ and $x \in \mathbb{R}$ then $A + x$ is defined as the set $\{a + x : a \in A\}$. The following proposition provides different facts about Vitali selectors of the real line.

Proposition 4.1.2 ([13, 6, 8]): Let Q be a countable dense subgroup of $(\mathbb{R}, +)$ and let V be a Vitali selector related to Q. Then the following statements hold:

(i) If $q_1, q_2 \in Q$ and $q_1 \neq q_2$ then $(V + q_1) \cap (V + q_2) = \emptyset$.

(ii) Any two sets in the collection $\{V + q : q \in Q\}$ are homeomorphic, and

$$\mathbb{R} = \bigcup \{V + q : q \in Q\}. \tag{4.1.2}$$

(iii) For each $x \in \mathbb{R}$, the set $V + x$ is also a Vitali selector related to Q.

The Vitali selectors defined above are, in general, unbounded subsets of $\mathbb{R}$. However, it is possible to define Vitali selectors which have additional properties of being bounded and dense in $\mathbb{R}$. In all cases the next theorem is valid.

Theorem 4.1.3 ([6, 8]): Any Vitali selector of $\mathbb{R}$ is not measurable in the Lebesgue sense.

The following statement is a more general result and it is about the non-Lebesgue measurability of a finite union of Vitali selectors.

Theorem 4.1.4 ([1]): If $\{V_\alpha : 1 \leq \alpha \leq m\}$ is a non-empty finite family of Vitali selectors related to a countable dense subgroup Q of $\mathbb{R}$, then the union $\bigcup\{V_\alpha : 1 \leq \alpha \leq m\}$ is not measurable in the Lebesgue sense.

The classical Banach Theorem given below together with the following lemmas, were used in the proof of Theorem 4.1.4, and they will be very useful in the construction of semigroups of sets made by sets that are not measurable in the Lebesgue sense to be considered in the following chapter. Let $\mathcal{B}_b(\mathbb{R})$ be the family of all bounded subsets of $\mathbb{R}$.

Theorem 4.1.5 (Banach Theorem [1]): Let $\mathcal{S}$ be a translation invariant ring of subsets of $\mathbb{R}$, satisfying the relations $\mathcal{S} \subseteq \mathcal{B}_b(\mathbb{R})$ and $[0,1) \in \mathcal{S}$, and let $\vartheta : \mathcal{S} \longrightarrow [0,+\infty)$ be a finitely additive translation invariant functional such that $\vartheta([0,1)) = 1$. Then there exists a finitely additive translation invariant functional $\eta : \mathcal{B}_b(\mathbb{R}) \longrightarrow [0,+\infty)$ such that η is an extension of ϑ.

Lemma 4.1.6 ([1]): Let ϑ be as in Theorem 4.1.5, and let $X \in \mathcal{B}_b(\mathbb{R})$ have the following property: There exists a bounded infinite sequence $\{h_k : k \in \mathbb{N}\}$ of elements of $\mathbb{R}$ such that the family $\{X + h_k : k \in \mathbb{N}\}$ is disjoint. If $X \in \mathrm{dom}(\vartheta)$ then necessarily $\vartheta(X) = 0$.

Lemma 4.1.7 ([1]): Let X be a bounded subset of a Vitali selector V. Then X has the property indicated in Lemma 4.1.6.

We remark that Theorem 4.1.4 does not need to hold for an infinite union of Vitali selectors. This observation follows from Proposition 4.1.2. It follows that from Theorem 4.1.3 any union of two Vitali selectors is not a measurable set.

Corollary 4.1.8: Suppose that V_1 and V_2 are Vitali selectors related to a countable dense subgroup Q of $\mathbb{R}$. Then at least one of the sets $V_1 \setminus V_2, V_2 \setminus V_1$ and $V_1 \cap V_2$ must be a non-measurable set in the Lebesgue sense.

Proof. It follows from Theorem 4.1.4 that the set $V_1 \cup V_2$ is not measurable in the Lebesgue sense. Note that $V_1 \cup V_2 = (V_1 \setminus V_2) \cup (V_2 \setminus V_1) \cup (V_1 \cap V_2)$ and sets in this union are disjoint. If all the sets in this union are Lebesgue measurable, then the set $V_1 \cup V_2$ will be a Lebesgue measurable set, and this will be a contradiction. □

Corollary 4.1.9: Suppose that V_1 and V_2 are Vitali selectors of $\mathbb{R}$. Then at least one of the sets $V_1 \Delta V_2$ and $V_1 \cap V_2$ must be non-measurable in the Lebesgue sense.

Proof. Suppose that V_1 and V_2 are Vitali subsets of $\mathbb{R}$ and all the sets $V_1 \setminus V_2, V_2 \setminus V_2$ and $V_1 \cap V_2$ are Lebesgue measurable. Note that $V_1 \cup V_2 = (V_1 \setminus V_2) \cup (V_2 \setminus V_1) \cup (V_1 \cap V_2)$. Since measurable sets form a σ-algebra of sets, and $V_1 \cup V_2$ is a disjoint union of measurable sets by the given assumption, the set $V_1 \cup V_2$ must be a measurable set. But this contradicts Theorem 4.1.4. Hence, one of the sets must be non-measurable.□

Let us point out that even if the union of two Vitali selectors is not Lebesgue measurable, the intersection, the difference, and the symmetric difference of two Vitali selectors can be a measurable set or a non-Lebesgue measurable set.

Theorem 4.1.10: Suppose that V_1 and V_2 are Vitali selectors of $\mathbb{R}$. If the intersection of V_1 and V_2 is measurable in the Lebesgue sense, then their difference is not measurable in the Lebesgue sense.

Proof. Suppose V_1 and V_2 are Vitali selectors of $\mathbb{R}$, and assume that $V_1 \cap V_2$ is a Lebesgue measurable set. Note that $V_1 = (V_1 \setminus V_2) \cup (V_1 \cap V_2)$. Since measurable sets form a σ-algebra, if $V_1 \setminus V_2$ is a measurable set, then the set V_1 will also be a measurable set, which is a contradiction. Hence, the set $V_1 \setminus V_2$ must be a non-Lebesgue measurable set. □

Corollary 4.1.11: Suppose that V_1 and V_2 are Vitali selectors of $\mathbb{R}$. If the intersection of V_1 and V_2 is measurable in the Lebesgue sense, then their difference cannot be a countable set.

Proof. Suppose V_1 and V_2 are Vitali selectors of $\mathbb{R}$, and assume that $V_1 \cap V_2$ is a Lebesgue measurable set. Note that $V_1 = (V_1 \setminus V_2) \cup (V_1 \cap V_2)$. If $V_1 \setminus V_2$ is a countable set, then it will be measurable. This is a contradiction to Theorem 4.1.10. □

Note that Theorem 4.1.10 can also be formulated by saying that if $V_1 \setminus V_2$ is measurable in the Lebesgue sense, then the intersection $V_1 \cap V_2$ is non-measurable in the Lebesgue sense.

Theorem 4.1.12: Suppose that V_1 and V_2 are Vitali selectors of $\mathbb{R}$. If $V_1 \triangle V_2$ is null measurable, then $V_1 \setminus V_2$ and $V_2 \setminus V_1$ are Lebesgue measurable and null.

Proof. Note that $V_1 \triangle V_2 = (V_1 \setminus V_2) \cup (V_2 \setminus V_1)$. If $V_1 \triangle V_2$ is null, i.e., $(\mu(V_1 \triangle V_2) = 0)$, Since $V_1 \setminus V_2$ and $V_2 \setminus V_1$ are subsets of $V_1 \triangle V_2$, then by monotonicity, it follows that $\mu(V_1 \setminus V_2) \leq \mu(V_1 \triangle V_2) = 0$ and $\mu(V_2 \setminus V_1) \leq \mu(V_1 \triangle V_2) = 0$. Hence, $\mu(V_1 \setminus V_2) = 0$ and $\mu(V_2 \setminus V_1) = 0$. Therefore, $V_1 \setminus V_2$ and $V_2 \setminus V_1$ are Lebesgue measurable and null sets. □

4.2 Bernstein subsets of the real line

A second example of subsets of $\mathbb{R}$ that are not measurable in the Lebesgue sense is known as the Bernstein sets [10]. The Bernstein sets on $\mathbb{R}$ constitute also an example of elements belonging to the family $\mathcal{L}^c(\mathbb{R})$, and they are constructed by using the method of transfinite induction. The construction of Bernstein sets can be found in [10] or in [9], and they can be defined in the following way:

Definition 4.2.1 (Bernstein set): [6, 10], A subset B of $\mathbb{R}$ is called a *Bernstein* set if it meets every uncountable closed subset of the real line but that contains none of them. In other words, a subset B of $\mathbb{R}$ is called a *Bernstein set* if $B \cap F \neq \emptyset$ and $(\mathbb{R} \setminus B) \cap F \neq \emptyset$ for each uncountable closed subset F of $\mathbb{R}$.

To represent the family of all Bernstein subsets of $\mathbb{R}$, we shall use $\mathcal{B}_E(\mathbb{R})$ in the subsequent text. Specifically, we note that $\mathcal{B}_E(\mathbb{R})$ is invariant under the action of the group $\mathcal{H}(\mathbb{R})$ of all homeomorphisms of $\mathbb{R}$ onto itself. This means if $B \in \mathcal{B}_E(\mathbb{R})$ and $h \in \mathcal{H}(\mathbb{R})$ then $h(B) \in \mathcal{B}_E(\mathbb{R})$. It follows that each element of $\mathcal{B}$ is also a Bernstein set on $\mathbb{R}$ as well.

The existence and the construction of Bernstein sets on $\mathbb{R}$ is based on the Method of Transfinite Recursion.

Proposition 4.2.2 ([9], [10]): Let B be a Bernstein subset of $\mathbb{R}$. Then the following statements hold.

(i) The complement $\mathbb{R} \setminus B$ of B is also a Bernstein set, and $\text{Int}(B) = \text{Int}(\mathbb{R} \setminus B) = \emptyset$.

(ii) Both sets B and $\mathbb{R} \setminus B$ are dense in $\mathbb{R}$ and $\text{Card}(B) = \text{Card}(\mathbb{R} \setminus B) = \mathfrak{c}$.

Lemma 4.2.3 ([10], [9]): Let A be a subset of $\mathbb{R}$. Then A is a Bernstein set if and only if $F \cap A \neq \emptyset$ and $F \setminus A \neq \emptyset$ for each uncountable closed subset F of $\mathbb{R}$.

Theorem 4.2.4 ([10]): Any Bernstein set B on $\mathbb{R}$ is not measurable in the Lebesgue sense. Indeed, every Lebesgue measurable subset of either B or $\mathbb{R} \setminus B$ has the Lebesgue measure zero.

Corollary 4.2.5 ([18]): If A is a Lebesgue measurable set with positive measure and B a Bernstein set, then the sets $A \cap B$ and $A \setminus B$ are not measurable in the Lebesgue sense.

We point out that there exist Bernstein subsets of $\mathbb{R}$ which have some additional algebraic structures for subgroups of the additive group $(\mathbb{R}, +)$ as it is indicated in the following statements.

Lemma 4.2.6 ([6]): There exists a subgroup B of $(\mathbb{R}, +)$ such that the factor group $\mathbb{R}/B$ is isomorphic to the group $(\mathbb{R}, +)$ and B is a Bernstein set in $\mathbb{R}$.

Theorem 4.2.7 ([6]): There exists two subgroups G_1 and G_2 of the additive group $(\mathbb{R}, +)$ such that $G_1 \cap G_2 = \{0\}$, and both G_1 and G_2 are Bernstein sets in $\mathbb{R}$.

Chapter 5

Semigroups of sets generated by Vitali selectors and Bernstein subsets of the real line

In this chapter, we consider families of sets having an algebraic structure of semigroup of sets and for which elements are not measurable in the Lebesgue sense. Those families are constructed by using Vitali selectors of the real line and Bernstein subsets of the real line.

5.1 Semigroups of non-Lebesgue measurable sets generated by Vitali selectors

Consider a countable dense subgroup Q of $(\mathbb{R}, +)$. For this subgroup, we can have many Vitali selectors that are related to it, depending on different choices made in the equivalence classes $E_\alpha(Q), \alpha \in I$, where $\mathrm{Card}(I)$ is the continuum, as discussed in Section 4.1. Let $\mathcal{V}(Q)$ be the family of all Vitali selectors related to Q, and consider the collection $\mathcal{S}_{\mathcal{V}(Q)} = \{\bigcup_{i=1}^{n} V_i : V_i \in \mathcal{V}(Q), n \in \mathbb{N}\}$ of all finite unions of elements of $\mathcal{V}(Q)$. It is clear that the family $\mathcal{S}_{\mathcal{V}(Q)}$ is closed under finite union, and hence it is a semigroup of sets with respect to the operation of union. It follows from Theorem 4.1.4 that each element of the family $\mathcal{S}_{\mathcal{V}(Q)}$ is not measurable in the Lebesgue sense.

The following statement provides examples of semigroups constructed by using Vitali selectors for which each element is not measurable in the Lebesgue sense.

Theorem 5.1.1 ([19]): Let Q be a countable dense subgroup of $(\mathbb{R}, +)$, and let $\mathcal{I}$ be an ideal of sets on $\mathbb{R}$. Then the following statements hold:

(i). The families $\mathcal{V}(Q)$ and $\mathcal{S}_{\mathcal{V}(Q)}$ are invariant under translation.

(ii). The families $\mathcal{S}_{\mathcal{V}(Q)} * \mathcal{I}$ and $\mathcal{I} * \mathcal{S}_{\mathcal{V}(Q)}$ are semigroups of sets such that $\mathcal{S}_{\mathcal{V}(Q)} \subseteq \mathcal{I} * \mathcal{S}_{\mathcal{V}(Q)} \subseteq \mathcal{S}_{\mathcal{V}(Q)} * \mathcal{I}$ for which elements are not measurable in the Lebesgue sense.

(iii). If $\mathcal{I}$ is invariant under translation then the semigroups $\mathcal{S}_{\mathcal{V}(Q)} * \mathcal{I}$ and $\mathcal{I} * \mathcal{S}_{\mathcal{V}(Q)}$ are invariant under translation.

In [19] and [15], it was proved that there is no countable dense subgroup Q of $(\mathbb{R}, +)$ such that the corresponding generated semigroup $\mathcal{S}_{\mathcal{V}(Q)}$ contains all others. So, we can consider a generalization of Theorem 5.1.1 in the following way:

Let $\mathcal{C}$ be the family of all countable dense subgroups of $(\mathbb{R}, +)$, and consider the family $\mathcal{V} = \{V : V \in \mathcal{V}(Q), Q \in \mathcal{C}\}$ of all Vitali selectors.

The family $\mathcal{S}(\mathcal{V})$ of all possible finite unions of elements of $\mathcal{V}$ was called in [19] the supersemigroup generated by Vitali selectors of $\mathbb{R}$. A more general result generalizing Theorem 4.1.4, was proved in [16]. The result shows that each element of the family $\mathcal{S}(\mathcal{V})$ is not measurable in the Lebesgue sense.

Theorem 5.1.2 ([16], [15]): Let $U = \bigcup_{i=1}^{n} V_i$ be a finite union of Vitali selectors of $\mathbb{R}$, where $V_i \in \mathcal{V}(Q_i)$ and each Q_i is an element of $\mathcal{C}$ for $i = 1, 2, \cdots, n$. Then the set U is not measurable in the Lebesgue sense.

Theorem 5.1.3 ([19]): Let $\mathcal{I}$ be an ideal of subsets of $\mathbb{R}$. Then the following statements hold.

(i). The families $\mathcal{V}$ and $\mathcal{S}(\mathcal{V})$ are invariant under translations.

(ii). The families $\mathcal{S}(\mathcal{V}), \mathcal{I} * \mathcal{S}(\mathcal{V})$ and $\mathcal{S}(\mathcal{V}) * \mathcal{I}$ are semigroups of sets such that $\mathcal{S}(\mathcal{V}) \subseteq \mathcal{I} * \mathcal{S}(\mathcal{V}) \subseteq \mathcal{S}(\mathcal{V}) * \mathcal{I}$ for which each element is not measurable in the Lebesgue sense.

(iii). If $\mathcal{I}$ is invariant under translations of $\mathbb{R}$, then the families $\mathcal{I} * \mathcal{S}(\mathcal{V})$ and $\mathcal{S}(\mathcal{V}) * \mathcal{I}$ are also invariant under translations of $\mathbb{R}$.

For $n = 2$, Theorem 5.1.2 implies the following statement.

Corollary 5.1.4: Suppose that V_1 and V_2 are Vitali selectors related to elements Q_1 and Q_2 respectively in $\mathcal{C}$. Then at least one of the sets $V_1 \setminus V_2, V_2 \setminus V_1$ and $V_1 \cap V_2$ must be a non measurable set in the Lebesgue sense.

Proof. It follows from Theorem 5.1.2 that the set $V_1 \cup V_2$ is not measurable in the Lebesgue sense. Note that $V_1 \cup V_2 = (V_1 \setminus V_2) \cup (V_2 \setminus V_1) \cup (V_1 \cap V_2)$ and sets in this union are disjoint. If all the sets in this union are Lebesgue measurable, then the set $V_1 \cup V_2$ will be a Lebesgue measurable set, and this will be a contradiction. □

The following theorem is a more general result than Theorem 5.1.2.

Theorem 5.1.5: Let $U = \bigcup_{i=1}^{n} V_i$ be a finite union of Vitali selectors of $\mathbb{R}$, where $V_i \in \mathcal{V}(Q_i)$ and each Q_i is an element of $\mathcal{C}$ for $i = 1, 2, \cdots, n$. Then the set U cannot contain any subset of strictly positive Lebesgue measure.

Proof. Suppose that there exists a Lebesgue measurable subset Y of $\mathbb{R}$ such that $\mu(Y) > 0$ and $Y \subseteq U$. Without loss of generality, we may assume that the set Y is bounded. Let ϑ be the restriction of μ to the ring of sets $\mathcal{B}_b(\mathbb{R}) \cap \mathrm{dom}(\mu)$. For this ϑ, there exists a functional η as in Theorem 4.1.5. Clearly, we have

$$0 < \vartheta(Y) = \eta(Y) = \eta(Y \cap U) = \eta\left[Y \cap \left(\bigcup_{i=1}^{n} V_i\right)\right] = \eta\left[\bigcup_{i=1}^{n} (Y \cap V_i)\right] \tag{5.1.1}$$

Inequality 5.1.1 implies that $\eta(Y \cap V_i) > 0$ for some $i \in \{1, 2, \cdots, n\}$. Since $Y \cap V_i$ is a bounded subset of the Vitali selector V_i, it follows from Lemma 4.1.7 that it has the property described in Lemma 4.1.6. According to Lemma 4.1.6, we must have the equality $\eta(Y \cap V_i) = 0$, but this a contradiction. We conclude that the set U cannot contain any Lebesgue measurable set with positive measure. □

5.2 Semigroups of non-Lebesgue measurable sets generated by Bernstein sets

In this section, we construct families of sets for which elements are not measurable in the Lebesgue sense, by using the Bernstein subsets of $\mathbb{R}$. We will first consider the case of one Bernstein set and then two Bernstein sets having additional algebraic structures of being subsgroups of $(\mathbb{R}, +)$.

5.2.1 Semigroups related to a Bernstein subgroup of $(\mathbb{R}, +)$

Let B be a Bernstein subset of $\mathbb{R}$ which has the algebraic structure of being a subgroup of $(\mathbb{R}, +)$ as in Lemma 4.2.6. Consider the collection $\mathbb{R}/B = \{B + x : x \in \mathbb{R}\}$ of all cosets of B. Without losing generality, we may assume that $\mathbb{R}/B$ consists of pairwise disjoint sets, and for simplicity the collection $\mathbb{R}/B$ will be denoted by $\mathcal{B}$. Hence, $\mathcal{B}$ is made of pairwise translated copies of B by real numbers. From [7], we observe that $\text{Card}(\mathcal{B}) \geq \aleph_0$, where $\aleph_0 = \text{Card}(\mathbb{N})$, and $\text{Card}(\mathcal{B})$ is the same as the cardinal of the set $\{\mathbb{R} \setminus Y : Y \in \mathcal{B}\}$. The family $\mathcal{B}_E(\mathbb{R})$ is invariant under the action of the group $\mathcal{H}(\mathbb{R})$. It follows that each element of $\mathcal{B}$ is also a Bernstein set on $\mathbb{R}$. Let $\mathcal{S}(\mathcal{B}) = \{\bigcup_{i=1}^{n} B_i : B_i \in \mathcal{B}, n \in \mathbb{N}\}$ be the semigroup of sets generated by $\mathcal{B}$. Evidently, the family $\mathcal{S}(\mathcal{B})$ is invariant under translation of $\mathbb{R}$ into itself.

Lemma 5.2.1: If U is an element of the semigroup $\mathcal{S}(\mathcal{B})$ then U is a Bernstein subset of $\mathbb{R}$. Consequently, the semigroup $\mathcal{S}(\mathcal{B})$ consists of sets which are not measurable in the Lebesgue sense.

Proof. Assume that $U \in \mathcal{S}(\mathcal{B})$. Then there exists $n \in \mathbb{N}$ such that $U = \bigcup_{i=1}^{n} B_i$ where $B_i \in \mathcal{B}$ for $i = 1, 2, \cdots, n$. Let F be an uncountable closed subset of $\mathbb{R}$.

Since each B_i is a Bernstein subset of $\mathbb{R}$ then we have $F \cap B_i \neq \emptyset$ for each $i = 1, 2, \cdots, n$. It follows that $F \cap U = F \cap (\bigcup_{i=1}^{n} B_i) = \bigcup_{i=1}^{n} (F \cap B_i) \neq \emptyset$.

Assume that $F \cap (\mathbb{R} \setminus U) = \emptyset$. Then $F \subseteq U$. Let B_k be an element of $\mathcal{B}$ for some $k \notin \{1, 2, \cdots, n\}$. Such an element exists, since $\text{Card}(\mathcal{B}) \geq \aleph_0$. Since each element of $\mathcal{B}$ is a Bernstein set, we must have $F \cap B_k \neq \emptyset$ but by construction, $B_k \cap U = B_k \cap (\bigcup_{i=1}^{n} B_i) = \emptyset$ implying that $\emptyset \neq F \cap B_k \subseteq U \cap B_k = \emptyset$. Hence the inclusion $F \subseteq U$ is impossible. Necessarily, we must have $F \cap (\mathbb{R} \setminus U) \neq \emptyset$.

We conclude that U is a Bernstein subset of $\mathbb{R}$. As a Bernstein set, U is not measurable in the Lebesgue. □

Proposition 5.2.2: The families $\mathcal{S}(\mathcal{B}) * \mathcal{N}_0$ and $\mathcal{N}_0 * \mathcal{S}(\mathcal{B})$ are semigroups of sets on $\mathbb{R}$ such that $\mathcal{S}(\mathcal{B}) \subseteq \mathcal{N}_0 * \mathcal{S}(\mathcal{B}) \subseteq \mathcal{S}(\mathcal{B}) * \mathcal{N}_0$. They are invariant under translation and they consist of sets which are not measurable in the Lebesgue sense.

Proof. The families are semigroups of sets by Proposition 2.1.17 and the inclusions follow from the same proposition.

Let $A \in \mathcal{S}(\mathcal{B}) * \mathcal{N}_0$ and assume that $A \in \mathcal{L}(\mathbb{R})$. Then $A = (U \setminus M) \cup N$ where $U \in \mathcal{S}(\mathcal{B})$ and $M, N \in \mathcal{N}_0$. Note that $A \setminus U \subseteq N$ and $U \setminus A \subseteq M$ and hence $A \Delta U \subseteq M \cup N$. It follows that $\mu\left((A \setminus U) \cup (U \setminus A)\right) = \mu(A \Delta U) \leq \mu(M \cup N) = 0$ and thus $\mu(A \Delta U) = 0$. Lemma 3.2.15 indicates that the set U must be measurable in the Lebesgue sense. However, U is a Bernstein set on $\mathbb{R}$ and thus it is not measurable in the Lebesgue sense. This is a contradiction.

The family $\mathcal{S}(\mathcal{B}) * \mathcal{N}_0$ is invariant under the action of the group $\Phi(\mathbb{R})$ since both families $\mathcal{S}(\mathcal{B})$ and $\mathcal{N}_0$ are invariant under the action of the group $\Phi(\mathbb{R})$. □

Lemma 5.2.3: Let Y be a bounded subset of a Bernstein set A in the collection $\mathcal{B}$. Then Y has the property indicated in Lemma 4.1.6.

Proof. Consider a Bernstein set $A \in \mathcal{B}$ and let Y be a bounded subset of A. Then we have $A = B + x_0$, for some $x_0 \in \mathbb{R}$, where B is a Bernstein set having an algebraic structure of being a subgroup of $(\mathbb{R}, +)$ as described in Lemma 4.2.6, and $Y + x \subseteq B + x_0 + x = B + y$ for $y = x_0 + x \in \mathbb{R}$. In view of the definition of $\mathcal{B}$, the family $\mathcal{B} = \mathbb{R}/B = \{B + x : x \in \mathbb{R}\}$ of all cosets is made by pairwise disjoint sets. Consequently, the family $\{Y + x : x \in \mathbb{R}\}$ consists of pairwise disjoint sets. Since every infinite set contains an infinitely countable subset [17], let Λ be an infinitely countable bounded subset of $\mathbb{R}$. The family $\{x_k : x_k \in \Lambda, k = 1, 2, \cdots\}$ can play the role of $\{h_k : k \in \mathbb{N}\}$ in Lemma 4.1.6. It follows that if $Y \in \mathrm{dom}(\vartheta)$ then $\vartheta(Y) = 0$, and this ends the proof.□

Proposition 5.2.4: Let B be a Bernstein set of $\mathbb{R}$ that has the algebraic structure of being a subgroup of $(\mathbb{R}, +)$. Any element U of the semigroup $\mathcal{S}(\mathcal{B})$ cannot contain any set of strictly positive Lebesgue measure.

Proof. Suppose that there exists a Lebesgue measurable subset Y of $\mathbb{R}$ such that $\mu(Y) > 0$ and $Y \subseteq U$. Since $U \in \mathcal{S}(\mathcal{B})$ then $U = \bigcup_{i=1}^{n} B_i$ with $B_i \in \mathcal{B}$ for each $i = 1, 2, \cdots, n$. Since $Y = \bigcup_{r=-\infty}^{\infty} (Y \cap [r, r+1))$ and $\mu(Y) > 0$ then we have $\mu\left(Y \cap [r, r+1)\right) > 0$ for some r. Without loss of generality, we may assume that Y is bounded. Let ϑ be the restriction of μ to the family $\mathcal{B}_b(\mathbb{R}) \cap \mathrm{dom}(\mu)$. Note that the family $\mathcal{B}_b(\mathbb{R}) \cap \mathrm{dom}(\mu)$ is a ring of sets on $\mathbb{R}$. So there exits a functional η as in Theorem 4.1.5 extending ϑ on $\mathcal{B}_b(\mathbb{R})$. Then

$$0 < \vartheta(Y) = \eta(Y) = \eta(Y \cap U) = \eta\left[Y \cap \left(\bigcup_{i=1}^{n} B_i\right)\right] = \eta\left[\bigcup_{i=1}^{n} (Y \cap B_i)\right] \qquad (5.2.1)$$

Inequality 5.2.1 implies that $\eta(Y \cap B_i) > 0$ for some $i \in \{1, 2, \cdots, n\}$. Since $Y \cap B_i$ is a bounded subset of the Bernstein set $B_i \in \mathcal{B} \subseteq \mathcal{S}(\mathcal{B})$ then it has the property described in Lemma 5.2.3. According to Lemma 5.2.3, we must have the equality $\eta(Y \cap B_i) = 0$, and this is a contradiction. $\square$

Corollary 5.2.5: Let B be a Bernstein set of $\mathbb{R}$ which has an algebraic structure of being a subgroup of $(\mathbb{R}, +)$. Any element of the family $\mathcal{S}(\mathcal{B}) * \mathcal{N}_0$ cannot contain any set of strictly positive Lebesgue measure.

Proof. Consider $A \in \mathcal{S}(\mathcal{B}) * \mathcal{N}_0$. Note that $A = (U \setminus M) \cup N \subseteq U \cup N$, where $U \in \mathcal{S}(\mathcal{B})$ and $M, N \in \mathcal{N}_0$. Assume that there exists $Y \subseteq A$ such that $\mu(Y) > 0$. Note that $Y = (Y \cap U) \cup (Y \cap N)$. This implies that $0 < \mu(Y) \leq \mu(Y \cap U) + \mu(Y \cap N) = \mu(Y \cap U)$. Hence the set $Y \cap U$ is a subset of U with a strictly positive Lebesgue measure $Y \cap U$, which is impossible by Proposition 5.2.4. $\square$

Observe that a contradiction in the proof of Corollary 5.2.5 can be obtained by considering Lemma 5.2.1 and Theorem 4.2.4.

5.2.2 Semigroups related to two Bernstein subgroups of $(\mathbb{R}, +)$

Let B_1 and B_2 be Bernstein sets having an algebraic structure of being subgroups of $(\mathbb{R}, +)$ as in Theorem 4.2.7, and consider the families $\mathcal{B}_1 = \mathbb{R}/B_1$ and and $\mathcal{B}_2 = \mathbb{R}/B_2$ of all disjoint translates (cosets) of B_1 and B_2, respectively. Let $\mathcal{S}(\mathcal{B}_1)$ and $\mathcal{S}(\mathcal{B}_2)$ be the semigroups of sets generated by $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively.

Lemma 5.2.6: Let B_1 and B_2 be Bernstein sets having an algebraic structure of being subgroups of $(\mathbb{R}, +)$, and let $U_1 \in \mathcal{S}(\mathcal{B}_1)$ and $U_2 \in \mathcal{S}(\mathcal{B}_2)$. Then the union $U = U_1 \cup U_2$ cannot contain any subset of strictly positive Lebesgue measure.

Proof. Assume that there exists a Lebesgue measurable set Y such that $\mu(Y) > 0$ and $Y \subseteq U = U_1 \cup U_2$, where $U_1 = \bigcup_{i=1}^{n} B_{1i}$ and $U_2 = \bigcup_{k=1}^{m} B_{2k}$ with $B_{1i} \in \mathcal{B}_1$ and $B_{2k} \in \mathcal{B}_2$ for $i = 1, 2, \cdots, n$ and $k = 1, 2, \cdots, m$. Without loss of generality, we may assume that the set Y is bounded. It follows from Proposition 5.2.4 that the set Y cannot be placed entirely in U_1 nor in U_2. Write $U = \bigcup_{i=1}^{n+m} X_i$ where $X_i = B_{1i}$ for $i = 1, 2, \cdots, n$ and $X_{n+k} = B_{2k}$ for $k = 1, 2, \cdots, m$. Accordingly, we have

$$0 < \mu(Y) = \mu(Y \cap U) = \mu\left[\bigcup_{i=1}^{n+m}(Y \cap X_i)\right] \leq \sum_{i=1}^{n+m}\mu(Y \cap X_i). \tag{5.2.2}$$

Inequality 5.2.2 implies that $\mu(Y \cap X_i) > 0$ for some index $i \in \{1, 2, \cdots, n+m\}$. Since $Y \cap X_i \in \mathcal{B}_b(\mathbb{R})$, let ϑ be the restriction of μ to the ring $\mathcal{B}_b(\mathbb{R}) \cap \mathrm{dom}(\mu)$. For this ϑ, there exists a functional η as in Theorem 4.1.5 which is an extension of ϑ. It follows that $0 < \mu(Y \cap X_i) = \vartheta(Y \cap X_i) = \eta(Y \cap X_i)$. We will have two cases to consider:

- If $X_i \in \mathcal{B}_1$ then we must have $\eta(Y \cap X_i) = 0$ by Lemma 5.2.3.
- If $X_i \in \mathcal{B}_2$ we must have $\eta(Y \cap X_i) = 0$ by Lemma 5.2.3.

We conclude that the set Y cannot exist and this ends the proof. □

Theorem 5.2.7: The family $\mathcal{S}(\mathcal{B}_1) \vee \mathcal{S}(\mathcal{B}_2)$ is a semigroup of sets on $\mathbb{R}$ which consists of non-Lebesgue measurable sets and it is invariant under the action of the group $\Phi(\mathbb{R})$.

Proof. The family $\mathcal{S}(\mathcal{B}_1) \vee \mathcal{S}(\mathcal{B}_2)$ is a semigroup by Lemma 2.1.14. Let $U \in \mathcal{S}(\mathcal{B}_1) \vee \mathcal{S}(\mathcal{B}_2)$ and assume that U is a Lebesgue measurable set. Then $U = U_1 \cup U_2$ where $U_1 \in \mathcal{S}(\mathcal{B}_1)$ and $U_2 \in \mathcal{S}(\mathcal{B}_2)$.

- Assume that $\mu(U) = 0$. Then $\mu(U_1) = \mu(U_2) = 0$. This is a contradiction of Theorem 4.2.4, since U_1 and U_2 are Bernstein sets by Lemma 5.2.1.
- The inequality $\mu(U) > 0$ is impossible by Lemma 5.2.6.

It follows that the set U is not Lebesgue measurable.

It is evident that for any $h \in \Phi(\mathbb{R})$ we have $h(U) = h(U_1 \cup U_2) = h(U_1) \cup h(U_2) \in \mathcal{S}(\mathcal{B}_1) \vee \mathcal{S}(\mathcal{B}_2)$, due to the fact that both families $\mathcal{S}(\mathcal{B}_1)$ and $\mathcal{S}(\mathcal{B}_2)$ are invariant under the action of the group $\Phi(\mathbb{R})$. □

Theorem 5.2.8: The families $\mathcal{N}_0 * (\mathcal{S}(\mathcal{B}_1) \vee \mathcal{S}(\mathcal{B}_2))$ and $(\mathcal{S}(\mathcal{B}_1) \vee \mathcal{S}(\mathcal{B}_2)) * \mathcal{N}_0$ are semigroups of sets on $\mathbb{R}$ and they satisfy the inclusions: $\mathcal{S}(\mathcal{B}_1) \vee \mathcal{S}(\mathcal{B}_2) \subseteq \mathcal{N}_0 * (\mathcal{S}(\mathcal{B}_1) \vee \mathcal{S}(\mathcal{B}_2)) \subseteq (\mathcal{S}(\mathcal{B}_1) \vee \mathcal{S}(\mathcal{B}_2)) * \mathcal{N}_0$. They are invariant under the action of the group $\Phi(\mathbb{R})$ and they consist of sets which are not measurable in the Lebesgue sense.

Proof. The families are semigroups by Proposition 2.1.17. The inclusions follow from the same proposition. The family $(\mathcal{S}(\mathcal{B}_1) \vee \mathcal{S}(\mathcal{B}_2)) * \mathcal{N}_0$ is invariant under the action of the group $\Phi(\mathbb{R})$ by Theorem 5.2.7 and the fact that the collection $\mathcal{N}_0$ is invariant under the action of the group $\Phi(\mathbb{R})$. The proof that each element of the family $(\mathcal{S}(\mathcal{B}_1) \vee \mathcal{S}(\mathcal{B}_2)) * \mathcal{N}_0$ is not measurable in the Lebesgue sense goes in the same line as in Proposition 5.2.2 taking into consideration Theorem 5.2.7. □

The family $\mathcal{S}(\mathcal{B}_1) * \mathcal{S}(\mathcal{B}_1)$ does not need to be a semigroup of sets, but the following statement shows that it consists of elements which are not measurable in the Lebesgue sense.

Corollary 5.2.9: Each element of the family $\mathcal{S}(\mathcal{B}_1) * \mathcal{S}(\mathcal{B}_2)$ is not measurable in the Lebesgue sense.

Proof. Let $A \in \mathcal{S}(\mathcal{B}_1) * \mathcal{S}(\mathcal{B}_2)$ and assume that A is a Lebesgue measurable set. Note that $A = (U_1 \setminus U_2) \cup U_3$ for some $U_1 \in \mathcal{S}(\mathcal{B}_1)$ and $U_2, U_3 \in \mathcal{S}(\mathcal{B}_2)$. Since U_3 is Bernstein set and $U_3 \subseteq A$ then the set A cannot have the Lebesgue measure zero. Assume that $\mu(A) > 0$. It follows that $A = (U_1 \setminus U_2) \cup U_3 \subseteq U_1 \cup U_3 \in \mathcal{S}(\mathcal{B}_1) \vee \mathcal{S}(\mathcal{B}_2)$. By Lemma 5.2.6, the set $U_1 \cup U_2$ can not contain any set of strictly positive Lebesgue measure. □

5.3 Semigroups of non-Lebesgue measurable sets generated by a combination of Bernstein sets and Vitali selectors

We now combine Bernstein sets and Vitali selectors of $\mathbb{R}$, to construct families of sets for which elements are not measurable in the Lebesgue sense.

Theorem 5.3.1: Let B be a Bernstein subset of $\mathbb{R}$ which has an algebraic structure of being a subgroup of $(\mathbb{R}, +)$, and let $\mathcal{S}(\mathcal{V})$ be the semigroup generated by the collection $\mathcal{V}$ all Vitali selectors of $\mathbb{R}$. Any union $U = U_1 \cup U_2$, where $U_1 \in \mathcal{S}(\mathcal{B})$ and $U_2 \in \mathcal{S}(\mathcal{V})$, cannot contain any subset of strictly positive Lebesgue measure.

Proof. Assume that there exists a Lebesgue measurable set Y such that $\mu(Y) > 0$ and $Y \subseteq U = U_1 \cup U_2$, where $U_1 = \bigcup_{i=1}^{n} B_i$ and $U_2 = \bigcup_{k=1}^{m} V_k$ for $B_i \in \mathcal{B}$ and $V_k \in \mathcal{V}$. Without loss of generality, we may assume that the set Y is bounded. It follows from

Proposition 5.2.4 and Theorem 5.1.5 that the set Y cannot stay entirely in U_1 nor in U_2. Write $U = \bigcup_{i=1}^{n+m} X_i$ where $X_i = B_i$ for $i = 1, 2, \cdots, n$ and $X_{n+k} = V_k$ for $k = 1, 2, \cdots, m$. Then we have

$$0 < \mu(Y) = \mu(Y \cap U) = \mu\left[\bigcup_{i=1}^{n+m} (Y \cap X_i)\right] \leq \sum_{i=1}^{n+m} \mu(Y \cap X_i). \tag{5.3.1}$$

It follows from Inequality 5.3.1 that $\mu(Y \cap X_i) > 0$ for some index $i \in \{1, 2, \cdots, n+m\}$. Since $Y \cap X_i \in \mathcal{B}_b(\mathbb{R})$, let ϑ be the restriction of μ to the ring of sets $\mathcal{B}_b(\mathbb{R}) \cap \operatorname{dom}(\mu)$. For this ϑ there exists a functional η as in Theorem 4.1.5 which is an extension of ϑ. So we have $0 < \mu(Y \cap X_i = \vartheta(Y \cap X_i) = \eta(Y \cap X_i)$.

- If X_i is an element of $\mathcal{V}$ then $\eta(Y \cap X_i) = 0$ by Lemma 4.1.7.
- If X_i is an element of $\mathcal{B}$ then $\eta(Y \cap X_i) = 0$ by Lemma 5.2.3.

As a conclusion the set Y cannot exist. □

Corollary 5.3.2: Let B be a Bernstein set of $\mathbb{R}$ which has an algebraic structure of being a subgroup of $(\mathbb{R}, +)$, and let $\mathcal{S}(\mathcal{V})$ be the semigroup generated by the collection $\mathcal{V}$ all Vitali selectors of $\mathbb{R}$. Then the semigroup $\mathcal{S}(\mathcal{B}) \vee \mathcal{S}(\mathcal{V})$ consists of sets which are not measurable in the Lebesgue sense, and it is invariant under the action of the group $\Phi(\mathbb{R})$.

Proof. Assume that there exists a Lebesgue measurable set U in $\mathcal{S}(\mathcal{B}) \vee \mathcal{S}(\mathcal{V})$. Then $U = U_1 \cup U_2$ where $U_1 \in \mathcal{S}(\mathcal{B})$ and $U_2 \in \mathcal{S}(\mathcal{V})$. Since $U_2 = \bigcup_{i=1}^{n} V_i$ where $V_i \in \mathcal{V}(Q_i)$, let V_k be a fixed Vitali selector in this union such that $V_k \in \mathcal{V}(Q_k)$. Since $\mathbb{R} = \bigcup\{V_k + q : q \in Q_k\}$ and $V_k \subseteq U_2 \subseteq U$ then we have $\mathbb{R} = \bigcup\{U + q : q \in Q_k\}$. Given that $\mu(U + q) = \mu(U)$ and $\mu(\mathbb{R}) > 0$ then we must have $\mu(U) > 0$, and this contradicts Theorem 5.3.1.

Since the two families $\mathcal{S}(\mathcal{V})$ and $\mathcal{S}(\mathcal{B})$ are invariant under the action of the group $\Phi(\mathbb{R})$. It follows that the family $\mathcal{S}(\mathcal{B}) \vee \mathcal{S}(\mathcal{V})$ is invariant under the action of the group $\Phi(\mathbb{R})$□

Corollary 5.3.3: Each element of the family $\mathcal{S}(\mathcal{B}) * \mathcal{S}(\mathcal{V})$ is not measurable in the Lebesgue sense.

Proof. Let $A \in \mathcal{S}(\mathcal{B}) * \mathcal{S}(\mathcal{V})$ and assume that A is a Lebesgue measurable set. Then $A = (U_1 \setminus U_2) \cup U_3$ for some $U_1 \in \mathcal{S}(\mathcal{B})$ and $U_2, U_3 \in \mathcal{S}(\mathcal{V})$. Since U_3 is a finite union of Vitali selectors and $U_3 \subseteq A$ the set A cannot have the Lebesgue measure zero. Assume that $\mu(A) > 0$. It follows that $A = (U_1 \setminus U_2) \cup U_3 \subseteq U_1 \cup U_3 \in \mathcal{S}(\mathcal{B}_1) \vee \mathcal{S}(\mathcal{V})$. But by Theorem 5.3.1, the set $U_1 \cup U_2$ cannot contain any set of strictly positive Lebesgue measure. □

Let us point out that the family $\mathcal{S}(\mathcal{B}) * \mathcal{S}(\mathcal{V})$ does not need to be a semigroup of sets.

Theorem 5.3.4: The families $\mathcal{N}_0 * (\mathcal{S}(\mathcal{V}) \vee \mathcal{S}(\mathcal{B}))$ and $(\mathcal{S}(\mathcal{V}) \vee \mathcal{S}(\mathcal{B})) * \mathcal{N}_0$ are semigroups of sets on $\mathbb{R}$ satisfying the inclusions: $\mathcal{S}(\mathcal{V}) \vee \mathcal{S}(\mathcal{B}) \subseteq \mathcal{N}_0 * (\mathcal{S}(\mathcal{V}) \vee \mathcal{S}(\mathcal{B})) \subseteq (\mathcal{S}(\mathcal{V}) \vee \mathcal{S}(\mathcal{B})) * \mathcal{N}_0$. They are invariant under the action of the group $\Phi(\mathbb{R})$ and they consist of sets which are not measurable in the Lebesgue sense.

Proof. The given families are semigroups of sets by Proposition 2.1.17, and the inclusions follow from the same statement. Let $A \in (\mathcal{S}(\mathcal{V}) \vee \mathcal{S}(\mathcal{B})) * \mathcal{N}_0$ and assume that A is measurable in the Lebesgue sense. Then $A = ((U_1 \cup U_2) \setminus M) \cup N$, where $U_1 \in \mathcal{S}(\mathcal{V}), U_2 \in \mathcal{S}(\mathcal{B})$ and $M, N \in \mathcal{N}_0$. Note that $A \setminus (U_1 \cup U_2) \subseteq N$ and $(U_1 \cup U_2) \setminus A \subseteq M$ and hence $A\Delta(U_1 \cup U_2) \subseteq M \cup N$. It follows that $\mu(A\Delta(U_1 \cup U_2)) \leq \mu(M \cup N) = 0$ and thus $\mu(A\Delta(U_1 \cup U_2)) = 0$. It follows from Lemma 3.2.15 that the set $U_1 \cup U_2$ must be measurable in the Lebesgue sense. But, Corollary 5.3.2 tells us that the set $U_1 \cup U_1$ is not measurable in the Lebesgue sense, and we have a contradiction. □

We have used the σ-ideal $\mathcal{N}_0$ in the construction of different semigroups for which elements are not Lebesgue measurable. All the statements remain valid by using an ideal of sets such that $\mathcal{I} \subseteq \mathcal{N}_0$, for example the ideal of all finite subsets of $\mathbb{R}$, and the ideal of all countable subsets of $\mathbb{R}$.

Conclusion and suggestions for future works

This part is about the concluding points and some suggestions which can be taken into consideration for the future work on the related topic.

Conclusion

In this thesis entitled "*Semigroups of sets generated by non-Lebesgue measurable subsets of the real line*", we have considered two types of classical subsets of the real line which are not measurable in the Lebesgue sense.

It is known that all open subsets and closed subsets of the real line are Lebesgue measurable, and the family of all Lebesgue measurable subsets of the real line is closed under countable unions and countable intersections. It is hard to imagine a subset of the real line that is not Lebesgue measurable. However, many such sets exist.

To study the algebraic structures among the subsets of $\mathbb{R}$, which are not measurable in the Lebesgue sense, we first developed the theory of semigroups of sets and ideals of sets on any given non-empty set. We then applied the developed theory on Vitali selectors and Bernstein sets. From here, we constructed different families of sets, having an algebraic structure of being semigroups, and which are invariant under the action of the group of all translations of $\mathbb{R}$ onto itself.

The construction of such families was done in three steps: In the first step, we used Vitali selectors of $\mathbb{R}$ and the $\mathcal{N}_0$. In the second step, we used a kind of Bernstein set having an additional structure of being a subgroup of the additive group $(\mathbb{R}, +)$ of real numbers and the $\mathcal{N}_0$. In the third step, we combined the Vitali selectors and Bernstein sets, and consequently, the obtained results were in some sense extending the results obtained in the first and second steps. In particular, we show that the family $(\mathcal{S}(\mathcal{B}) \vee \mathcal{S}(\mathcal{V})) * \mathcal{N}_0 := \{((U_1 \cup U_2) \setminus N) \cup M : U_1 \in \mathcal{S}(\mathcal{B}), U_2 \in \mathcal{S}(\mathcal{V}), N, M \in \mathcal{N}_0\}$ is a semigroup of sets, invariant under translations, and consists of sets which are not measurable in the Lebesgue sense. Here, $\mathcal{S}(\mathcal{B})$ is the collection of all finite unions of some type of Bernstein subsets of $\mathbb{R}$; $\mathcal{S}(\mathcal{V})$ is the collection of all finite unions of Vitali selectors of $\mathbb{R}$. Apart from the σ-ideal $\mathcal{N}_0$ any subideal of $\mathcal{N}_0$ can be used to produce semigroups of sets for which elements are not measurable in the Lebesgue sense.

Recommendations

After this thesis, some special points have to be enumerated as suggestions for future works in the same direction.

- Consider other types of non-measurable sets (nonmeasurable sets associated with Hamel bases) and examine the algebraic structures, from the set-theoretic point of view, among these classes of sets.
- It would be interesting to extend the study to other measures and abstract spaces. A non-measurable set in the Lebesgue sense, may have nice properties with respect to other measures.
- It would be interesting to extend the results in finite-dimensional Euclidean spaces to generalize the obtained results on the real line.
- It would be better to find other types of structures that can be satisfied by non-Lebesgue measurable sets.
- Introduce other types of binary operations between families of sets in order to extend the theory of semigroups and ideal of sets.
- Because a master's degree in science was planned to last only two years, I encourage that the University of Rwanda can make some effort so that the next intake will not face the same delay.

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