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DEPARTMENT OF MATHEMATICS

REPRESENTATION OF BOUNDED C-QUASILINEAR FUNCTIONAL DEFINED ON THE SET OF H-CONTINUOUS INTERVAL VALUED FUNCTIONS

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Declaration

I, Jérôme Mwizerwa, certify that the research thesis " REPRESENTATION OF BOUNDED C-QUASILINEAR FUNCTIONAL DEFINED ON THE SET OF H-CONTINUOUS INTERVAL VALUED FUNCTIONS " is my own work . It has never before submitted or published in other academic institution.

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Approval

We, the undersigned, certify that we have read and hereby recommend for acceptance by the University of Rwanda, a dissertation entitled "Representation of bounded c-quasilinear functional defined on the set of H-continuous interval valued functions " in partial fulfillment of the requirements for the award of the Master of Science in Applied Mathematics.

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Dedication

I dedicate this dissertation to my family since I appreciate their support and encouragement during my time as a student.

Acknowledgment

Many people provided assistance and support in order for this job to be completed. Several people and organizations contributed significantly and incalculably to the completion of this dissertation. I give a debt of gratitude to everyone who helped and supported me but who is too numerous to list here.

I want to express my gratitude to the All-Powerful God for always keeping an eye on me and providing me with the extra push I need to complete my task.

I sincerely appreciate my family members contributions to the accomplishment of this project.

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May the God of all power bless everyone!

Mwizerwa Jérôme

Abstract

Every bounded linear functional from the set $C[0, 1]$ of continuous functions on $[0, 1]$, endowed with the supremum norm, can be written as a Riemann-Stieltjes integral on $[0, 1]$, according to a valuable statement proposed and proven by F. Riesz in 1909. This result is now known as Riesz's representation theorem.

The above result has been generalized from various class $C(X)$ of continuous functions on various topological spaces X.

Moreover, it has been generalized in the set of p- integrable real valued functions defined on X. The goal of this work is to extend the Riesz's representation theorem taking into account that the set of continuous real valued functions $C(X)$ is replaced by the set of interval valued functions and the notion of linear functional is replaced by the notion of quasilinear functional. It has been shown that the bounded convex quasilinear functional defined on the set of Hausdorff continuous functions on $[0, 1]$ can be represented as Henstock-Stieltjes integral of interval valued function on $[0, 1]$.

Contents

CHAPTER 1

Introduction

1.1 Background of the study

The Riesz's representation theorem is a useful result which describes the bounded linear functional acting on the space $C(K)$ of continuous functions defined on a set K. Surprisingly, all functionals are just integrals and vice versa. If K is a closed interval of real numbers then any such functional is represented by Riemann-Stieltjes integral as follows.

Theorem 1.1.1. *[1] Assume that $f : C[0,1] \longrightarrow \mathbb{R}$ is a bounded linear functional . There exists $w : [0,1] \longrightarrow \mathbb{R}$ a function of bounded variation such that*

$$f(x) = \int_0^1 x(t)dw(t)\ \forall x \in C[0,1]. \tag{1.1}$$

The integral in this case is known as the $[0,1]$ Riemann-Stieltjes integral.

The next Hahn-Banach extension theorem is one of the premises for the previous theorem's proof.

Theorem 1.1.2. *[1] (Hahn-Banach Theorem for Normed Space)*
If f is a bounded linear functional on subspace Z of a normed space X,
then there exists a bounded linear functional $\tilde{f}$ on X such that $\tilde{f}$ is an extension of f to X and $||\tilde{f}||_X = ||f||_Z$.

In [2], Theorem 1.1.2 has been extended to interval valued functions from a real normed space to the set $\mathbb{IR} = \{[\underline{a}, \overline{a}] : \underline{a}, \overline{a} \in \mathbb{R}, \underline{a} \leq \overline{a}\}$.

Below, we give some properties of the set $\mathbb{IR}$ given above.

Given $a = [\underline{a}, \overline{a}] \in \mathbb{IR}$, $w(a) = \overline{a} - \underline{a}$ is the width of a, while $|a| = max\{|\underline{a}|, |\overline{a}|\}$ is the modulus of a.

In [3] an element $a \in \mathbb{IR}$ is called proper interval if $w(a) > 0$ and point interval if $w(a) = 0$.

Identifying $a \in \mathbb{R}$ with the point interval $a = [a, a] \in \mathbb{IR}$, we consider $\mathbb{R}$ as a subset of $\mathbb{IR}$.

A partial order which extends the total order on $\mathbb{R}$ can be defined on $\mathbb{IR}$ in more than one way. However, it will prove useful to consider on $\mathbb{IR}$ the partial order $\leq$ defined in [3], [4]

$$[\underline{a}, \overline{a}] \leq [\underline{b}, \overline{b}] \Longleftrightarrow \underline{a} \leq \underline{b}, \overline{a} \leq \overline{b}, \tag{1.2}$$

where $[\underline{a}, \overline{a}]$, $[\underline{b}, \overline{b}] \in \mathbb{IR}$.

For example, $[1, 2] \leq [2, 3]$.

In [4], the inclusion on $\mathbb{IR}$ is defined by:

$$[\underline{a}, \overline{a}] \subseteq [\underline{b}, \overline{b}] \Leftrightarrow \underline{b} \leq \underline{a} \leq \overline{a} \leq \overline{b}, \tag{1.3}$$

where $[\underline{a}, \overline{a}]$, $[\underline{b}, \overline{b}] \in \mathbb{IR}$.

For example, $[-1, 2] \subseteq [-2, 3]$.

In [4], interval addition and multiplication by scalar are defined a point wise as usually:

$$[\underline{a}, \overline{a}] + [\underline{b}, \overline{b}] = [\underline{a} + \underline{b}, \overline{a} + \overline{b}], \tag{1.4}$$

$$\alpha.[\underline{a}, \overline{a}] = \begin{cases} [\alpha\underline{a}, \alpha\overline{a}] \text{ for } \alpha \geq 0 \\ [\alpha\overline{a}, \alpha\underline{a}] \text{ for } \alpha < 0, \end{cases} \tag{1.5}$$

where $[\underline{a}, \overline{a}]$, $[\underline{b}, \overline{b}] \in \mathbb{IR}$ and $\alpha \in \mathbb{R}$.

For example, let $a = [-1, 2], b = [-2, 3] \in \mathbb{IR}$ and $\alpha = 2$.

Then $a + b = [-1, 2] + [-2, 3] = [-1 - 2, 2 + 3] = [-3, 5]$ and $\alpha.a = 2.[-1, 2] = [-2, 4]$.

In [5], it has been shown that $(\mathbb{IR}, +, ., \leq)$ is quasilinear space over $\mathbb{R}$.

A more general definition of a quasilinear space is given below.

Definition 1.1.1. *[6], [2] An algebraic system $(X, +, ., \preceq)$ is a quasilinear space (briefly;QLS) over $\mathbb{R}$ if the following conditions hold for any elements $x, y, z, v \in X$ and any $\alpha, \beta \in \mathbb{R}$,*

$$x \preceq x,$$

$$x \preceq z \textit{ if } x \preceq y \textit{ and } y \preceq z,$$

$$x = y \textit{ if } x \preceq y \textit{ and } y \preceq x,$$

$$x + y = y + x,$$

$$x + (y + z) = (x + y) + z,$$

$$\textit{there exists an element (zero) } \theta \in X \textit{ such that } x + \theta = x,$$

$$\alpha.(\beta.x) = (\alpha\beta).x,$$

$$\alpha.(x + y) = \alpha.x + \alpha.y,$$

$$1.x = x,$$

$$0.x = \theta,$$

$$(\alpha + \beta).x \preceq \alpha.x + \beta.x,$$

$$x + z \preceq y + v \textit{ if } x \preceq y \textit{ and } z \preceq v,$$

$$\alpha.x \preceq \alpha.y \textit{ if } x \preceq y.$$

Note that every linear space is quasilinear space.

Definition 1.1.2. *[6], [2] Let $X(, +, ., \preceq)$ be quasilinear space over $\mathbb{R}$. A function $||.||_X : X \to \mathbb{R}$ is called a norm on X if for all $x, y \in X$ and $\alpha \in \mathbb{R}$:*

$$||x||_X > 0 \textit{ if } x \neq 0,$$

$$||x + y||_X \leq ||x||_X + ||y||_X,$$

$$||\alpha.x||_X = |\alpha|||x||_X,$$

$$\textit{if } x \preceq y, \textit{ then } ||x||_X \leq ||y||_X,$$

if for any $\varepsilon > 0$ *there exists an element* $x_\varepsilon \in X,$ *such that* $x \preceq y + x_\varepsilon$ *and* $||x_\varepsilon||_X \leq \varepsilon,$ *then* $x \preceq y.$

A quasilinear space X with a norm defined on it is called normed quasilinear space (briefly, normed QLS).

Note that every normed linear space is a normed quasilinear space.

In [7], it has been shown that $(\mathbb{IR}, +, ., \leq)$ is a normed quasilinear space on $\mathbb{R}$ with norm $||a|| = |a| = max\{|\underline{a}|, |\overline{a}|\}, \forall a = [\underline{a}, \overline{a}] \in \mathbb{IR}$.

Definition 1.1.3. *[6], [2] Let* $(X, +, ., \preceq)$ *be quasilinear space over* $\mathbb{R}$*.*
A mapping $f : X \to \mathbb{IR}$ *is called a convex-quasilinear functional (briefly, c-quasilinear functional) if for all* $x_1, x_2, x \in X$ *and* $\alpha \in \mathbb{R}$*, we have*

$$f(x_1 + x_2) \leq f(x_1) + f(x_2),$$

$$f(\alpha x) = \alpha f(x),$$

$$if x_1 \preceq x_2 \text{ then } f(x_1) \leq f(x_2). \tag{1.6}$$

Definition 1.1.4. *[6], [2] Let* X *be normed quasilinear space over* $\mathbb{R}$ *with norm* $||.||_X$*.*
A quasi-linear functional $f : X \to \mathbb{IR}$ *is called a bounded c-quasilinear functional if there exists a number* $k > 0$ *such that* $|f(x)| \leq k||x||_X$ *for any* $x \in X$*.*

As we said, Theorem 1.1.2 has been extended to the interval valued functions as follows.

Theorem 1.1.3. *[2] Let* Z *be a subspace of real normed space* X *and assume that* $f : Z \to \mathbb{IR}$ *is a bounded c-quasilinear functional on* Z*. Then there exists a bounded c-quasilinear extension* $\tilde{f} : X \to \mathbb{IR}$ *of* f *such that* $||\tilde{f}||_X = ||f||_Z$*,*

$$\text{where } ||\tilde{f}||_X = \sup_{x \in X, ||x||=1} |\tilde{f}(x)| \text{ and } ||f||_Z = \sup_{x \in Z, ||x||=1} |\tilde{f}(x)|.$$

1.2 Problem statement

In this work, instead to consider bounded linear functional from $C[0,1]$, we will consider the bounded c-quasilinear functional acting from the space $\mathbb{H}_b[0,1]$ of bounded Hausdorff continuous interval valued functions defined on $[0,1]$. We will prove that every bounded c-quasilinear functional acting from a space $\mathbb{H}_b[0,1]$ to the set $\mathbb{IR}$ can be represented as the kind of integral of interval valued functions which is an extension of Riemann-Stieltjes integral. This establishes a new version of Riesz's representation theorem on the set of bounded interval valued functions $\mathbb{H}_b[0,1]$.

1.3 Motivation of the study

For extending Theorem 1.1.1 to interval valued functions we have the following four motivations.

The first motivation of this work is that the concept of Hausdorff continuous for interval valued functions generalizes the concepts of continuous for real valued functions.

The second motivation is that bounded linear functionals are the particular cases of the bounded quasilinear functionals.

The third motivation is the Hahn Banach Extension theorem for bounded linear functionals, which is useful for proving Riesz's representation theorem, can be extended to the set of interval valued functions.

The last motivation is that the Henstock -Stieltjes integrals for interval valued functions generalizes the Riemann-Stieltjes integrals for real valued functions.

1.4 Objective

1.4.1 General objective

Theorem 1.1.1 states that any bounded linear functional on $C[0,1]$ can be expressed as Riemann -Stieltjes integral. The objective of this work is to extended Theorem 1.1.1 so that any bounded convex quasilinear functional on the set $\mathbb{H}_b(\Omega)$ of bounded Hausdorff continuous interval valued functions on $[0,1]$ can be written as Henstock-Stieltjes integral for interval valued functions.

1.4.2 Specific objectives

(1) To explore the application of Hahn-Banach extension theorem for interval valued functions.

(2) To display some important properties of Hausdorff continuous functions .

(3) To establish that the result related to the Riemann-Stieltjes integral for real valued functions can be extended to the result known as the Henstock -Stieltjes integrals for interval valued functions.

1.5 Methodology

The proof of main theorem is based on the extension of the results of Hahn Banach extension theorem to the set of interval valued functions. Start by reviewing the existing literature on Theorem 1.1.1 and its generalizations. Analyze how the Riemann-Stieltjes integral is applied to bounded linear functionals on $C[0,1]$. Develop the theoretical framework to generalize this theorem to interval-valued functions. This involves studying the properties of convex quasilinear functionals and their representation as integrals. Formulate and prove the extension of Theorem 1.1.1 to include Henstock-Stieltjes integrals for interval-valued functions. Determine and investigate the key features of Hausdorff continuous functions, especially their boundedness, reliability, and how they behave under different operations. Describe how the integration of interval-valued functions in the Henstock-Stieltjes integral extends the area it covers compared to the Riemann-Stieltjes integral. The same methodology that will be used is that every Hausdorff continuous interval valued functions can be approximated by interval step functions using Hausdorff distance between functions in the set of Hausdorff continuous functions defined on $[0,1]$.

1.6 The structure of the work

In chapter 2, we give preliminaries notions and results including Hausdorff continuous functions and Henstock-Stieltjes integral which are useful in Chapter 3. Chapter 3 gives the main result. We state and prove this main result. In chapter 4, we give conclusion and recommendation.

CHAPTER 2

Preliminaries

We provide some concepts and results in this chapter that will be used in Chapter 3. Particularly, we will discuss the concept of Hausdorff continuous interval valued functions and present a few theorems that are helpful in demonstrating the primary result. In addition, the concept of Henstock-Stieltjes integrals for interval valued functions will be covered, together with the demonstration that the Riemann-Stieltjes integral is a particular case of the Henstock-Stieltjes integral.

2.1 Hausdorff continuous interval valued functions

In this section, we consider interval valued function from an open set $\Omega \subseteq \mathbb{R}^n$ to the set $\mathbb{IR}$. In [7], [8], real or interval function f on Ω is locally bounded if for every $x \in \Omega$ there exist $\delta > 0$ and $M \in \mathbb{R}$ such that $|f(y)| \leq M$, for all $y \in B_\delta(x)$,
where $B_\delta(x) = \{y \in \Omega : ||x - y|| < \delta\}$ and $||.||$ is a norm on Ω .
We denoted:

$$\mathbb{A}(\Omega) = \{f : \Omega \to \mathbb{IR} : f \text{ is locally bounded on } \Omega\},$$

$$\mathcal{A}(\Omega) = \{f : \Omega \to \mathbb{R} : f \text{ is locally bounded on } \Omega\}.$$

Since $\mathbb{R}$ is the subset of $\mathbb{IR}$ and every continuous real valued function is locally bounded, we have $C(\Omega) \subseteq \mathcal{A}(\Omega) \subseteq \mathbb{A}(\Omega)$,
where $C(\Omega)$ is the set of all continuous functions from Ω to $\mathbb{R}$.
Let $f \in \mathbb{A}(\Omega)$. For every $x \in \Omega$, the value of f is an interval $[\underline{f}(x), \overline{f}(x)] \in \mathbb{IR}$.
Hence the function f can be written in the form

$$f = [\underline{f}, \overline{f}], \text{ where } \underline{f}, \overline{f} \in \mathcal{A}(\Omega) \text{ and } \underline{f}(x) \leq \overline{f}(x), x \in \Omega.$$

The partial order induced in $\mathbb{A}(\Omega)$ by (1.2) in a point wise way such that

$$f \leq g \Leftrightarrow \underline{f}(x) \leq \underline{g}(x), \overline{f}(x) \leq \overline{g}(x), x \in \Omega, \tag{2.1}$$

where $f = [\underline{f}, \overline{f}], g = [\underline{g}, \overline{g}] \in \mathbb{A}(\Omega)$.

The inclusion induced in $\mathbb{A}(\Omega)$ by (1.3) as follows:

$$f \subseteq g \Leftrightarrow \underline{g}(x) \leq \underline{f}(x) \leq \overline{f}(x) \leq \overline{g}(x), x \in \Omega, \tag{2.2}$$

where $f = [\underline{f}, \overline{f}], g = [\underline{g}, \overline{g}] \in \mathbb{A}(\Omega)$.

Definition 2.1.1. *[7] Let Ω be an open subset of $\mathbb{R}^n$ and D be dense subset of Ω .*
The functions $I(D, \Omega, .); S(D, \Omega, .) : \mathbb{A}(D) \to \mathcal{A}(\Omega)$ defined for $f = [\underline{f}, \overline{f}] \in \mathbb{A}(D)$ and $x \in \Omega$ by :

$$I(D, \Omega, f)(x) = \sup_{\delta > 0} \inf\{z \in f(y) : y \in B_\delta(x) \cap D\}, \tag{2.3}$$

$$S(D, \Omega, f)(x) = \inf_{\delta > 0} \sup\{z \in f(y) : y \in B_\delta(x) \cap D\}, \tag{2.4}$$

are called, respectively, lower and upper Baire operators.

For every $f \in \mathbb{A}(D)$, $x \in \Omega$, we have $I(D, \Omega, f)(x) \leq f(x) \leq S(D, \Omega, f)(x)$, $x \in \Omega$.

Definition 2.1.2. *[7] Let Ω be an open subset of $\mathbb{R}^n$ and D be dense subset of Ω .*
The function $F(D, \Omega, .) : \mathbb{A}(D) \to \mathbb{A}(\Omega)$ defined for $f \in \mathbb{A}(D)$, $x \in \Omega$ by
$F(D, \Omega, f)(x) = [I(D, \Omega, f)(x), S(D, \Omega, f)(x)]$
is called the graph completion operator.

In [7], for $D = \Omega$ we write, $I(f) = I(\Omega, \Omega, f), S(f) = S(\Omega, \Omega, f), F(f) = F(\Omega, \Omega, f)$. In [7], using end-point presentation of functions:$f = [\underline{f}, \overline{f}] \in \mathbb{A}(\Omega)$, we can write $I(D, \Omega, f) = I(D, \Omega, \underline{f}), S(D, \Omega, f) = S(D, \Omega, \overline{f}), F(D, \Omega, f) = [I(D, \Omega, \underline{f}), S(D, \Omega, \overline{f})]$.

The next theorem combines some characteristics of the graph completion operator and the lower and upper Baire operators.

Theorem 2.1.1. *[7]*

(a) *Let Ω be a subset of $\mathbb{R}^n$ and D be a dense subset of Ω .*
If $f, g \in \mathbb{A}(D)$, then

$$f(x) \leq g(x), x \in D \Rightarrow \begin{cases} I(D,\Omega,f)(x) \leq I(D,\Omega,g)(x) & x \in \Omega \\ S(D,\Omega,f)(x) \leq S(D,\Omega,g)(x) & x \in \Omega \\ F(D,\Omega,f)(x) \leq F(D,\Omega,g)(x) & x \in \Omega. \end{cases}$$

(b) *Let Ω be a subset of $\mathbb{R}^n$ and D be a dense subset of Ω .*
If $f, g \in \mathbb{A}(D)$, then

$$f(x) \subseteq g(x), x \in D \Rightarrow F(D,\Omega,f)(x) \subseteq F(D,\Omega,g)(x), x \in \Omega.$$

(c) *Let Ω be a subset of $\mathbb{R}^n$ and D_1, D_2 be dense subsets of Ω.*
If $f \in \mathbb{A}(D_1 \cup D_2)$, then $D_1 \subseteq D_2 \Rightarrow F(D_1,\Omega,f)(x) \subseteq F(D_2,\Omega,f)(x), x \in \Omega$.
In particular for any dense subset D of Ω and $f \in \mathbb{A}(D)$,
we have $F(D,\Omega,f)(x) \subseteq F(f)(x), x \in \Omega$.

(d) *Let Ω be a subset of $\mathbb{R}^n$ and D_1 , D_2 be dense subsets of Ω .*
If $D_1 \subseteq D_2$, then

$$I(D_2,\Omega,.) \circ I(D_1,\Omega,.) = I(D_1,\Omega,.),$$

$$S(D_2,\Omega,.) \circ S(D_1,\Omega,.) = S(D_1,\Omega,.),$$

$$F(D_2,\Omega,.) \circ F(D_1,\Omega,.) = F(D_1,\Omega,.).$$

In particular for $D_1 = D_2 = \Omega$ and $f \in \mathbb{A}(\Omega)$, we have

$$I(I(f)) = I(f),$$

$$S(S(f)) = S(f),$$

$$F(F(f)) = F(f).$$

Next, we give the relationship between semi-continuous real valued functions and lower and upper Baire operators.
First, recall the definition of semi-continuous real valued functions.

Definition 2.1.3. *[9]*

(a) *Let Ω be a subset of $\mathbb{R}^n$.*
A real-valued function $f : \Omega \to \mathbb{R}$ is upper semi-continuous at a point $x_0 \in \Omega$ if, the function values for arguments near x_0 are either close to $f(x_0)$ or less than $f(x_0)$.
We denote $USC(\Omega) = \{f : \Omega \to \mathbb{R} : f$ is upper semi-continuous at each point of $\Omega\}$

(b) *Let Ω be a subset of $\mathbb{R}^n$.*
A real-valued function $f : \Omega \to \mathbb{R}$ is lower semi-continuous at a point $x_0 \in \Omega$ if, the function values for arguments near x_0 are either close to $f(x_0)$ or greater than $f(x_0)$.
We denote $LSC(\Omega) = \{f : \Omega \to \mathbb{R} : f$ is lower semi-continuous at each point of $\Omega\}$

Note that $C(\Omega) = USC(\Omega) \cap LSC(\Omega)$.

Theorem 2.1.2. *[7],[10] Let D be a dense subset of Ω and Ω be a subset of $\mathbb{R}^n$. Then we have $I(D, \Omega, f) \in LSC(\Omega)$ and $S(D, \Omega, f) \in USC(\Omega)$.*

Theorem 2.1.3. *[7] Let D be a dense subset of Ω, $f \in \mathbb{A}(D)$, and Ω be a subset of $\mathbb{R}^n$.*

(a) *If $g(x) \in LSC(\Omega)$ and $g(x) \leq f(x), x \in D$, then $g(x) \leq I(D, \Omega, f)(x), x \in \Omega$.*

(b) *If $g(x) \in USC(\Omega)$ and, $f(x) \leq g(x), x \in D$, then $S(D, \Omega, f)(x) \leq g(x), x \in \Omega$.*

From Theorem 2.1.3, we have the following:

$$\text{If } f \in LSC(\Omega), \text{ then } I(f) = f. \tag{2.5}$$

$$\text{If } f \in USC(\Omega), \text{ then } S(f) = f. \tag{2.6}$$

Now we want to introduce 2 concepts of continuity of interval valued functions generalizing the concepts of continuity of usual real valued functions.

Definition 2.1.4. *[7] Let Ω be the subset of $\mathbb{R}^n$. A function $f \in \mathbb{A}(\Omega)$ is S-continuous, if $F(f) = f$. We denote $\mathbb{F}(\Omega) = \{f \in \mathbb{A}(\Omega) : F(f) = f\}$.*

Definition 2.1.5. *[7] Let Ω be a subset of $\mathbb{R}^n$. A function $f \in \mathbb{A}(\Omega)$ is called Hausdorff continuous, (H-continuous), if and only if for every function $g \in \mathbb{A}(\Omega)$,*
we have $g(x) \subseteq f(x), x \in \Omega \Rightarrow F(g)(x) = f(x), x \in \Omega$.
We denote $\mathbb{H}(\Omega) = \{f \in \mathbb{A}(\Omega) :$ f is Hausdorff continuous function on $\Omega\}$.

It is easy to prove that $C(\Omega) \subseteq \mathbb{H}(\Omega) \subseteq \mathbb{F}(\Omega)$.

In fact, first prove that $C(\Omega) \subseteq \mathbb{H}(\Omega)$. Let $f \in C(\Omega)$. Since f is continuous function then f is both lower and upper semi-continuous means that $C(\Omega) = LS(\Omega) \cap US(\Omega)$, using (2.5) and (2.6) we have $F(f) = [I(f), S(f)] = [f, f] = f$ and this shows that $C(\Omega) \subseteq \mathbb{F}(\Omega)$.

Furthermore , if $g \in \mathbb{A}(\Omega)$ such that $g(x) \subseteq f(x)$, $x \in \Omega$, then $g(x) = f(x)$, $x \in \Omega$, because $f(x)$ is point interval for an $x \in \Omega$. Hence $F(g)(x) = F(f)(x)$, $x \in \Omega$ and the Hausdorff continuity of f follows for interval continuous of Hausdorff continuous functions.

Second, prove that $\mathbb{H}(\Omega) \subseteq \mathbb{F}(\Omega)$.

If $f \in \mathbb{H}(\Omega)$, then for an inclusion $f(x) \subseteq g(x)$, $x \in \Omega$ and definition of Hausdorff continuous functions it follows that $F(f)(x) = f(x)$, $x \in \Omega$. Thus $f \in \mathbb{F}(\Omega)$.

We remark that there exist a S-continuous functions which is not a H-continuous functions as shown by the following example.

Example 2.1.1. *Consider the following example, when* $\Omega = [1, 2]$.
Then, the function f*, defined by :*

$$f(x) = \begin{cases} [-1, 1], x = 1 \\ 1, 1 < x \leq 2 \end{cases}$$

is S-continuous at $\Omega = [1, 2]$ *since* $F(f) = f$ *on* Ω *but it is not H-continuous on* $\Omega = [1, 2]$*. In fact, consider the function on* $[1, 2]$

$$g(x) = \begin{cases} [0, 1], x = 1 \\ 1, 1 < x \leq 2. \end{cases}$$

We have $g(x) \subseteq f(x)$ *for every* $x \in [1, 2]$ *and* g *is S-continuous on* $[1, 2]$ *But*$F(g)(1) = g(1) = [0, 1] \neq [-1, 1] = f(1)$ *so* f *cannot be H-continuous function on* $[1, 2]$ *according to Definition 2.1.5.*

However, we have the following theorem showing additionally conditions to S-continuous functions to be Hausdorff continuous functions.

Theorem 2.1.4. *[7] Let Ω be a subset of $\mathbb{R}^n$.*
A function $f \in \mathbb{A}(\Omega)$ is H-continuous if and only if f satisfies the following two conditions:

(a) f is S-continuous on Ω,

(b) For every S-continuous function g, the inclusion
$g(x) \subseteq f(x), x \in \Omega \Longrightarrow g(x) = f(x), x \in \Omega.$

The following theorem shows how to construct S-continuous and H-continuous functions.

Theorem 2.1.5. *[7] Let Ω be a subset of $\mathbb{R}^n$.*
If $\underline{f} \in LSC(\Omega)$ and $\overline{f} \in USC(\Omega)$ such that $\underline{f} \leq \overline{f}$ on Ω, then $f = [\underline{f}, \overline{f}] \in F(\Omega)$. Furthermore, if the set $\{\varphi \in \mathbb{A}(\Omega) : \underline{f} \leq \varphi \leq \overline{f}\} = \{\underline{f}, \overline{f}\}$, then $f = [\underline{f}, \overline{f}] \in \mathbb{H}(\Omega)$.

The necessary and sufficient conditions for a function $f \in \mathbb{A}(\Omega)$ to be a Hausdorff continuous function are provided by the following theorem.

Theorem 2.1.6. *[7] Let Ω be a subset of $\mathbb{R}^n$ and $f = [\underline{f}; \overline{f}] \in \mathbb{A}(\Omega)$.*
The following conditions are equivalent :

(a) $f \in \mathbb{H}(\Omega)$,

(b) $F(\underline{f}) = F(\overline{f}) = f$,

(c) $S(\underline{f}) = \overline{f}, I(\overline{f}) = \underline{f}$, and $f \in \mathbb{F}(\Omega)$.

With any interval valued function we can construct Hausdorff continuous functions as follows.

Theorem 2.1.7. *[7] Let Ω be a subset of $\mathbb{R}^n$. If $f \in \mathbb{A}(\Omega)$, then $F(I(S(f)))$ and $F(S(I(f)))$ are H-continuous and we have $F(S(I(f))) \leq F(I(S(f)))$.*

We use the following example to demonstrate the previous theorem.

Example 2.1.2. *Consider the function $f \in \mathbb{A}(\mathbb{R})$ given by*

$$f(x) = \begin{cases} [-2, 2], x \in \mathbb{Z} \\ 0, x \in (-\infty, 0) \backslash \mathbb{Z} \\ [0, 2], x \in (0, \infty) \backslash \mathbb{Z}, \end{cases}$$

where $\mathbb{Z}$ denotes the set of integers. We have $F(f) = f$ meaning that f is S-continuous. We have the H- continuous functions $F(S(I(f)))(x) = 0, x \in \mathbb{R}$ and

$$F(I(S(f)))(x) = \begin{cases} 0, x \in (-\infty, 0) \\ [0,2], x = 0 \\ 2, x \in (0, \infty). \end{cases}$$

The following theorem gives the way to extend a Hausdorff continuous functions.

Theorem 2.1.8. *[7] Let Ω be a subset of $\mathbb{R}^n$ and D be dense subset of Ω. If $\varphi \in \mathbb{H}(D)$, then there exists $f \in \mathbb{H}(\Omega)$, such that $f(x) = \varphi(x), x \in D$. In fact, $f = F(D, \Omega, \varphi)$.*

Corollary 2.1.9. *[7] Let Ω be a subset of $\mathbb{R}^n$ and D is dense subset of Ω.*

(a) If $f(x) \leq g(x), x \in D,$ then $f(x) \leq g(x), x \in \Omega$

(b) If $f(x) = g(x), x \in D,$ then $f(x) = g(x), x \in \Omega$.

In general, $\mathbb{H}$-continuity is not preserved on subsets of the domain Ω. Precisely, if $K \subset \Omega$ and $f \in \mathbb{H}(\Omega)$, then we cannot conclude that $f|_K \in \mathbb{H}(K)$, where $f|_K$ is the restriction of f to K.

Example 2.1.3. *Consider $\Omega = \mathbb{R}, K = [1, +\infty)$ and f defined on $\mathbb{R}$ by*

$$f(x) = \begin{cases} -1, x < 1 \\ [-1,1], x = 1 \\ 1, x > 1. \end{cases}$$

Obviously, $f \in \mathbb{H}(\mathbb{R})$. The function $f|_K$ is defined on K as

$$f|_K(x) = \begin{cases} [-1,1], x = 1 \\ 1, x > 1. \end{cases}$$

To prove that $f|_K \notin \mathbb{H}(K)$, we consider the interval valued functions g defined on K by

$$g(x) = \begin{cases} [0,1], x = 1 \\ 1, x > 1. \end{cases}$$

We have $g(x) \subseteq f|_K(x)$ for every $x \in K$ and $g \in \mathbb{F}(K)$. But $F(g)(1) = g(1) = [0,1] \neq [-1,1] = f|_K(1)$, so $f|_K$ is not Hausdorff continuous function in K although $K \subseteq \mathbb{R}$ according to Definition 2.1.5.

Hausdorff continuous functions are the restriction of Hausdorff continuous functions on an open subset of Ω, as demonstrated by the next theorem.

Theorem 2.1.10. *[7] Let D be an open subset of Ω and Ω a subset of $\mathbb{R}^n$.*
If $f|_D \in \mathbb{H}(D)$, then $f \in \mathbb{H}(\Omega)$.

If D is a dense subset of Ω and $f \in \mathbb{H}(\Omega)$, then f resembles standard continuous functions on D. It was shown in [7] that for every $f \in \mathbb{H}(\Omega)$ the set

$$W_f = \{x \in \Omega : w(f(x)) \geq 0\}.$$

So

$$D_f = \{x \in \Omega : w(f(x)) = 0\} = \Omega - W_f$$

is dense in Ω and f is continuous on D_f.
Since a finite or countable union of sets of first Baire category is also a set of first Baire category we have that for every finite or countable set $\mathcal{F}$ of Hausdorff continuous functions on Ω the set

$$D_{\mathcal{F}} = \{x \in \Omega : w(f(x)) = 0, f \in \mathcal{F}\} = \Omega - \cup_{f \in \mathcal{F}} W_f$$

is dense in Ω.
Following [11], the point wise addition and scalar multiplication on $\mathbb{H}(\Omega)$ are defined for:

$$f = [\underline{f}, \overline{f}], g = [\underline{g}, \overline{g}] \in \mathbb{H}(\Omega) \text{ and } \alpha \in \mathbb{R},$$

by

$$(f+g)(x) = [\underline{f}(x) + \underline{g}(x), \overline{f}(x) + \overline{g}(x)], x \in \Omega. \tag{2.7}$$

$$(\alpha * f)(x) = \begin{cases} [\alpha \underline{f}(x), \alpha \overline{f}(x)], \alpha \geq 0 \\ [\alpha \overline{f}(x), \alpha \underline{f}(x)], \alpha \leq 0. \end{cases} \tag{2.8}$$

The below example shows the set $\mathbb{H}(\Omega)$ is not closed under the addition given in (2.7)

Example 2.1.4. *For $f, g \in \mathbb{H}(\mathbb{R})$ given by*

$$f(x) = \begin{cases} 0, \; \textit{if}\, x < 2 \\ [0,1], \; \textit{if}\, x = 2 \\ 1, \; \textit{if}\, x > 2 \end{cases} \tag{2.9}$$

and

$$g(x) = (-1) * f(x) = \begin{cases} 0, \; \textit{if}\, x < 2 \\ [-1,0], \; \textit{if}\, x = 2 \\ -1, \; \textit{if}\, x > 2 \end{cases}$$

For the sum we have

$$(f+g)(x) = f(x) + g(x) = \begin{cases} 0, \; \textit{if}\, x < 2 \; \textit{or}\; x > 2 \\ [-1,1] \; \textit{if}\, x = 2. \end{cases}$$

It is clear that $(f+g) \notin \mathbb{H}(\mathbb{R})$.

For $h(x) = \begin{cases} 0, x < 2, x > 2 \\ [-1,0], x = 2 \end{cases}$

It is clear that $h(x) \subseteq f(x) + g(x)$ but $h(2) = [-1,0] \neq [-1,1] = (f+g)(2)$

We need to introduce $\mathbb{H}$- addition denoted by $\oplus$ in the set $\mathbb{H}(\Omega)$. For that the following theorem is helpful.

Theorem 2.1.11. *[11]*

For every x in Ω, there is a unique function $h \in \mathbb{H}(\Omega)$ such that $h(x) \subseteq (f+g)(x)$.

We now define, via interval operations, the $\mathbb{H}$-addition of Hausdorff continuous functions $f, g \in \Omega$.

Definition 2.1.6. *[11] Let f, g belong to $\mathbb{H}(\Omega)$. The function $f \oplus g$ is a unique Hausdorff continuous function $p(x)$ as defined by Theorem 2.1.11, that is satisfying $(f \oplus g)(x) \subseteq (f+g)(x)$,$x \in \Omega$.*

Example 2.1.5. *For the H-sum of Heaviside step functions f and g given on $\Omega = \mathbb{R}$ as*

$$f(x) = \begin{cases} 0, \ \text{if } x < 2 \\ [0,1], \ \text{if } x = 2 \\ 1, \ \text{if } x > 2 \end{cases} \quad \text{and } g = (-1) * f(x) = \begin{cases} 0, \ \text{if } x < 2 \\ [-1,0], \ \text{if } x = 2 \\ -1, \ \text{if } x > 2, \end{cases}$$

we have $(f \oplus g)(x) = 0, x \in \mathbb{R}$.

In [11], it has been shown that the point wise addition given in (2.7) and H-addition given in Definition 2.1.6 coincide when one of the summand is real valued functions. More precisely, for $f, g \in \mathbb{H}(\Omega)$ we have $(f \oplus g)(x) = (f + g)(x)$, $w(f(x)) = 0$ or $w(g(x)) = 0$, $x \in \Omega$.
The following theorem gives an other alternative definition of H-addition.

Theorem 2.1.12. *[11] The sequences $(f_k)k \in \mathbb{N} \subseteq \mathbb{H}(\Omega)$ and $(g_k)k \in \mathbb{N} \subseteq \mathbb{H}(\Omega))$ converge to $f, g \in \mathbb{H}(\Omega)$. This means that the sequence $(f_k \oplus g_k)k \in \mathbb{N}$ converges to an S-continuous function h, i.e., the only H-continuous function satisfying the inclusion $\phi(x) \subseteq h(x), x \in \Omega$, is $\phi = f \oplus g$. Additionally, if $h \in \mathbb{H}(\Omega)$, then $h = f \oplus g$.*

The following 4 theorems will be used in chapter 3 for proving the main result.

Theorem 2.1.13. *[7] Let Ω be a subset of $\mathbb{R}^n$. $(\mathbb{H}(\Omega), \oplus, *)$ is a linear space over $\mathbb{R}$ and $C(\Omega)$ is a subspace of $\mathbb{H}(\Omega)$.*

Let $\mathbb{H}_b(\Omega)$ be the set of bounded Hausdorff continuous functions over Ω.

Theorem 2.1.14. *[10] Let Ω be a subset of $\mathbb{R}^n$. The set $\mathbb{H}_b(\Omega)$ is a subspace of $\mathbb{H}(\Omega)$. Moreover, the set $C_b(\Omega)$ of bounded continuous real valued functions on Ω is the subspace of $\mathbb{H}_b(\Omega)$.*

The usually supremum norm on $C(\Omega)$ can be extended to the set $\mathbb{H}_b(\Omega)$ as follow. For $f \in \mathbb{H}_b(\Omega)$, in [7], the supremum norm on $\mathbb{H}_b(\Omega)$ can be defined as usually by

$$||f|| = \sup_{x \in \Omega} |f(x)|. \tag{2.10}$$

Theorem 2.1.15. *[7] Let Ω be a subset of $\mathbb{R}^n$. The mapping $||.|| : \mathbb{H}_b(\Omega) \longrightarrow \mathbb{R}$ given in (2.10) is a norm on the linear space $\mathbb{H}_b(\Omega)$.*

Theorem 2.1.16. *[7] $(\mathbb{H}(\Omega), \oplus, *, ||.||)$ is a normed space, where $||.||$ is given in (2.10). Moreover, $\mathbb{H}_b(\Omega)$ is a subspace of $\mathbb{H}(\Omega)$.*

Below, we would like to show the connections between Hausdorff continuous functions and Hausdorff distance. Here, let Ω be the subset of $\mathbb{R}^n$.

Definition 2.1.7. *[11] Hausdorff distance (H-distance) $\rho(f, g)$ between two functions $f, g \in \mathbb{A}(\Omega), \Omega \in \mathbb{R}^n$, is defined as the distance between their completed graphs $F(f)$ and $F(g)$ considered as closed subsets of $\Omega \times \mathbb{R}$.*
More precisely we have, $\rho(f, g) = \max\{\sup_{\mathbf{A}\in F(f)} \inf_{\mathbf{B}\in F(g)} ||A - B||, \sup_{\mathbf{B}\in F(g)} \inf_{\mathbf{A}\in F(f)} ||A - B|\}$.

It has been shows in [11] that if $f = [\underline{f}, \overline{f}] \in \mathbb{F}(\Omega)$, then $f \in \mathbb{H}(\Omega) \Leftrightarrow \rho(\underline{f}, \overline{f}) = 0$.

The approximation of continuous function by a step function on a real functional can be extended to interval valued functions.Here we approximate the Hausdorff continuous functions by interval step function.

Definition 2.1.8. *[11]*

A function $f \in \mathbb{H}(\Omega)$ is called a step function if there exists a collection $\{U_1, U_2....U_k\}$ of open subsets of Ω with the following properties:

(a) $U_i \cap U_j = \phi$ *for* $i \neq j$,

(b) The set $V = \bigcup_{i=1}^{k} U_i$ *is dense in* Ω,

(c) For every $i \in \{1, 2, 3...k\}$ f *is a real constant on* U_i.

Furthermore, we observe that a linear subspace of $\mathbb{H}(\Omega)$ is the set of step functions. In fact, both the product of a step function and a real integer, as well as the sum of step functions, are step functions. We establish certain approximation properties of the step functions in the following theorem which will be used for proving the main result.

Theorem 2.1.17. *[11] Let $f \in \mathbb{H}(\Omega)$. For every $\epsilon > 0$ there exists a step function φsuch that $\rho(f; \varphi) < \epsilon$.*

In the special case of a real argument, the interval step-functions have a simple representation in terms of the Heaviside step function $h(x)$ given by

$$h(x) = \begin{cases} 0, -\infty < x < 0 \\ [0, 1], x = 0 \\ 1, 0 < x < \infty \end{cases}$$

Since in chapter 3, we will consider $\Omega = [0, 1]$, then we show that the Heaviside steps we defined on $[0, 1]$.

Indeed, when $\Omega = [0, 1]$, the sets $U_i, i = 1, ..., k$, associated with an interval step function k in terms of Definition 2.1.8 are open intervals of the form (d_{i-1}, d_i), where $d_0 = 0$, $d_k = 1$, and $d_1, d_2, ..., d_{k-1}$ is a finite increasing sequence of reals.

Let $k(x) = ci$ for $x \in (d_{i-1}, d_i)$. A familiar rectangular pulse on the interval $[d_{i-1}, d_i], i = 1, ..., k-1$, is represented as $f(x - d_{i-1}) - f(x - d_i)$.

Then the step function $k(x)$ is given by:

$$k(x) = c_1(1-f(x-d_1)) \oplus c_2(f(x-d_1)-f(x-d_2)) \oplus ... \oplus c_{k-1}(f(x-d_{k-2})-f(x-d_{k-1})) \oplus c_k(x-d_{k-1})$$

$$k(x) = c_1 \oplus \sum_{i=1}^{n} (c_{i-1} - c_i) f(x - d_i).$$

Note that f is discontinuous only at the points $d_1, ..., d_{k-1}$ where it assumes interval values. More precisely, we have

$$f(d_i) = \begin{cases} [c_i, c_{i+1}], c_i < c_{i+1}, \\ [c_{i+1}, c_i], c_i > c_{i+1}. \end{cases} \quad (2.11)$$

We conclude this section by stating applications of Hausdorff continuous functions to partial differential equations. In [12], it has been shown the solution by the order completion methods of larges classes of non linear partial differential equations can be associated with Hausdorff continuous functions.

Moreover in [13], the theory of viscosity solutions for real valued functions can be extended to the Hausdorff continuous functions.

Finally, the space of solutions of conservation law is a subspace of Hausdorff continuous functions, see [14].

2.2 Henstock-Stieltjes integrals

This section begins with a definition and some examples of the Riemann-Stieltjes integral for real valued functions.

Second, we define the Henstock-Stieltjes integral for functions with real values and demonstrate how it is a generalization of the Riemann-Stieltjes integral.

In closing, we define the Henstock-Stieltjes integral for interval valued functions and demonstrate how this type of integral is an extension of Henstock-Stieltjes integral as well Riemann -Stieltjes integral.

Definition 2.2.1. *[1] Let $f : [a, b] \to \mathbb{R}$ be a function.*

(i) The total variation of f is given by

$$\sum_{j=1}^{n} |f(t_j) - f(t_{j-1})|, \text{ where } a = t_0 < t_1 < ... < t_n = b,$$

(ii) f is bounded variation function if

$$Var(f) = \sup \sum_{j=1}^{n} |f(t_j) - f(t_{j-1})| < \infty$$

the supremum being taken over all partitions $a = t_0 < t_1 < ... < t_n = b$.

We denote that $BV[a, b]$ is the set of all functions of bounded variation on $[a, b]$.
The following Theorem is given in [15], seen Theorem 2.2.

Theorem 2.2.1. *Let $f : [a, b] \longrightarrow \mathbb{R}$ is increasing on $[a, b]$, then f is of bounded variation on $[a, b]$ and $V(f, [a, b]) = f(b) - f(a)$.*

Now we define the Riemann-Stieltjes integrals for real valued functions.

Definition 2.2.2. *[1] Consider the function $f : [a, b] \longrightarrow \mathbb{R}$.*
Let $w : [a, b] \longrightarrow \mathbb{R}$ be bounded variation function.
Let P_n be any partition of $[a, b]$ such that $a = x_0 < x_1 < x_2 < ... < x_n = b$.
Let $\eta(P_n)$ be the length of the largest interval $[x_{i-1}, x_i]$,
where $\eta(P_n) = \max\{x_1 - x_0, x_2 - x_1, ..., x_n - x_{n-1}\}$.
For every P_n of $[a, b]$ we consider the Riemann -Stieltjes sum:

$$S(P_n) = \sum_{i=1}^{n} f(x_i)[w(x_i) - w(x_{i-1})]. \tag{2.12}$$

If a real number I exist with the property that there is a $\delta > 0$ such that for any $\epsilon > 0$,

$$\eta(P_n) < \delta \Longrightarrow |I - S(P_n)| < \epsilon,$$

then I is called the Riemann -Stieltjes integral of f over $[a, b]$ with respect to w and it is denoted by

$$\int_a^b f(x)dw(x). \tag{2.13}$$

Hence we can obtain (2.13) as the limit of the Riemann-Stieltjes sum (2.12) for a sequence (P_n) of partitions of $[a, b]$ satisfying:

$$\eta(P_n) \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Remark 2.2.1. *The Riemann-Stieltjes integral extended the Riemann integral by allowing the integration to be performed with respect to the function $w(x) = x$.*

Example 2.2.1. *Let $\alpha(x) = x$ be a bounded variation function on $[0, 1]$, and let $f(x) = 2x$ be a real valued function.*

Now, let's compute the Riemann-Stieltjes integral of $f(x) = x$ over $[0, 1]$. This is equal to

$$\int_0^1 2 * xd(x) = \int_0^1 2 * xdx = 1.$$

Using Riemann-Stieltjes sum.

we divide the interval $[0, 1]$ into n sub-intervals $[x_{i-1}, x_i]$ then Riemann-Stieltjes sum is

$$S_n = \sum_{i=1}^{n} f(x_i)[w(x_i) - w(x_{i-1})].$$

Let $x_i = \dfrac{i}{n}$,

when we replace x_i and t_i in Riemann-Stieltjes sum we get

$$S_n = \sum_{i=1}^{n} 2(\frac{i}{n})[\alpha(\frac{i}{n}) - \alpha(\frac{i-1}{n})]$$

$$= \sum_{i=1}^{n} 2(\frac{i}{n})[\frac{i}{n} - \frac{i-1}{n}]$$

$$= 2\sum_{i=1}^{n} \frac{i}{n}[\frac{i}{n} - \frac{i}{n} + \frac{1}{n}]$$

$$= 2\sum_{i=1}^{n}\frac{i}{n^2} = \frac{2}{n^2}\sum_{i=1}^{n} i = \frac{n^2+n}{n^2} = 1 + \frac{1}{n}.$$

When $n \longrightarrow \infty$ then $S_n \longrightarrow 1$ as we see in the below limit notation.

$$\lim_{n\longrightarrow\infty} S_n = \lim_{n\longrightarrow\infty}(1+\frac{1}{n}) = 1.$$

Next, we define Henstock-Stieltjes integral for real valued functions. Before we define δ-fine division of interval.

Definition 2.2.3. *[16] A Henstock partition of $[a,b]$ is a finite collection $P = \{([c_{i,d_i}], t_i) : 1 \le i \le n\}$ such that $\{([c_{i,d_i}], t_i) : 1 \le i \le n\}$ is a non-overlapping family of sub-intervals of $[a,b]$ covering $[a,b]$ and $t_i \in [c_i, d_i]$ for each $1 \le i \le n$.*

Definition 2.2.4. *[16] A gauge on $[a,b]$ is a function $\delta : [a,b] \longrightarrow (0,\infty)$.*
A Henstock partition $P = \{([c_{i,d_i}], t_i) : 1 \le i \le n\}$ is subordinate to a gauge δ if $[c_i, d_i] \subset (t_i - \delta(t_i), t_i + \delta(t_i))$ for each $1 \le i \le n$

Definition 2.2.5. *[16], [17], [18], [19], [20] Let $\alpha : [a,b] \longrightarrow \mathbb{R}$ be an increasing function on $[a,b]$.*
A function $f : [a,b] \longrightarrow \mathbb{R}$ is said to be Henstock-Stieltjes integrable to $I \in \mathbb{R}$ with respect to α on $[a,b]$ if for every $\varepsilon > 0$ there exists a positive function δ on $[a,b]$ such that

$$|\sum_{k=1}^{n} f(t_i)(\alpha(d_i) - \alpha(c_i)) - I| < \varepsilon$$

whenever $P = \{([c_{i,d_i}], t_i) : 1 \le i \le n\}$ is Henstock partition of $[a,b]$ subordinate to δ.
We write

$$(H)\int_a^b f(x)d\alpha = I \text{ and } f \in H_\alpha[a,b].$$

Remark 2.2.2. *For real valued functions, the Riemann-Stieltjes integral is a particular case of the Henstock-Stieltjes integral.*

Example 2.2.2. *Let $f(x) = 2x$ and let $\alpha(x) = 4x^2$.*
The Henstock-Stieltjes of $f(x)$ on $[0,1]$ is

$$\int_0^1 2xd(4x^2) = \int_0^1 2x.8xdx = \int_0^1 16.x^2dx = \frac{16}{3}.$$

By using Henstock -Stieltjes sum let us consider $n = 2$ means that $P = \{0, \frac{1}{2}, 1\}$.

Then

$$S(P,f,\alpha)=\sum_{i=1}^{2}f(t_i)[\alpha(x_i)-\alpha(x_{i-1})]$$

$$=f(t_1)[\alpha(x_1)-\alpha(x_0)]+f(t_2)[\alpha(x_2)-\alpha(x_1)]$$

$$=f(\frac{1}{4})[4\frac{1}{4}-0]+f(\frac{3}{4})[4.1-4.\frac{1}{4}]=5,$$

when $n\longrightarrow\infty$ *then the limit of* $S(P,f,\alpha)\longrightarrow\frac{16}{3}$.

Now we define Henstock-Stieltjes integral for interval valued functions.

Definition 2.2.6. *[16] Let* $\alpha:[a,b]\longrightarrow\mathbb{R}$ *be an increasing function on [a, b].*
An interval-valued function $f:[a,b]\longrightarrow\mathbb{IR}$ *is Henstock-Stieltjes integrable to* $I\in\mathbb{IR}$ *with respect to* α *on* $[a,b]$ *if, for every* $\varepsilon>0$, *there exists a positive function* δ *such that*

$$d(\sum_{i=1}^{n}(f(t_i))[\alpha(d_i)]-\alpha(c_i),I)<\varepsilon, \tag{2.14}$$

whenever $P=\{([c_{i,d_i}],t_i):1\leq i\leq n\}$ *is Henstock partition of* $[a,b]$ *subordinate to* δ. *Here,* d *given in Equation(2.14) is defined as* $d(a,b)=\max\{|\underline{a}-\underline{b}|,|\overline{a}-\overline{b}|\}$
for $a=[\underline{a},\overline{a}]$,$b=[\underline{b},\overline{b}]\in\mathbb{IR}$.
We write

$$(IH)\int_a^b f(x)d\alpha=I \text{ and } f\in IH_\alpha[a,b].$$

Remark 2.2.3. *Henstock-Stieltjes integrals for real valued functions is a particular case of Henstock -Stieltjes integral for interval valued functions .*

Le $f:[0,1]\to\mathbb{R}$ be a function. Consider interval valued function $f=[f,f]$. Applying this to Henstock-Stieltjes integral for interval valued functions we obtain Henstock -Stieltjes integral for real valued functions.

CHAPTER 3

Extended Riesz's Theorem to interval valued functions

3.1 Introduction

In this chapter, we give and prove the main theorem.

The main theorem states that every bounded convex quasilinear functional from the set $\mathbb{H}_b[0,1]$ to $\mathbb{IR}$ can be represented as Henstock-Stieltjes integral defined on $[0,1]$.

This theorem is proven using the same method as Riesz's representation for real valued functions, which says that any bounded linear functional on $C[0,1]$ may be represented as a Riemann-Stieltjes integral on $[0,1]$.

Note that the classical Riesz's theorem can be proved by using Hahn Banach extension theorem for bounded linear functional. For proving the new result, first we use Hahn Banach extension theorem for interval valued functions, see Theorem 1.1.3 .

Next we construct an increasing function using step interval valued function .

Next we construct the step interval valued functions using new H-addition and using Theorem 2.1.7 which can approximate an interval valued function using Hausdorff distance.

Finally, the image of the extended step functions will be approximated the Henstock-Stieltjes integral.

3.2 Main Result

As we said in introduction, we state the main result and prove it.

Theorem 3.2.1. *Let* $f : \mathbb{H}_b([0,1]) \to \mathbb{IR}$ *be a bounded c-quasilinear functional on* $\mathbb{H}_b([0,1])$ *.There exists* $\alpha : [0,1] \to \mathbb{R}$ *increasing function such that*

$$f(\varphi) = \int_0^1 \varphi(t) d\alpha(t) \; \forall \varphi \in \mathbb{H}_b([0,1]). \tag{3.1}$$

Here the integral is give in the sense of Henstock-Stieltjes integral .

Proof. Step 1. Extension of bounded c-quasilinear functional.
Since the space $\mathbb{H}_b[0,1]$ of bounded Hausdorff continuous interval valued functions on $[0,1]$ is the subspace of $\mathbb{H}[0,1]$ the space of Hausdorff continuous intervals valued functions on $[0,1]$ with norm given in equation (2.10), see Theorem 2.1.13 the bounded convex-quasilinear functional f defined on $\mathbb{H}_b[0,1]$ can be extended to bounded convex quasilinear functional $\tilde{f}$ defined on $\mathbb{H}[0,1]$ using Theorem 1.1.3.
Step 2. Definition of an increasing function $\alpha : [0,1] \longrightarrow \mathbb{R}$
Define α as follows:
For $t \in [0,1]$ define the function $\beta_t : [0,1] \longrightarrow \mathbb{IR}$ by

$$\beta_t(s) = \begin{cases} 0, s \in [0,t) \\ [0,1], s = t \\ 1, s \in (t,1]. \end{cases} \tag{3.2}$$

For $t \in [0,1]$, it is clear that $\beta_t \in \mathbb{H}[0,1]$ since its behavior is like the behavior of the H-continuous function f given in Example 2.1.1 given in $[1,2]$.
Let $t_1, t_2 \in [0,1]$. By the construction of β_t, if $t_1 \leq t_2$ then $\beta_{t_1} \leq \beta_{t_2}$.
Since $\tilde{f}$ is convex quasilinear functional on $[0,1]$ and $\beta_{t_1} \leq \beta_{t_2}$, we have

$$\tilde{f}(\beta_{t_1}) \leq \tilde{f}(\beta_{t_2}).$$

implying that

$$I(\tilde{f}(\beta_{t_1})) \leq I(\tilde{f}(\beta_{t_2})), \tag{3.3}$$

$$S(\tilde{f}(\beta_{t_1})) \leq S(\tilde{f}(\beta_{t_2})). \tag{3.4}$$

Now define $\alpha : [0,1] \longrightarrow \mathbb{R}$ be defined by $\alpha(t) = I(\tilde{f}(\beta_t)), t \in [0,1]$.

From the equation 3.3, if $t_1 \leq t_2$, we have $\alpha(t_1) \leq \alpha(t_2)$.

So α is an increasing function .

Step 3: Approximation of any function $\varphi \in \mathbb{H}[0,1]$ by a step interval function.

Let $\varphi \in \mathbb{H}_b[0,1]$.

Consider the following partition of $[0,1]$.

Let $d_0 = 0, d_1 = \frac{1}{n}, d_2 = \frac{2}{n}, ..., d_n = \frac{n}{n} = 1$.

Let $U_1 = (d_0, d_1), U_2 = (d_1, d_2), U_3 = (d_2, d_3), .., U_n = (d_{n-1}, d_n)$.

Let $V = U_1 \cup U_2 \cup ... \cup U_n$.

For every $j \in \{1, 2, 3, .., n\}$ there exists $t_j \in U_j$ such that $\varphi(t_j) \in \mathbb{R}$.

Define $\psi(t) = f(t_j)$ for $t \in U_j$, where $j = 1, 2, 3, ..., n$.

Then $\varphi_n = F(V, \Omega, \psi)$ and $\varphi_n \in \mathbb{H}[0,1]$.

By Theorem 2.1.16 for every $\varepsilon > 0$, $\rho(\varphi, \varphi_n) < \varepsilon$,where ρ is the Hausdorff distance on $\mathbb{H}[0,1]$.

The function φ_n can be written as

$$\varphi_n = \sum_{j=1}^{n} \varphi(t_j)(\alpha(t_j) - \alpha(t_{j-1})),$$

where the used sum is defined according to Definition (2.1.6) .

Step 4. Proof formula (3.1)

By using the definition of α and the fact that $\tilde{f}$ is quasilinear functional, it can be proved that for every $\varepsilon > 0$, $\rho(\tilde{f}(\varphi_n), \tilde{f}(\varphi)) < \varepsilon$, where ρ is the Hausdorff distance.

Now, $\tilde{f}(\varphi_n)$ can be approximated by

$$\tilde{f}(\varphi_n) = \tilde{f}(\sum_{j=1}^{n} \varphi(t_j)(\alpha(t_j) - \alpha(t_{j-1}))). \tag{3.5}$$

Since the sum (3.5) converges to Henstock-Stieltjes integral

$$\int_0^1 \varphi(t) d\alpha(t),$$

we have $\tilde{f}(\varphi_n)$ can be also approximated by

$$\int_0^1 \varphi(t) d\alpha(t).$$

Since $\rho(\tilde{f}(\varphi), \tilde{f}(\varphi_n))\varepsilon$, we have

$$f(\varphi) = \tilde{f}(\varphi) = \int_0^1 \varphi(t) d\alpha(t). \tag{3.6}$$

□

CHAPTER 4

Conclusion and Recommendations

4.1 Conclusion

The main purpose of this work was to extend the usual Riesz's representation theorem to interval valued functions. As Theorem 1.1.1 states that every bounded linear functional on $C[0, 1]$ can be represented as Riemann-Stieltjes integral, we extended this theorem to interval valued functions. Namely, we stated and proved the result saying that every bounded convex quasilinear functional from the set of bounded Hausdorff continuous functions defined on $[0, 1]$ can be represented as Henstock-Stieltjes integral for interval valued function. For the proof we followed the strategy for proving the usual Riesz's representation for bounded linear functionals which is based to Hahn-Banach extension theorem. We note that this result is the contribution in the theory of Hausdorff continuous interval valued functions.

4.2 Recommendations

For the recommendations, it will be good to continue to analyse properties of Hausdorff continuous interval valued functions in order to look at how those functions can be used in functionals analysis.

Moreover, it is recommended that bounded convex quasilinear functionals can play an important role in functional analysis in order to replace the usual bounded linear functionals in the theory of functional analysis.

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