COLLEGE OF SCIENCE AND TECHNOLOGY

SCHOOL OF SCIENCE

DEPARTMENT OF MATHEMATICS

Stochastic Partial differential equations and their applications

by

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Student number: 220017400

Submitted in partial fulfillment of the requirements

for the degree of Master of Science

in

Applied Mathematics

(Mathematical Modeling and Scientific Computing)

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Declaration

I, **Jean de Dieu MUTIMA**, attest that the research thesis titled **"Stochastic Partial Differential Equation and their applications"** is an entirely original work of mine. It has never before been submitted or published for academic recognition by another person at a university or other higher education establishment. **Assoc. Prof. BANZI Wellars** from the University of Rwanda-College of Science and Technology served as the thesis supervisor.

Signature: .... Date: ....

Jean de Dieu MUTIMA

I attest that this dissertation was presented under my guidance and supervision.

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Dedication

I commend this work to my family and friends.

Abstract

The research investigates the existence and uniqueness of solutions in investment forward performance stochastic partial differential equations (SPDEs). It employs sophisticated mathematical methods, such as the Picard iteration approach, to establish the well-posedness of the SPDE and the convergence of iterative solution techniques. The study delves into advanced mathematical concepts to analyse the behaviour of these complex equations. The study drew upon the seminal works of researchers like **Mykhaylo Shkolnikov**, **Ronnie Sircar**, and **Thaleia Zariphopoulou**. They formulated equations for forward investment strategies and optimal feedback portfolios. Their contributions deepened the comprehension of optimal portfolio selection in incomplete market scenarios based on criteria for forward investment performance. The investigation put forth several key ideas to explain how certain processes function. These concepts, consistency, limits, continuity and memorability of complex processes, aimed to describe the conduct of the systems analyzed. The hypotheses played a vital role in understanding the intricate workings of the processes under examination. This work focuses solely on theoretical aspects concerning the existence and uniqueness of solutions to given equation, rather than covering estimation or calibration methods for determining the parameters of the SPED model, which would require specialized data, models, and statistical techniques. The choice of parameters will be random in certain applications discussed here. The core emphasis lies on exploring theorems and methodologies related to investigating the solution's existence and uniqueness for a given equation. The thesis emphasizes the importance of key assumptions and initial conditions for ensuring the well-posedness of the SPDE and the convergence of iterative solution methods. It also highlights the rigorous reasoning used in establishing the uniqueness of solutions with the strict application of mathematical techniques like Gronwall's lemma and Itô's formula. Modeling population dynamics and crack propagation in composite materials are two applications of the research's findings.

Key words: Stochastic Partial Differential Equation (SPDE), forward performance processes, Ito process, Gelfand triples, and portfolio choice.

Acknowledgment

I am delighted to express my gratitude and appreciation to my colleagues, among others, for their moral support and special care during the completion of my project. I extend heartfelt thanks to **Assoc. Prof. Wellars Banzi**, a lecturer at the University of Rwanda , College of Science and Technology (CST). Despite his numerous tasks and responsibilities, he graciously supervised this project, and his guidance and accessibility proved crucial for its success.

I also wish to thank the **International Program for Mathematical Sciences-Partial Differential Equations and Applications (IPMS PDEAAP)** for financially supporting me. I sincerely thank everyone who contributed in various ways to make this research effective, with special mention to my Lecturers, classmates, and my wife.

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SPDE: Stochastic Partial differential Equation.
SDE: Stochastic Differential Equation.
HJB: Hamilton Jacobi Bellman.
a.e: Almost everywhere.
a.s Almost surely.

Chapter 1

Introduction

1.1 Background

Stochastic Partial Differential Equations (SPDEs) are versatile mathematical models used to describe systems that blend randomness and continuity. These equations are applicable in various fields such as physics, biology, finance, economics, and climate modeling. In portfolio selection, SPDEs are harnessed to address optimal investment problems. As Musiela and Zariphopoulou in 2009 provided solutions for SPDEs, shedding light on the optimal investment strategies. They have developed equations for two approaches to the investment problem: the traditional method and the newly suggested forward formulation. In the first method, their primary focus was on maximizing the expected utility concerning final wealth. However, their contribution extends to sub-optimal market conditions. They formally define the highest achievable and expected utility as

$$V(x,t,T) = \sup_{\pi \in \mathcal{A}_T} \mathbb{E}_P[u_T(X_T^\pi)|\mathcal{F}_t], \tag{1.1}$$

where a function u_T represents the financier's utility capacity that satisfies the standard Inada condition.

X_T^π indicates the asset after trading period T if the investor follows strategy π.

$\mathcal{F}_t$: Contains all events and data accessible up to t.

x: This represents the current wealth at time t. It is a given value that serves as an input to the value function $V(x,t,T)$.

$\mathcal{A}_T$ is the set of permissible strategies within the interval $[0,T]$,

$$\text{Where } V(x,t) = \sup_{\pi \in \mathcal{A}_T} \mathbb{E}_P(V(X_s^\pi, s; T) \mid \mathcal{F}_t, X_t = x) \tag{1.2}$$

is anticipated to take after the Energetic Programming Rule as a result of tending to a stochastic optimization issue, for $t \leq s \leq T$. This speaks to a essential result in ideal control and has been illustrated in different optimization scenarios. The Stochastic Partial Differential Equation (SPDE) related to the portfolio choice issue in its in reverse definition presented by Musiela is of the form

$$dV(x,t) = \frac{1}{2}\frac{\left|V_x(x,t)\lambda_t + \sigma_t\sigma_t^T a_x(x,t)\right|^2}{V_{xx}(x,t)}dt + a(x,t)dW, \tag{1.3}$$

where σ_t represents the volatility matrix associated with resource costs, σ^T denotes the Moore-Penrose pseudo-inverse of the matrix σ_t, and λ_t signifies the market price of risk. The function $a(x,t)$ captures the volatility of the system.

In a different strategy, the investor is not bound to a fixed risk preference but can adjust his/her risk preferences continuously throughout the trading periods. In this case the future investment performance is modeled by:

$$dU(x,t) = \frac{1}{2}\frac{\left|U_x(x,t)\lambda_t + \sigma_t\sigma_t^T a_x(x,t)\right|^2}{U_{xx}(x,t)}dt + a(x,t)dW, \tag{1.4}$$

where $a(x,t)$ represents volatility and W denotes a standard d-dimensional Wiener process (Brownian motion). The process

$$U(x,t) = \sup_{\pi \in \mathcal{A}} \mathbb{E}_P\left(U(X_s^\pi, s) \mid \mathcal{F}_t, X_t^\pi = x\right), \quad \text{for} \quad t \leq s \leq T$$

is considered the solution to the above equation if it meets specific conditions that will be discussed further.

1.1.1 Conditions for forward performance

Definition 1.1. *A process $U(x,t)$ is called continuous forward performance which is measured with regard to the filter $\mathcal{F}_t$ if it meets the following conditions:*

i) $U(x,t)$ is strictly concave and increasing with respect to x for any $t \geq 0$.

ii) For each $\pi \in \mathcal{A}$, $\mathbb{E}[U(X_t^\pi, t)]$ is finite and non-negative, while $\mathbb{E}[U_s(X_s^\pi \mid \mathcal{F}_t)] \leq U_t(X_t^\pi)$ when $s \geq t$.

iii) For any $s \geq t$, $\mathbb{E}[U(X_s^{\pi^*}, s) \mid \mathcal{F}_t] = U_t(X_t^{\pi^*}, t)$ exists , where π^* denotes an optimal policy or strategy that maximizes the expected utility .

iv) $U(x,0) = u_0(x)$.

The study referenced in [1] considers a procedure $U(x,t)$ that progresses measurability with respect to $\mathcal{F}_t$ and meets requirement (i) of the definition. It assumes that $U(x,t)$ is sufficiently smooth to figure out $U(X_t^\pi)$ for any $\pi \in \mathcal{A}$ employing the Ito-Ventzell formula.

Moreover, the study assumes the positivity and finiteness of $\mathbb{E}[U(X_t^\pi, t)]$ for every $t \geq 0$. Based on these presumptions, the function $U(x,t)$ is considered to adhere to the Stochastic Partial Differential Equation (SPDE):

$$dU(x,t) = b(x,t)dt + a(x,t)dW. \tag{1.5}$$

Here, $a(x,t)$ signifies volatility that is measured gradually in relation to $\mathcal{F}_t$, and λ_t represents the market price of risk. The drift coefficient in equation (1.4) is denoted as

$$b(x,t) = \frac{1}{2}\frac{\left|U_x(x,t)\lambda_t + \sigma_t\sigma_t^T a_x(x,t)\right|^2}{U_{xx}(x,t)}.$$

1.1.2 Classifications of forward performance processes

The procedure of progressive performance $U(\cdot,t)$ for $t \geq 0$ is a random procedure that aligns with the financiers 's information flow. It has the property that, almost surely, all functions mapping x to $u(x,t)$ meet the criteria specified in definition 1.1.

To date, the literature has identified three distinct categories of forward performance processes. Time-monotone progressive performance procedures, homothetic progress performance procedures, and factorized forward performance processes within the context of complete markets. The three forms of progressive performance procedures arise with significant simplifications of the stochastic partial differential equation (SPDE) described in equation (1.4) under particular and specialized conditions [2].

a) **Time-monotone progressive performance procedures.**

These procedures exhibit a consistent trend over time, either increasing or decreasing without significant reversals. These processes were initially developed by Musiela and Zariphopoulou in 2003 [2]. By setting the function $a(x,t)$ to zero ,then solve the resultant partial differential equation (PDE). time-monotone progressive performance processes can be derived. When volatility $a(x,t) \equiv 0$, the stochastic partial differential equation 1.4 transforms into

$$d\mathrm{U}\,(x,t) = \frac{1}{2}\left|\lambda_t\right|\frac{\mathrm{U}_x(x,t)^2}{\mathrm{U}_{xx}(x,t)}dt. \tag{1.6}$$

In [1], it is demonstrated that its solution can be expressed as a time-monotonic process

$$U(x,t) = u(x,\mathcal{A}_t).$$

The operation u is specified over the domain $\mathbb{R}^+ \times [0,\infty)$ and is characterized by being monotonically increasing and strictly concave in terms of the variable

x. It adheres to the conditions set forth by the fully nonlinear expression:

$$u_t = \frac{1}{2}\frac{u_x^2}{u_{xx}} \tag{1.7}$$

with $u(x,0) = u_0(x)$. Here, $\mathcal{A}_t$, for $t \geq 0$, is the input process from the market, defined as

$$\mathcal{A}_t = \int_0^t |\lambda_s|^2 \, ds.$$

[1]

b) Homothetic forward performance processes

Homothetic forward performance processes arise when U is chosen to follow a product structure, featuring a power function relationship with respect to x.

In [3], the authors seek initial conditions and volatility processes that yield clearly defined solutions, denoted as $U(x,t)$, to equation 1.4 with a specific characteristic or property:

$$U(kx,t) = \mathcal{K}^{\alpha} u(x,t) \tag{1.8}$$

For any $t \geq 0$ and $\mathcal{K} \in \mathbb{R}^{+}$, $0 < \alpha < 1$. It becomes clear that the advancing process requires a nonlinear structure.

$$U(x,t) = \frac{x^{\alpha}}{\alpha}\mathcal{K}_t. \tag{1.9}$$

The permissible initial conditions follow this particular format:

$$u_0(x) = \frac{x^{\alpha}}{\alpha}\mathcal{K}_0.$$

For additional information, readers are encouraged to refer to [3].

c) Progressive performance mechanisms of factor structure in complete markets

Forward performance processes characterized by a factor structure within complete markets are obtained by simplifying equation 1.4 into a Hamilton-Jacobi-Bellman (HJB) relations, It can be rendered linear in the whole market environment by using the Fenchel-Legendre transform. More information about this process can be found in references [4, 5].

1.1.3 Introduction to Wiener Process

Wiener processes, also known as Brownian motion, are mathematical models used in physics, economics, and engineering to analyse and simulate random events. It details the unpredictable motion of particles within fluids or gases.

Definition 1.2. *A Wiener process, denoted as $W(t)$ for $t \geq 0$, is a stochastic process defined by the function $W\colon \Omega \times T \to \mathbb{R}$ and has the following properties:*

1. *It starts at zero with probability one: $\mathbb{P}[W(0) = 0] = 1$ (also stated as $W(0) = 0$ almost surely).*
2. *It has continuous paths, meaning the process is a continuous function of time with probability one. While discontinuous jumps are possible, their probability is zero.*
3. *The increments are independent and normal distribution with an average of zero and a variance matching the time increment:*

$$dW(t) = W(t + dt) - W(t) \sim \mathcal{N}(0, dt), \quad t \geq 0.$$

These properties imply that the Wiener process is Markovian, meaning it has no memory of past states [6].

Definition 1.3. *An unpredictable process $W(t)$, taking values in a set U, defined over the time interval $[0, T]$ on the probabilistic field $(\Omega, \mathcal{F}, P)$, is termed a (standard) Q-Wiener process if:*

- *$W(0)$ starts at the origin,*
- *W has almost sure continuity in its paths, as guaranteed by the probability measure P,*
- *The steps of W are empirically unrelated. It implies that the random parameters $W(t_1)$, $W(t_2) - W(t_1)$, $\ldots$, $W(t_n) - W(t_{n-1})$ are independent for all time intervals $0 \leq t_1 < \ldots < t_n \leq T$ and $n \in \mathbb{N}$,*
- *These processes obey Gaussian variations with predefined parameters. The rule of the increase $W(t) - W(s)$ adheres to a Gaussian distribution with mean 0 and variance of $(t - s)Q$ for every $0 \leq s \leq t \leq T$. Where Q is represents a covariance operator.*

[7].

1.1.4 Cylindrical Wiener processes

This procedure is commonly linked with Gaussian distributions and is employed for representing random variations or irregularities within a Hilbert space. It finds frequent application in the realm of stochastic analysis and the investigation of randomly generated partial differential equations . The term "Wiener" is often employed to underscore the Gaussian characteristics of this process.

For more information about Sample path properties and statistic properties the reader can see the reference [6].

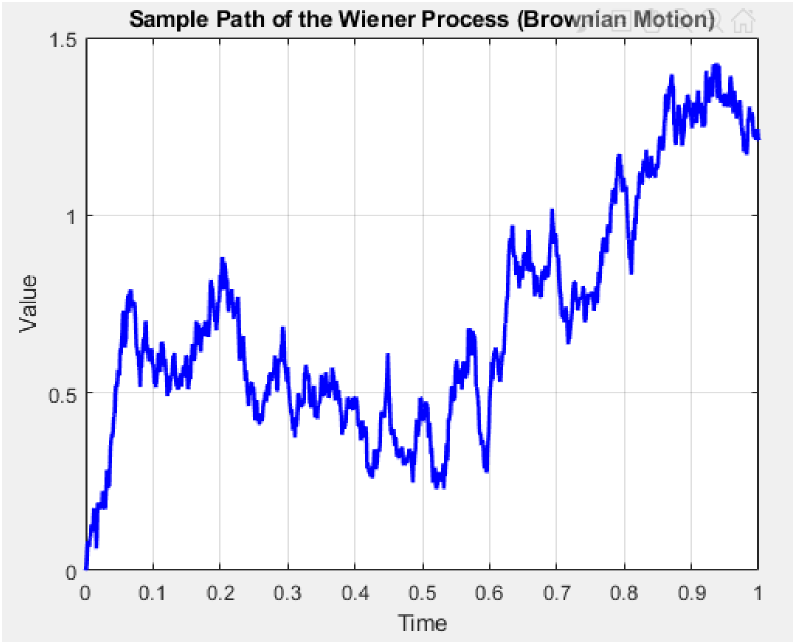

Figure 1.1: Trajectories of the random process.

An instance path of a random process, sometimes referred to as Brownian motion, is shown in the image. The figure extends from 0 to 1.0 along the horizontal axis, which represents time, and corresponds to the total time T given in the code. The Wiener process value at each time point is indicated by the vertical axis. The Wiener process's trajectory is shown by the blue line that runs across the plot; each point on the line represents the process's value at a particular point in time. The plot's overlaying grid lines make it easier to understand how the process's values change over time. All things considered, the picture provides a visual representation of the evolution of a Wiener process over time, emphasizing the randomness and continuous routes that are inherent to the process.

1.2 Statement of the Problem

Musiela and T. Zariphopoulou [8] developed an equation involving both randomness and partial derivatives connected with future investment success . The resulting SPDE is expressed as follows:

$$dU(x,t) = \frac{1}{2}\frac{\left|U_x(x,t)\lambda_t + \sigma_t\sigma_t^T a_x(x,t)\right|^2}{U_{xx}(x,t)}dt + a(x,t)dW. \tag{1.10}$$

Where W corresponds to a Brownian motion, with the asset price adjustment aligned with the filtration associated with this motion

σ_t: represents the volatility matrix linked to the asset prices.

σ^T: signifies the Moore-Penrose pseudo-inverse of the matrix σ_t.

λ_t: denotes the market price of risk.

The function $a(x,t)$ captures the volatility of the system.

They looked at families with zero volatility as well as families with non-zero volatility. They did not delve into examining the solution's existence and uniqueness in the context of this investment performance SPDE.

Our task is to examine the existence and uniqueness, of the SPDE that governs the formulation of portfolio choice in the forward direction. The methods and strategies employed to address this challenge play a crucial role in developing reliable prediction models, depending on the outcomes of existence and distinctiveness analysis.

1.3 Objectives

1.3.1 Main Objective

The fundamental purpose of this research project is to investigate the Stochastic Partial Differential Equation (SPDE) detailed in Equation 1.10, and paying attention to demonstrating both the existence and uniqueness of its solution.

1.3.2 Specific Objectives

1. The first specific objective is to illustrate the conditions that guarantee the existence of solutions to the SPDE. 1.10.

2. The second specific objective is to examine the uniqueness of the solution.

3. The final objective is to apply these findings to practical scenarios.

1.4 Limitation

This thesis will not cover the estimation or calibration of the SPDE model's parameters, such as volatility (σ), price of risk in the market (λ), and the volatility function ($a(x,t)$). Estimating these parameters requires data, models, and statistical techniques. The thesis will focus on theorems and methodologies to explore the existing and Inimitability of the solutions. For practical applications, parameters will be chosen at random.

Chapter 2

Basic concepts

2.1 Stochastic Differential Equations (SDEs)

This section introduces differential equations involving random fluctuations, which are a more complex version of ordinary differential equations (ODEs). SDEs involve the Itô integral, and their theory is more detailed and extensive compared to ODEs [9]. One way to formulate this kind of equations is to introduce random elements at the right hand side of the equation. These random elements' dependence on the solution can vary. Particular instances, such as those involving the heat equation and reaction-diffusion equation, can be represented in a particular developing form:

$$dX_t = A(t, X_t)dt + B(t, X_t)dW_t; \quad X_0 = x; \tag{2.1}$$

Here, X is considered as variable that can take values within a suitable infinite-dimensional Hilbert space H. Additionally, W represents a Brownian motion taking values in a Hilbert space U. The function B takes values in the space of linear mapping from U to H, denoted by $L(U; H)$. To establish a meaningful framework, several steps need to be taken:

1. Define the notion of a Brownian motion with values in U, particularly when U has endless dimension.

2. Announce the notion of integral involving randomness within infinite-dimensional spaces.

3. Provide clarity regarding what is meant by a solution to equation (2.1).

4. Determine the prerequisites for A and B.that ensure the existence and uniqueness of solutions for the equation.

These steps are important to successfully apply stochastic calculus in unlimited dimensional spaces [9]. Table below lists a few examples of SDEs along with their deterministic counterparts.

Deterministic	Stochastic	Name
$dx = \mu dt$	$dX(t) = \mu dt + \sigma dW(t)$	Arithmetic Brownian Motion (ABM)
$dx = \mu x dt$	$dX(t) = \mu X(t)dt + \sigma X(t)dW(t)$	Geometric Brownian Motion (GBM)
$dx = \alpha(\mu - x)dt$	$dX(t) = \alpha(\mu - x)dt + \sigma\sqrt{X(t)}dW(t)$	Square root (Cox-Ingersoll-Ross, CIR)

Table 2.1: Examples of stochastic differential equations and their deterministic counterparts.

The solutions and Matlab implementation of the above SDEs can be found in reference [10].

Definition 2.1. *The Hilbert space U is considered separable if it has a countable dense subset $D \subseteq U$ in which, for each $u \in U$, there exists a sequence $(d_n) \subseteq D$ such that:*

$$\lim_{n\to\infty} \|u - d_n\| = 0$$

In simpler terms, the property of being separable of U suggests the presence of a denumerable set D in which, for each u in U, the sequence (d_n) exist from D and converges to u with regard to the norm [11].

Definition 2.2. *Let H and U be Hilbert spaces, and $W(t)$ represent a conventional real-valued Wiener mechanism defined on the probability space (Ω, Σ, P). A U-valued Wiener process $X(t), t \geq 0$ is a stochastic process that meets each of the following characteristics:*

1. ***Stationarity:*** *$X(0) = 0$ almost surely (P-a.s.), where 0 represents the zero element of the Hilbert space U.*

2. ***Increment Independence:*** *For $0 \leq t_1 < t_2 < \ldots < t_n$, the changes in $X(t_1), X(t_2) - X(t_1), \ldots, X(t_n) - X(t_{n-1})$ are unrelated U-valued random variables.*

3. ***Gaussian Increments:*** *The increments $X(t) - X(s)$ are Gaussian distributed U-valued random variables for any $s < t$.*

4. ***Continuous Paths:*** *$X(t)$ has continuous paths, meaning that for almost every ω in Ω, the sample path $t \mapsto X(t, \omega)$ is continuous.*

5. ***Zero Mean and Covariance:*** *For any $s < t$, the U-valued increments $X(t) - X(s)$ have zero mean and covariance operator given by:*

$$Cov(X(t) - X(s)) = (t - s)I,$$

I symbolises the identity operator in the Hilbert space U.

6. ***Quadratic Variation:*** *The quadratic variation of $X(t)$ is given by:*

$$[X,X](t)=\int_0^t \|\mathrm{d}X(s)\|^2,$$

where $\|\mathrm{d}X(s)\|$ represents the norm of the U-valued increment $\mathrm{d}X(s)$, and $[X,X](t)$ is the U-valued quadratic variation of $X(t)$.

[12].

2.2 Stochastic Integrals

We are now prepared to delve into stochastic integration and its various associated features. This establish the basis for addressing stochastic differential equations and explaining the filtering problem.

Definition 2.3. *Consider a Hilbert space H, an H-valued Wiener process denoted by $W(t)$, and a stepwise function $S(t)$ mapping from the set T to the space of $\mathcal{L}(H,H)$. In simpler terms, we have a collection of time points $t_0 < t_1 < \ldots < t_n$ in the set T, and corresponding operators S_j for $j=1,2,\ldots,n$ in the space $\mathcal{L}(H,H)$ such that for each t within these intervals $[t_{j-1},t_j]$ (where j ranges from 1 to n):*

$$S(t)=\begin{cases}0, & \textit{if } t<t_0\\ S_j, & \textit{if } t_{j-1}\le t<t_j, \quad j=1,2,\ldots,n\\ 0, & \textit{if } t\ge t_n\end{cases}$$

Next, we define the stochastic integral of $S(t)$ in relation to $dW(t)$, denoted as $\int S(t)\,dW(t)$, as follows:

$$\int S(t)dW(t)=\sum_{j=1}^{n} S_j[W(t_j)-W(t_{j-1})]$$

[13].

The stochastic integral is crucial for studying and solving stochastic differential equations (SDEs) and finds within applications different domains, including economics, physics, and engineering, where randomness and uncertainty are fundamental characteristics of the systems under study [14].

2.2.1 The Itô Formula

Theorem 2.1. *Consider an Itô process X_t such that $dX_t = U_t dt + V_t dW_t$. Take $g(x)\in C^2(\mathbb{R})$ be a continuous, twice differentiable function. Assuming $g(X_t)\in L^2$,*

$Y_t = g(X_t)$ is similarly an Itô process. such that

$$dY_t = \frac{\partial g}{\partial x}(X_t)\, dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(X_t)\,(dX_t)^2. \tag{2.2}$$

By substituting the values of $dX_t = U_t dt + V_t dW_t$ in (2.2), then Itô formula may be expressed as

$$dY_t = \left(\frac{\partial g}{\partial x}(X_t)U_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(X_t)V_t^2\right) dt + \left(\frac{\partial g}{\partial x}(X_t)\right) V_t dW_t. \tag{2.3}$$

As a result, we may observe that the set of Itô processes stays constant when exposed to transformations that are twice repeatedly differentiable.

The proof of this theorem can be found in reference [15].

2.3 Gelfand Triples and Coefficient Requirements

Let H represent A divisible Hilbert space. equipped with the inner product $<.,.>_H$, and let H^* denote its dual space. Additionally, consider a Banach space V continuously and densely embedded within H.

In this context, the dual space H^* of H can be continuously and densely embedded within the dual space V^* of V.

This relationship can be succinctly expressed as:

$$V \subset H \subset V^* \tag{2.4}$$

In essence, V is a subspace of H, and simultaneously, H can be considered a subspace of the dual space V^*, with both embeddings becoming uninterrupted and dense [9]. A Gelfand triple, denoted as (V, H, V^*), exhibits the following characteristics. It's important to note that H is constantly and tightly integrated into V^* implies the separability of V^* and, consequently, the separability of V. Additionally, the operator space $B(V)$ is formed by V^*, while $B(H)$ is generated by H^*. Moreover, Kuratowski's theorem establishes that V is an element of $B(H)$, and H is an element of $B(V^*)$. To elaborate further:

- The separability of V^* implies the separability of V.
- V^* generates $B(V)$, and H^* generates $B(H)$.
- According to Kuratowski's theorem, V is within $B(H)$, and H is within $B(V^*)$.

Shifting our focus to stochastic differential equations defined on the Hilbert space H, these equations

Take the structure:

$$dX(t) = A(t, X(t))dt + B(t, X(t))dW(t), \quad X(0) = x \in H \tag{2.5}$$

We have the conditions for functions A and B:

1. A mapping $A : V \to V^*$ is definable.
2. A mapping $B : V \to L^2(U; H)$ is measurable, with U being a real separable Hilbert space.
3. W is a cylindrical Q-Wiener process with $Q = \text{id}$ in $L(U)$.
4. X takes values in H.

We assign these conditions on A and B:

- **(H1) Hemicontinuity:** For all u, v, and x in V, the function $\lambda \mapsto V^* \langle A(u + \lambda v), x \rangle v$ is in $C(\mathbb{R}, \mathbb{R})$.
- **(H2) Weak Monotonicity:** There is a fixed $c_1 \in \mathbb{R}$ such that for any u and v in V, the expression $2V^* \langle A(u) - A(v), u - v \rangle V + ||B(u) - B(v)||^2_{L^2(U;H)}$ is bounded above by $c_1 \|u - v\|^2$ in H.
- **(H3) Coercivity:** There exist constants $\alpha \in (1, \infty)$, $c_1 \in \mathbb{R}$, $c_2 \in (0, \infty)$, and $c_3 \in \mathbb{R}$ such that for all $v \in V$,

  $$2\|A(v), v\|_V + \|B(v)\|^2_{L^2(U,H)} \leq c_1 \|v\|^2_H - c_2 \|v\|^\alpha_V + c_3.$$

  This inequality holds true.
- **(H4) Boundedness:** There exist constants c_4 and c_5 such that for all $v \in V$, the following inequality holds:

  $$\|A(v)\|_{V^*} \leq c_4 \|V\|^{\alpha-1}_V + c_5$$

  Here, α corresponds to the value specified in condition (H_3) [9].

2.3.1 Examples of Stochastic Partial Differential Equations

Random Partial Differential Equations are a flexible family of equations used in a variety of domains, including physics, biology, and finance. Here are some instances:

1. **Heat Equation with Random Perturbations**

   The heat equation models the diffusion of heat in a material. When stochastic fluctuations are introduced, it leads to an SPDE, commonly used in describing the unpredictable fluctuations of stock values in finance, among other applications.

Let's focus on a specific instance of the stochastic heat equation. Several algorithms will be employed to gain insights into the solution's nature, albeit with some degree of rigor. Further refinement may be required to ensure full mathematical rigor.

Recalling the heat equation as:

$$\frac{\partial u}{\partial t} = \Delta u, \quad u : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R} \tag{2.6}$$

where $u_0 : \mathbb{R}^n \to \mathbb{R}$ is any bounded and continuous initial condition, and u is the only solution satisfying $u(0, x) = u_0(x)$ for all $x \in \mathbb{R}^n$, given by:

$$u(t, x) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} u_0(y)\, dy. \tag{2.3}$$

For simplicity, let's denote this as $u(t, \cdot) = e^{-t} u_0$, analogous to the solution for a linear equation $\frac{\partial u}{\partial t} = Au$.

Introducing a forcing term f into (2.6), we have:

$$\frac{\partial u}{\partial t} = \Delta u + f, \quad u : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R} \tag{2.7}$$

The solution to (2.7) can be expressed using the variations of constants formula:

$$u(t, \cdot) = e^{\Delta t} u_0 + \int_0^t e^{\Delta(t-s)} f(s, \cdot)\, ds. \tag{2.8}$$

If f denotes time-space , the stochastic heat equation becomes:

$$\frac{\partial u}{\partial t} = \Delta u + \xi, \quad u : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R} \tag{2.9}$$

For the case $u_0 = 0$, the solution is:

$$u(t, x) = \int_0^t \int_{\mathbb{R}^n} \frac{1}{(4\pi|t-s|)^{n/2}} e^{-\frac{|x-y|^2}{4(t-s)}} \xi(s, y)\, dy\, ds. \tag{2.10}$$

2. Representation of a Stochastic Polymer Chain

Consider $N+1$ particles in a fluid, connected by elastic springs and subjected to external forces. The equations governing their motion, when forces dominate over inertia, are:

$$\frac{du_0}{dt} = k(u_1 - u_0) + F(u_0),$$

$$\frac{du_n}{dt} = k(u_{n+1} + u_{n-1} - 2u_n) + F(u_n), \quad n = 1, \ldots, N-1,$$

$$\frac{du_N}{dt} = k(u_{N-1} - u_N) + F(u_N).$$

In the continuous restrict with $k \approx \nu N^2$ and $\sigma \approx \sqrt{N}$, This process is adequately characterized by the solution to a random partial differential equation.

$$\frac{\partial u(x,t)}{\partial t} = \nu \frac{\partial^2 u(x,t)}{\partial x^2} dt + F(u(x,t))dt + dW(x,t), \tag{2.11}$$

With respect to limit constraints $\frac{\partial u(0,t)}{\partial x} = \frac{\partial u(1)}{\partial x} = 0$.

2.3.2 Ensure the existence and singularity of SPDE solutions.

The possibility of finding a solution for an SPDE depends on the specific equation involved and the characteristics of its coefficients. Typically, confirming the existence of a solution relies on certain assumptions regarding these coefficients, such as continuity, boundedness, and growth conditions.

Consider that A and B in 2.5 match the constraints $(H1)$-$(H4)$. Assume $T > 0$ and designate the Lebesgue measure on the interval $[0, T]$ as λ_T.

Definition 2.4. *A continuous sequence $X(t)$ for t in the interval $[0, T]$, which accepts values from the space H and adapts to the filter $\mathbb{F}$, is regarded as a solution to equation (2.5) under specific conditions. In particular, for the equivalence class $\hat{X}$ of processes valued in V^* with respect to the measure $\lambda_T \otimes \mathbb{P}$, the following conditions must hold:*

$$\hat{X} \in L^\alpha([0, T] \times \Omega, \lambda_T \otimes \mathbb{P}; V) \cap L^2([0, T] \times \Omega, \lambda_T \otimes \mathbb{P}; H),$$

where α is greater than 1.

Moreover, under the probability measure $\mathbb{P}$, it is almost certain that:

$$X(t) = X(0) + \int_0^t A(\bar{X}(s))\, ds + \int_0^t B(\bar{X}(s))\, dW(s), \quad t \in [0, T],$$

where $\bar{X}$ represents any predictable modification of $\hat{X}$ with values in V.

A continuous process $X(t)_{t \in [0,T]}$ with values in space H, where the equivalence class $\hat{X}$ with respect to the measure $\lambda_T \otimes \mathbb{P}$ belongs to $L^\alpha([0, T] \times \Omega, \lambda_T \otimes \mathbb{P}; V)$, possesses a predictable modification in space V. This can be illustrated by defining the set $\mathcal{R} := \{(s, w) \in [0, T] \times \Omega : X(s, w) \in V\}$, which is predictable as it is the pre-image of the set $V \in \mathcal{B}(H)$ under the continuous process X with values in H. Then, $\bar{X}(s, w)$ is defined as follows:

$$\bar{X}(s, w) := \begin{cases} X(s, w) & \text{if } (s, w) \in \mathcal{R}, \\ 0 & \text{otherwise.} \end{cases}$$

With this definition, $\bar{X}$ possesses the stated property [16].

In contrast to the definition in [17], where the term "predictable" is replaced with "progressively measurable," our Definition 2.4 offers several advantages. Firstly, it eliminates the requirement to incorporate the progress sigma algebra altogether, guaranteeing that our integrands in stochastic integrals are foreseeable, as necessary. Thus, there is no necessity to argue for permitting progressive integration. Furthermore, our definition allows for more flexibility in generalizing to more complex integrators, potentially with jumps, where allowing general progressive integrands is not feasible. Conversely, in [17], the coefficients are permitted to be time-dependent , and in such a scenario, Requiring them to be progressive is a less strict condition than regularity.

Theorem 2.2. *Assuming that the conditions for A and B specified in H_1 through H_4 are met, and considering a function f belonging to the space $L^{p/2}([0,T]\times\Omega; dt\otimes P)$ where p is greater than or equal to $\beta+2$, and further assuming the existence of a constant C, subject to the following conditions for all t in the interval $[0,T]$ and v in the set V:*

$$\|B(t,v)\|^2_{L^2(U,H)} \le C(f(t)+\|v\|_H^2),$$
$$\rho(v) \le C(1+\|v\|_V^\alpha)(1+\|v\|_H^\beta), \tag{2.12}$$

Then, for any initial value X_0 in the space $L^p(\Omega;\mathcal{F}_0;\mathbb{P};H)$, the equation (2.5) has a unique solution $X(t)$ for all t within the interval $[0,T]$, with the initial condition $X(0)=X_0$, and this solution satisfies the following inequality:

$$\mathbb{E}\left(\sup_{t\in[0,T]}\|X(t)\|_H^P\right)<\infty.$$

Moreover, if $A(t,.)(w)$ and $B(t,.)(w)$ do not vary with respect to t in the interval $[0,T]$, and for all $\omega\in\Omega$, then the solution $X(t)_{t\in[0,T]}$ of equation (2.5) behaves as a Markov process [17].

Prior to presenting and demonstrating the primary outcome regarding the existence and uniqueness of solutions for equation (2.5), we express a form of Itô's formula.

Theorem 2.3. *Suppose $\alpha>1$, and let $X_0\in L^2(\Omega,\mathcal{F}_0,\mathbb{P};H)$. Furthermore, assume that Y is an element of $L^{\frac{\alpha}{\alpha-1}}([0,T]\times\Omega,\lambda_T\otimes\mathbb{P};V^*)$, and Z belongs to $L^2([0,T]\times\Omega,\lambda_T\otimes\mathbb{P};L^2(U,H))$, with the condition that both Y and Z are predictable. We define a continuous process taking values in the dual space of V as follows:*

$$X(t)=X_0+\int_0^t Y(s)\,ds+\int_0^t Z(s)\,dW(s),\quad t\in[0,T]$$

If the equivalence class $\hat{X}$ of X, as defined under the measure $\lambda_T\otimes\mathbb{P}$, belongs to *the space $L^\alpha([0,T]\times\Omega, \lambda_T\otimes\mathbb{P}; V)$, and if the expected value of $\|X(t)\|_H^2$ is finite for almost all $t\in[0,T]$ under the measure λ_T, then X is both adapted and a continuous procedure of extracting values from the Hilbert space H. Furthermore, the square norm of X for every point in time t is provided by:*

$$\|X(t)\|_H^2 = \|X_0\|_H^2 + \int_0^t \left(2_V \cdot \langle Y(s), \bar{X}(s)\rangle_V + \|Z(s)\|_{L^2(U,H)}^2\right) ds+$$
$$2\int_0^t \langle \bar{X}(s), Z(s)\, dW(s)\rangle_H$$

Where $\bar{X}$ represents any predictable modification of X with values in the space V.

To establish uniqueness, Let us suppose that both X and Y are solutions of equation (2.5). with their respective initial conditions X_0 and Y_0. In other words, we have:

$$X(t) = X(0) + \int_0^t A(\bar{X}(s))\, ds + \int_0^t B(\bar{X}(s))\, dW(s), \quad t\in[0,T],$$
$$Y(t) = Y(0) + \int_0^t A(\bar{Y}(s))\, ds + \int_0^t B(\bar{Y}(s))\, dW(s), \quad t\in[0,T].$$

Using the product rule, Itô's equation from hypothesis (2.3) and H_2, we obtain for $t\in[0,T]$.

$$e^{-\int_0^t (f(s)+\rho(\bar{Y}(s)))\mathrm{d}s}\|X(t)-Y(t)\|_H^2 \le \|X_0-Y_0\|_H^2$$
$$+ 2\int_0^t e^{-\int_0^s (f(r)+\rho(\bar{Y}(r)))\mathrm{d}r}\langle X(s)-Y(s), (B(s,\bar{X}(s)) - B(s,\bar{Y}(s)))\mathrm{d}W(s)\rangle_H.$$

According to Proposition in in [[17],pg 225], the martingale at a local level on the right-hand side is a real-valued martingale. Therefore, when we take the expected value, We get the following outcome.:

The expectation of the quantity specified below is bounded by the expectation of another quantity for every value of t inside the range $[0,T]$:

$$E\left[e^{-\int_0^t (f(s)+\rho(\bar{Y}(s)))\mathrm{d}s}\|X(t)-Y(t)\|_H^2\right] \le E\,\|X_0-Y_0\|_H^2$$

If the starting circumstances, X_0 and Y_0 are equal, the expectation of the aforementioned quantity becomes zero for all t in the interval $[0,T]$.

Since Y belongs to the set $\mathcal{M}$, equation (2.12) suggests that the following integral is finite with probability one:

$$\int_0^T (f(s)+\rho(\bar{Y}(s)))\mathrm{d}s < \infty \quad \text{almost surely}$$

As a consequence, for every $t \in [0, T]$, $X(t) = Y(t)$ almost surely. Therefore, the path wise uniqueness is established due to the continuity of the paths of X and Y in the Hilbert space H [17].

The following lemma is useful in proving that nonlinear stochastic partial differential equations have unique solutions. Gronwall's Lemma, a fundamental concept inside the differential equations theory, gives the solution to a differential inequality an upper bound, expressed as an integral involving the same inequality.

Lemma 2.1. ***Gronwall's Lemma:*** *Let $f(t)$ and $g(t)$ be non-negative continuous functions on $[0, T]$, with α being a non-negative constant. If the inequality*

$$f(t) \leq \alpha + \int_0^t g(s)f(s)ds$$

holds for every t in the interval $[0, T]$, then

$$f(t) \leq \alpha \exp\left(\int_0^t g(s)ds\right)$$

also holds for all t in the interval $[0, T]$.

In simpler terms, Gronwall's Lemma offers an exponential upper restricted to the solution of the differential inequality in terms of the integral of the inequality. This result finds numerous significant applications across various branches of mathematics and science and is frequently utilized in differential equation analysis [18].

2.4 Picard Iteration for SPDE

For certain types of random partial differential equations, the Picard iteration technique is commonly used to establish solution existence. Let's consider establishing the solution existence for the nonlinear wave propagation equation on $[0, T] \times \mathbf{R}$ influenced by space-time Gaussian noise:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = \sigma(u(t, x))\dot{W}(t, x), \quad (30) \tag{2.13}$$

with the starting conditions given by:

$$u(0, \cdot) = \frac{\partial u}{\partial t}(0, \cdot) \equiv 0. \tag{2.14}$$

The solution follows the steps:

1. **Define the Picard Iteration Scheme:** The Picard iteration scheme involves recursively defining an approximate solution. For each iteration $k = 1, 2, ...$, the solution is updated based on the previous iteration's solution.

2. **Establish L^2-Convergence using Gronwall's Lemma:** To demonstrate L^2-convergence, it's necessary to show that the iterative sequence converges in the mean square sense. Gronwall's Lemma is often utilized to prove such convergence results for integral equations.

3. **Demonstrate Existence of Higher Moments using Burkholder's Inequality:** Burkholder's Inequality relates the moments of stochastic processes and is often used to establish the existence of higher moments for such processes.

4. **Confirm ρ-Hölder Continuity for $\rho \in (0, \frac{1}{2})$:** Hölder continuity addresses the regularity of the solution. To show ρ-Hölder continuity, it's essential to control the growth of the solution as the spatial variable x changes [19].

Chapter 3

Methodology and Techniques

This Section is dedicated to investigating The existing and distinctiveness of remedies of nonlinear stochastic partial differential equations (SPDEs) through the application of Picard's iteration scheme. It explores how this mathematical technique contributes to verifying both the presence of solutions and their uniqueness in the domain of nonlinear SPDEs.

This part outlines the fundamental steps involved in applying the Picard's iteration method within the context of nonlinear stochastic partial differential equations (SPDEs).

Picard's iteration scheme entails iteratively refining an initial guess to generate a sequence of approximations and demonstrating that this sequence converges to a unique solution. This method is pivotal in the theory of SPDEs and is frequently employed to establish the presence and individuality of solutions within the framework of random processes and functional spaces [20].

3.1 Ensuring Existence and Uniqueness of Itô Solutions

In [21], the authors demonstrated the presence and individuality of a solution, following Itô's approach, for a broad class of SPDEs characterized by nonlinear monotone operators and incorporating delays. Specifically, they address the stochastic parabolic equation of the form:

$$\begin{cases} dx(t) + [A(t, x(t)) + B(t, x(\tau(t))) + f(t)]\, dt = [C(t, x(\rho(t))) + g(t)]\, dW(t), & t > 0 \\ x(0) = x_0, \end{cases} \tag{3.1}$$

Here, $A(t, \cdot)$, $B(t, \cdot)$, and $C(t, \cdot)$ represent groups of functions acting in Hilbert spaces, possibly nonlinear, and adhering to a monotonicity requirement. The symbol $W(t)$ denotes a Hilbert-valued Brownian motion, and ρ and τ correspond to delay functions [21].

The document employs the monotonicity method, a significant technique for resolving nonlinear partial differential equations (PDEs). Furthermore, a modified version of this approach is applied to address a different class of nonlinear monotone equations under the conditions where $B = 0$ and there are no delays.

When no delays are present ($\tau(t) = \rho(t) = t$), researchers have examined equation (3.1) under the condition where $B = C = 0$, considering A as a nonlinear factor.

In the paper [22] authors explored a range of mathematical methods to prove both the presence of solutions to stochastic partial differential equations and their uniqueness with nonlinear monotone operators and delays. Key techniques utilized in the proof encompass Picard's Scheme, Gronwall's Lemma, Stochastic Integration, the Monotonicity Method, and the Coercivity Condition.

It builds upon the well-established theory of stochastic integrals in Hilbert spaces, as outlined in prior literature. The approach relies on classical real separable Hilbert spaces V and H, satisfying the condition $V \subseteq H$, with H considered as its own dual space and the dual of V denoted as V_0. This establishes the relationships $V \subseteq H \equiv H_0 \subseteq V_0$. Norms in V, H, and V_0 are represented by $\|\cdot\|_V$, $\|\cdot\|_H$, and $\|\cdot\|_{V_0}$ respectively, with the dual product of V_0 and V denoted by $\langle\cdot,\cdot\rangle$, and $(\cdot,\cdot)$ representing the Scalar product in H-space.

Considering a Brownian motion W_t defined on the entire probability space, the paper introduces it as a martingale relative to the generated σ-algebra. W_t is represented as a sum involving a collection of orthonormal eigenvectors for the covariance operator W, with the incremental covariance σ_i^2 for each process β_t^i, which are mutually independent real Wiener processes.

Departing from conventional notation, the paper uses $\|\cdot\|$ to denote the norm within continuing linear operator space $L(K; H)$. Additionally, it defines $I_p(0, T, V)$, where $p > 1$, as the V-valued processes' space satisfying certain measurability and integrability conditions.

Authors established a closed subspace $I^p(0, T; V)$ within a larger space, denoted $L^p(\Omega \times [0, T]; \mathcal{F} \times B([0, T]); dP \otimes dt; V)$. It further introduces nonlinear operators $A(t, \cdot) : V \to V_0$, defined almost everywhere in time, under certain concepts:

(a.1) **Coercivity:** Positive constants α and λ exist such that

$$2\langle A(t, x), x\rangle + \lambda\|x\|^2 \geq \alpha\|x\|^p, \quad \forall x \in V, \text{ a.e.t }.$$

(a.2) **Monotonicity:** For all $x, y \in V$, there exists λ such that

$$2\langle A(t, x) - A(t, y), x - y\rangle + \lambda\|x - y\|^2 \geq 0, \quad \text{a.e.t }.$$

(a.3) **Boundedness:** A positive constant β exists such that

$$\|A(t,x)\|_v \leq \beta \|x\|^{p-1}, \quad \forall x \in V, \text{ a.e.t .}$$

(a.4) **Hemicontinuity:** The function $\theta \in \mathbb{R} \mapsto \langle A(t, x + \theta y), z \rangle$ is continuous for all $x, y, z \in V$, a.e.t .

(a.5) **Measurability:** The mapping $t \in (0,T) \mapsto A(t,x) \in V_0$ is Lebesgue-measurable for all $x \in V$, a.e.t .

Let $B(t,\cdot) : H \to H$ be a group of functions defined almost everywhere in time, subject to the following conditions:

(b.1) $B(t,0) = 0$

(b.2) Lipschitz condition: Almost everywhere in time, there is a constant k_1 such that for every $x, y \in H$, $|B(t,x) - B(t,y)| \leq k_1|x-y|$.

(b.3) Measurement: For any $x \in H$, the translation $t \in (0,T) \longrightarrow B(t,x) \in H$ is Lebesgue-measurable.

Similarly, let $C(t,\cdot) : H \to L(K,H)$ be another family defined almost everywhere in time, satisfying:

(c.1) $C(t,0) = 0$

(c.2) Lipschitz condition: For any $x, y \in H$, practically everywhere in time, there exists a constant k_2 such that $|C(t,x) - C(t,y)| \leq k_2|x-y|$.

(c.3) Measurement: For any $x \in H$, the mapping $t \in (0,T) \mapsto C(t,x) \in L(K,H)$ is Lebesgue-measurable.

Furthermore, two measurable delay functions $\rho, \tau : [0,T] \to [0,T]$ are taken into consideration, fulfilling

$$0 \leq \rho(t), \tau(t) \leq t \text{ for all } t \in [0,T]. \tag{3.2}$$

For the functions f and g, we make the assumption that

$$f \in I^2(0,T;H), g \in I^2(0,T;L(K,H)).$$

The starting value $x_0 \in L^2(\Omega, \mathcal{F}_0; P; H)$ is where we end up, at last. Next, we pose the following issue:

$$\begin{cases} \text{Find a process } x \in I_p(0,T;V) \cap L^2(\Omega; C(0,T;H)) \text{ satisfying:} \\ x(t) + \int_0^t [A(s,x(s)) + B(s,x(\tau(s))) + f(s)]\, ds \\ = x_0 + \int_0^t [C(s,x(\rho(s))) + g(s)]\, dW_s, \quad P \text{ a.s.}, \quad \forall t \in [0,T]. \end{cases} \tag{3.3}$$

Remark 3.1. *We note that if $x \in L^2(0,T;H)$, then according to conditions (b.1)–(b.3), $B(x) \in L^2(0,T;H)$, where $B(x)(t) = B(t, x(t))$. Furthermore, the mapping $x \in L^2(0,T;H) \to B(x) \in L^2(0,T;H)$ is continuous and hence measurable. Considering $x \in H \to B(t,x) \in H$ is continuous almost everywhere, if $x(t)$ is an H-valued stochastic process and $\mathcal{F}_t$-adapted, then $B(t,x(t))$ also possesses these properties. Additionally, if $x \in L^2(\Omega\times(0,T);H)$, then $B(x) \in L^2(\Omega\times(0,T);H)$. Lastly, if x^n is a bounded sequence in $L^2(\Omega\times(0,T);H)$, $B(x^n)$ is also bounded. Similar observations can be made from conditions (c.1)–(c.3) for $C : L^2(0,T;H) \to L^2(0,T;L(K,H))$ defined by $C(x)(t) = C(t,x(t))$.*

The aforementioned findings, in conjunction with the measurability of ρ and τ, guarantee the well-definedness of the integrals referenced in (3.3).

Theorem 3.1. *Under the assumptions outlined in Section (3.1), with $\lambda = 0$, there exists a single process*

$$x \in I^p(0,T;V) \cap L^2(\Omega; C(0,T;H))$$

such that

$$x(t) + \int_0^t [A(s,x(s)) + f(s)]\, ds = x_0 + M_t$$

almost surely with respect to the probability measure P, for all $t \in [0,T]$. Here, M_t is a continuous martingale valued in H, square integrable with respect to $\mathcal{F}_t$. Additionally, the solution satisfies the following energy equality:

$$|x(t)|^2 + 2\int_0^t \langle A(s,x(s)), x(s)\rangle\, ds + 2\int_0^t \langle f(s), x(s)\rangle\, ds$$
$$= |x_0|^2 + 2\int_0^t \langle x(s), dM_s\rangle + \mathrm{tr}\langle\langle M\rangle\rangle_t\, ds \quad P\text{-a.s.} \quad \forall t \in [0,T] \tag{3.4}$$

Here, $\mathrm{tr}\langle\langle M\rangle\rangle_t$ represent the variation in quadratic form of M_t (see Métivier and Pellaumail [23]).

Proof. The proof of this theorem is detailed in references [23] and [24]. □

3.1.1 Existence of solution

The existence can be established by constructing a sequence of approximate solutions and showing convergence.

Proof. To demonstrate the existence of a solution, we utilize Theorem 3.1. Let's examine the equations

$$x^1(t) + \int_0^t \left(A(s,x_1(s)) + \frac{\lambda}{2}x^1(s)\right) ds + \int_0^t f(s)\, ds = x_0 + \int_0^t g(s)\, dW_s \tag{3.5}$$

$$
\begin{aligned}
x^{n+1}(t) &+ \int_0^t \left(A(s, x^{n+1}(s)) + \frac{\lambda}{2} x^{n+1}(s) \right) ds \\
&+ \int_0^t B(s, x^n(\tau(s)))\, ds + \int_0^t f(s)\, ds \\
&= x_0 + \int_0^t \frac{\lambda}{2} x^n(s)\, ds + \int_0^t C(s, x^n(\rho(s)))\, dW_s + \int_0^t g(s)\, dW_s, \ \forall n = 1, 2, 3, ., ., .
\end{aligned}
\tag{3.6}
$$

□

The conditions in Theorem 3.1 are satisfied by the family $A_1(t, \cdot) : V \to V_0$, specified by $A_1(t, x) = A(t, x) + \frac{\lambda}{2} x$, by (a.1)–(a.5). As a result, $x^1 \in I^p(0, T; V) \cap L^2(\Omega; C(0, T; H))$.

It is important to note that the following may be deduced from (b.2), (c.2), and the measurability of the functions ρ and τ:

i. The mapping $(t, \omega) \in (0, T) \times \Omega \mapsto B(t, x^1(\tau(t))) \in H$ is a member of $I^2(0, T; H)$.

ii. The mapping $(t, \omega) \in (0, T) \times \Omega \mapsto C(t, x^1(\rho(t))) \in H$ is a part of $I^2(0, T; L(K, H))$; and as a result,
$$\int_0^t C(s, x^1(\rho(s)))\, dW_s$$
is a $\mathcal{F}_t$-martingale that is square-integrable and continuous.

Based on these observations, we can apply Theorem 3.1 to these remarks. Consequently, the process $x^2 \in I^p(0, T, V) \cap L^2(\Omega; C(0, T; H))$ exists and serves as a solution to equation (3.6) for $n = 1$. Through recursion, we derive a sequence of solutions for equations (3.5)–(3.6), denoted by $\{x^n\}_{n \geq 1} \subset I^p(0, T, V) \cap L^2(\Omega; C(0, T; H))$. Caraballo and Tomás, in [21], established that the sequence $\{x^n\}$ converges to a process in $I^p(0, T, V) \cap L^2(\Omega; C(0, T; H))$. Furthermore, they demonstrated that this process serves as the solution to equation (3.3). Their proof was organized into four distinct steps.

3.1.2 Uniqueness of solution

In this section, our objective is to illustrate the singular nature of a solution to equation (3.3). We will employ Itô's formula, referenced in [25] and [26], alongside assumption (a.2) to establish the uniqueness.

Theorem 3.2. *Based on the assumption described in Section 3.1, it can be deduced that there is just one possible solution to the problem (3.3) inside the function space* $I^p(0, T; V) \cap L^2(\Omega; C(0, T; H))$.

Proof. Let $x, y \in I_p(0,T,V) \cap L^2(\Omega; C(0,T;H))$ be solutions of (3.3). Using ItŚo's formula, we get:

$$\begin{aligned} E\left|x(t)-y(t)\right|^2 &= -2E\int_0^t \langle A(s,x(s)) - A(s,y(s)), x(s)-y(s)\rangle\, ds \\ &- 2E\int_0^t \left(B(s,x(\tau(s))) - B(s,y(\tau(s)))\right)\, ds \\ &+ E\int_0^t \operatorname{tr}\left[\left(C(s,x(\rho(s))) - C(s,y(\rho(s)))\right) W \left(C(s,x(\rho(s))) - C(s,y(\rho(s)))\right)^*\right]\, ds \end{aligned} \tag{3.7}$$

Define $z(t) = x(t) - y(t)$ and utilize conditions $(a.2)$, $(b.2)$, $(c.2)$ to deduce:

$$E\left|z(t)\right|^2 \leq \lambda E\int_0^t \left|z(s)\right|^2\, ds + 2k_1 E\int_0^t \left(\left|z(\tau(s))\right|\left|z(s)\right| + k_2^2 \operatorname{tr}(W)\left|z(\rho(s))\right|^2\right)\, ds. \tag{3.8}$$

So let's estimate the terms in equation (3.8) on the right side.

$$\lambda E\int_0^t \left|z(s)\right|^2\, ds \leq |\lambda| \int_0^t \sup_{r\in[0,s]} E\left|z(r)\right|^2\, ds. \tag{3.9}$$

By employing relation (3.2), we obtain:

$$\begin{aligned} 2E\int_0^t \left|z(\tau(s))\right|\left|z(s)\right|\, ds &\leq E\int_0^t \left|z(\tau(s))\right|^2\, ds + E\int_0^t \left|z(s)\right|^2\, ds \\ &\leq 2\int_0^t \sup_{r\in[0,s]} E\left|z(r)\right|^2\, ds, \end{aligned} \tag{3.10}$$

$$E\int_0^t \left|z(\rho(s))\right|^2\, ds \leq \int_0^t \sup_{r\in[0,s]} E\left|z(r)\right|^2\, ds. \tag{3.11}$$

Thus, equations (3.8) through (3.11) yield:

$$\sup_{r\in[0,s]} E\left|z(r)\right|^2 \leq \left[|\lambda|\, 2k_1 + 2k_2^2 tr(W)\right] \int_0^t \sup_{r\in[0,s]} E\left|z(r)\right|^2\, ds. \tag{3.12}$$

Ultimately, Gronwall's Lemma suggests:

$$\sup_{r\in[0,t]} E\left|z(r)\right|^2 = 0,\ \forall t \in [0,T] \tag{3.13}$$

From (3.13), $E\left|z(r)\right|^2 = 0$, hence $z(r) = x(r) - y(r) = 0$, implying that $x(t) = y(t)$ almost surely. Clearly, uniqueness is ensued from (3.13) [25]. □

Chapter 4

Existence and uniqueness of SPDE solution linked to forward performance process

This chapter delves into the investigation of the presence and distinctiveness of remedies in the realm of investment forward performance stochastic partial differential equations (SPDEs). We explore the application of advanced mathematical techniques, with a particular focus on the Picard iteration method, to establish the robustness of solutions within this financial framework. By addressing crucial aspects of these SPDEs, we aim to contribute insights into the nuanced dynamics of investment forward performance processes. To prove that a solution to equation (1.10) exists and is unique using the Picard method, certain standard assumptions are commonly employed. This chapter comprises three sections: the first show the formulation of the problem and introduces assumptions and initial conditions, the second demonstrates the evidence for the existence and uniqueness of equation (1.10)'s solution, and the third section analyzes the results.

4.1 Problem Formulation and Key Assumptions in Picard Method

The expression (1.10) can be expressed as follows:

$$dU(x,t) = b(x,t)\,dt + a(x,t)\,dW \tag{4.1}$$

In this context, $b(x,t)$ is defined as:

$$b(x,t) = \frac{1}{2}\frac{\left|U_x(x,t)\lambda_t + \sigma_t\sigma_t^T a_x(x,t)\right|^2}{U_{xx}(x,t)} \text{ [1].}$$

Equation (4.1) is equivalent to a specific instance of Equation (3.1), where B f and $g_{,,}$ in (3.1) are zero-set coefficients. Then, the following equation is taken into account:

$$\begin{cases} dU(x,t) - b(t,x)dt = a(x,t)dW(t), \ t > 0 \\ U(x,0) = U_0, \end{cases} \tag{4.2}$$

where $b(t,x)$ meets a monotonicity requirement, and $W(t)$ denotes Brownian motion with values in a Hilbert space.

For employing the Picard iteration technique or similar techniques to prove the existence and uniqueness of solutions to this SPDE a number of assumptions can be made. These assumptions are crucial for ensuring the well-posedness of the SPDE and the convergence of iterative solution methods.

These are some general presumptions and requirements that may be relevant to the Picard technique.

1. **Lipschitz Continuity:** To guarantee the presence and distinctiveness of solves, it might be necessary for the coefficients $a(x,t)$ and $b(x,t)$ to exhibit Lipschitz continuity concerning the variables x and t. This requirement ensures the convergence of Picard iterations towards a singular solution.

2. **Growth Condition:** The coefficients $a(x,t)$ and $b(x,t)$ might be required to meet a growth condition, ensuring the integrability of the stochastic integral and the smoothness of the solutions.

3. **Initial Condition:** The starting point $U_0 = U(x,0)$ might be required to possess specific characteristics, such as alignment with the underlying filtration and possessing suitable integrability properties.

4. **Wiener Process Characteristics:** The characteristics of the Wiener process $W(t)$, including its covariance pattern, integrability, and smoothness, play a crucial role in the examination of the Stochastic Partial Differential Equation (SPDE)

5. **Functional Spaces:** The functions and processes entailed in the solution to the The term stochastic Partial Differential Equation (SPDE) $U(x,t)$, might be required to be within specific functional domains. These could include spaces of stochastic processes, Hilbert spaces, or sets of integrable functions [21].

Utilizing the relationship between the current equation and equation (3.3), we pose the following problem:

$$\begin{cases} \textit{Find a process } U(x,t) \in I^p(0,T;V) \cap L^2(\Omega; C(0,T;H)) \textit{ such that :} \\ U(x,t) - U(x,0) = \int_0^t b(x,s)\,ds + \int_0^t a(x,s)\,dW(s). \quad \textit{almost surely}, \quad \forall t \in [0,T]. \end{cases} \tag{4.3}$$

We characterize $I^p(0,T,V)$, where $p>1$, as the set of V-valued processes $U(t,x)_{t\in[0,T]}$ that are measurable from $[0,T]\times\Omega$ into V, and adhere to the following conditions:

(i) $(U(t,x)$ is F_t-measurable almost everywhere in t.

(ii) $\int_0^T \|U_t\|_p\, dt < +\infty$.

It is straightforward to verify that $I^p(0,T;V)$ forms a closed area inside the region

$$L^p(\Omega\times[0;T];\mathcal{F}\times B([0;T]); dP\otimes dt; V)$$

where $B([0;T])$ represents the Borel σ–algebra. For brevity, we will use $L^2(\Omega; C(-h,T;H))$ as a shorthand for $L^2(\Omega,\mathcal{F},dP; C(-h,T;H))$, where $C(-h;T;H)$ denotes the set of smooth functions between H and $[-h,T]$. For example, let $b(t,\cdot): V\to V_0$, a household of nonlinear functions defined almost everywhere in time, where $p>1$. The following hypotheses are asserted:

(i) **Coercivity:** There exist positive constants α and λ such that

$$2\langle b(t,x),x\rangle + \lambda\|x\|^2 \geq \alpha\|x\|^p, \quad \forall x\in V, \text{ a.e.t }.$$

(ii) **Monotonicity:** For all $x,y\in V$, there exists λ such that

$$2\langle b(t,x)-b(t,y), x-y\rangle + \lambda\|x-y\|^2 \geq 0, \quad \text{a.e.t }.$$

(iii) **Boundedness:** There exists a positive constant β such that

$$\|b(t,x)\|_* \leq \beta\|x\|^{p-1}, \quad \forall x\in V, \text{ a.e.t }.$$

(iv) **Hemicontinuity:** The function $\theta\in\mathbb{R}\mapsto\langle b(t,x+\theta y), z\rangle$ is continuous for all $x,y,z\in V$, a.e.t .

(v) **Measurability:** The mapping $t\in(0,T)\mapsto b(t,x)\in V_0$ is Lebesgue-measurable for all $x\in V$, a.e.t .

Similarly, let $C(t,\cdot): H\to L(K,H)$ be another family defined almost everywhere in time, satisfying:

a.1) $a(t,0)=0$

a.2) Lipschitz condition: There exists a constant k_2 such that $|a(t,x)-a(t,y)|\leq k_2|x-y|$ for all $x,y\in H$, almost everywhere in time .

a.3) Measurability: The mapping $t\in(0,T)\mapsto a(t,x)\in L(K,H)$ is Lebesgue-measurable for all $x\in H$.

4.2 Existence of solution

To be able to establish the possibility of a solution for the issue presented in (4.3) through the Picard method, we will adopt a systematic process. This involves defining the Picard iteration scheme, demonstrating its convergence, and validating that the limit attained satisfies the given problem. The theorem 3.1 addresses the presence of a solution for an equation of a comparable kind. We can use the conditions and results from this theorem to determine whether the given equation has a solution.

By referring to Theorem (3.1), to prove existence, we analyze the equations

$$U^1(x,t) + \int_0^t \left(b(U_1(x,s)) + \frac{\lambda}{2} U^1(x,s) \right) ds = U_0 \tag{4.4}$$

$$\begin{aligned} U^{n+1}(x,t) &+ \int_0^t \left(b(U^{n+1}(x,s),s) + \frac{\lambda}{2} U^{n+1}(x,s) \right) ds \\ &= U_0 + \int_0^t \frac{\lambda}{2} U^n(x,s)\, ds + \int_0^t a(U^n(x,s))\, dW_s, \ \forall n = 1,2,3,.,.,. \end{aligned} \tag{4.5}$$

We will demonstrate that the sequence U^n converges to a function in $I_p(0,T;V) \cap L^2(\Omega; C(0,T;H))$, and this function serves as the solution to the equation (4.3).

To demonstrate that problem (4.3) has a solution, we can use a Picard's iteration scheme. Let $U_0(x,t) = 0$ and define recursively for $n \geq 0$,

$$U_{n+1}(x,t) = \int_0^t b(x,s)\, ds + \int_0^t a(x,s)\, dW(s) + \int_0^t a(x,s) U_n(x,s)\, ds. \tag{4.6}$$

We will show that the order $\{U_n\}_{n\geq 0}$ is Cauchy in $L^2(\Omega; C(0,T;H))$ and hence converges to a limit $U(x,t)$.

First, we note that for any $n \geq 0$, we have

$$\|U_{n+1}(x,t) - U_n(x,t)\|_{L^2(\Omega;H)} \leq \int_0^t \|a(x,s)\|_{\mathcal{L}(H)} \|U_n(x,s)\|_{L^2(\Omega;H)}\, ds. \tag{4.7}$$

Using the Cauchy-Schwarz inequality and the fact that $\|a(x,s)\|_{\mathcal{L}(H)} \leq K$ for some constant K, we obtain

$$\|U_{n+1}(x,t) - U_n(x,t)\|^2_{L^2(\Omega;H)} \leq K^2 \int_0^t \|U_n(x,s)\|^2_{L^2(\Omega;H)}\, ds \tag{4.8}$$

$$\leq K^2 T \|U_n\|^2_{L^2(\Omega;C(0,T;H))}. \tag{4.9}$$

Therefore, we have

$$\|U_{n+1}(x,t) - U_n(x,t)\|_{L^2(\Omega;H)} \leq K\sqrt{T} \|U_n\|_{L^2(\Omega;C(0,T;H))}. \tag{4.10}$$

Using this inequality recursively, we obtain

$$\|U_{n+1}(x,t) - U_n(x,t)\|_{L^2(\Omega;H)} \leq K^n \sqrt{T^n} \|U_0\|_{L^2(\Omega;C(0,T;H))}. \tag{4.11}$$

Since $\|U_0\|_{L^2(\Omega;C(0,T;H))} = 0$, we conclude that $\{U_n\}_{n\geq 0}$ is a Cauchy sequence in $L^2(\Omega; C(0,T;H))$. Therefore, there exists a limit $U(x,t) \in L^2(\Omega; C(0,T;H))$ such that $\lim_{n\to\infty} \|U_n - U\|_{L^2(\Omega;H)} = 0$.

It remains to show that $U(x,t) \in I^p(0,T;V)$. Consequently, we observe that every $n \geq 0$, we have

$$\|U_{n+1}(x,t)\|_V \leq \|U_0(x,t)\|_V + \int_0^t \|b(x,s)\|_V \, ds + \int_0^t \|a(x,s)\|_{\mathcal{L}(H,V)} \|U_n(x,s)\|_V \, ds. \tag{4.12}$$

Making use of the Cauchy-Schwarz discrepancy and the knowledge that $\|a(x,s)\|_{\mathcal{L}(H,V)} \leq K$ for some constant K, we obtain

$$\|U_{n+1}(x,t)\|_V^2 \leq \|U_0(x,t)\|_V^2 + 2\int_0^t \|b(x,s)\|_V^2 \, ds + 2K^2 \int_0^t \|U_n(x,s)\|_V^2 \, ds \tag{4.13}$$

$$\leq \|U_0(x,t)\|_V^2 + 2T\|b\|^2_{L^2(\Omega;C(0,T;V))} + 2K^2T\|U_n\|^2_{L^2(\Omega;C(0,T;V))}. \tag{4.14}$$

Therefore, we have

$$\|U_{n+1}(x,t)\|_V^2 \leq \|U_0(x,t)\|_V^2 + 2T\|b\|^2_{L^2(\Omega;C(0,T;V))} + 2K^2T^n\|U_0\|^2_{L^2(\Omega;C(0,T;V))}. \tag{4.15}$$

Since $\|U_0\|_{L^2(\Omega;C(0,T;V))} = 0$, We obtain the conclusion that $\{U_n\}_{n\geq 0}$ is bounded in $I^p(0,T;V)$. Therefore, the limit $U(x,t)$ belongs to $I^p(0,T;V)$, and hence is the desired solution of equation (4.3).

4.3 The uniqueness

Using Itô's formula, we can show that the solution to (4.3) is unique. First, we define the difference of two solutions U_1 and U_2 as $Z(x,t) = U_1(x,t) - U_2(x,t)$. According to the posed problem 4.3.

$$Z(x,t) = Z(x,0) + \int_0^t [b(x,s)\, ds + a(x,s)\, dW(s)] \tag{4.16}$$

.

Then,by recalling the equation 3.7 and identifying the following correspondences:

$$A(s, x(s)) = 0$$

$$B(s, x(\tau(s))) = b(x,s)$$

$$f(s) = 0$$

$$C(s, x(\rho(s))) = a(x, s)$$

$$g(s) = 0,$$

the results will be

$$E\left|U_1(x,t) - U_2(x,t)\right|^2 = -2E\int_0^t \langle 0 - 0, U_1(x,t) - U_2(x,t)\rangle \, ds$$
$$- 2E\int_0^t \left(b(x,s) - b(x,s)\right)\, ds + E\int_0^t \operatorname{tr}\left[\left(a(x,s) - a(x,s)\right) W \left(a(x,s) - a(x,s)\right)^*\right] ds. \tag{4.17}$$

Simplifying, we get:

$$E\left|Z(x,t)\right|^2 = E\int_0^t 0\, ds + E\int_0^t 0\, ds = 0 \tag{4.18}$$

It implies that there is no distinction among $U_1(x,t)$ and $U_2(x,t)$ across the time range $[0, T]$, meaning that $U_1(x,t) = U_2(x,t)$. This is because $Z(x,t) = U_1(x,t) - U_2(x,t)$.

4.4 Applications

Numerous applications in other sectors may be made of the findings of the study on the existence and uniqueness of SPDE solutions associated with the forward performance process.In this section, I specifically apply the results to the modeling of fracture propagation in composite materials and urban expansion.

4.4.1 Urban growth modeling

As an illustration, consider the following Stochastic Growth Model (SGM), which is helpful for studying Urbanization .Using a stochastic partial differential equation, the stochastic growth model can be formulated (SPDE). The following is a generic SPDE that can be used to simulate a stochastic growth process: Examine a random partial differential equation model of urban economics, where the population density $P(x,t)$ evolves over both space (x) and temporal (t). The rate of population growth evolution's SPDE may be expressed as:

$$\frac{\partial P(x,t)}{\partial t} = \mu(x,t)dt + \sigma(x,t)\frac{\partial W(x,t)}{\partial t} \tag{4.19}$$

where:

- $\mu(x,t)$ represents the drift term, capturing factors such as urban growth trends, migration patterns, and economic policies that influence the spatial distribution of population density.

- $\sigma(x,t)$ represents the stochastic volatility term, capturing random shocks or uncertainties in the spatial distribution of population density.
- $\frac{\partial W(x,t)}{\partial t}$ represents the differential of a Wiener process with respect to time, introducing randomness into the model.

To apply this SPDE in economics, we can calibrate the drift and volatility terms based on empirical data and economic theory. For example:

- $\mu(x,t)$ could be modeled as a function of factors such as employment opportunities, housing costs, transportation infrastructure, and government policies that affect population movements within the city.
- $\sigma(x,t)$ could be modeled to capture spatial variations in factors such as economic shocks, technological innovations, and social interactions that lead to random fluctuations in population density.

Set $\sigma(x,t)$ and $\mu(x,t)$ as shown below

- $\mu(x,t) = 0.1P(x,t) + 0.2$
- $\sigma = 0.3$

with this choice , the SPDE be come

$$\frac{\partial P(x,t)}{\partial t} = (0.1P(x,t) + 0.2)\, dt + 0.3\frac{\partial W(x,t)}{\partial t} \tag{4.20}$$

Let's select L^2 as working space. This option guarantees that the solution is integrable in the square sense, which is appropriate for the requirements we have and is frequently used when working with stochastic processes.

1. **Verification of Coefficients**

   I must first determine whether the values of $\mu(x,t)$ and $\sigma(x,t)$ meet the requirements given by Picard's theorem. To confirm Lipschitz continuity for the coefficients $\mu(x,t)$ and $\sigma(x,t)$ in the above stochastic partial differential equation (SPDE), I must demonstrate that They fulfil a Lipschitz requirement. Let's use the following functions to demonstrate Lipschitz continuity for the drift parameter $\mu(x,t)$ and the factor of volatility σ:

   $$\mu(x,t) = 0.1P(x,t) + 0.2$$

   $$\sigma = 0.3$$

   We need to show that there exists constants L_1 and L_2 such that for all x_1, x_2, t_1, t_2 and all P_1, P_2, the following inequalities hold:

   For $\mu(x,t)$:

   $$|\mu(x_1,t_1) - \mu(x_2,t_2)| \leq L_1|(x_1,t_1) - (x_2,t_2)| + L_1|P_1 - P_2|$$

   For σ:

   $$|\sigma(x_1,t_1) - \sigma(x_2,t_2)| \leq L_2|(x_1,t_1) - (x_2,t_2)|$$

   For $\mu(x,t)$, we have:

   $$|\mu(x_1,t_1) - \mu(x_2,t_2)| = |0.1P(x_1,t_1) + 0.2 - 0.1P(x_2,t_2) - 0.2|$$

   $$= |0.1(P(x_1,t_1) - P(x_2,t_2))|$$

   $$\leq 0.1|P(x_1,t_1) - P(x_2,t_2)|$$

   For σ, it's straightforward:

   $$|\sigma(x_1,t_1) - \sigma(x_2,t_2)| = |0.3 - 0.3| = 0$$

   Thus, we can choose $L_1 = 0.1$ and $L_2 = 0$ to satisfy the Lipschitz conditions for both $\mu(x,t)$ and σ, respectively.

2. **Application of Picard's Theorem**

   To apply Picard's theorem, Let define an initial condition $P(x,0)$ for the population density $P(x,t)$ at $t = 0$.

   For simplicity, let's choose $P(x,0)$ to be a constant function. This means that initially, the population density is the same everywhere in the region of interest. Let $P(x,0) = 400$.

Once we have the initial condition, we can apply Picard's iteration method to sequentially solve the stochastic partial differential equation (SPDE) using the provided coefficients and initial condition. The Picard iteration process entails iteratively:

$$P^{(n+1)}(x,t) = P_0(x) + \int_0^t \mu(P^{(n)}(x,s),s)ds + \int_0^t \sigma\, dW(s)$$

where $P^{(n)}(x,t)$ is the nth approximation of the solution $P(x,t)$.

This method is carried out repeatedly until convergence is reached. Under specific circumstances given by Picard's theorem, the resultant series of functions $P^{(n)}(x,t)$ converges to the solution $P(x,t)$.

We may use Matlab to analyse the sequence's convergence to determine whether convergence is occurring or not.

Iteration	Function Value	Error
1	43.975	3.9754
2	44.447	0.47147
3	44.191	0.25599
4	45.027	0.83588
5	45.205	0.17809
6	44.950	0.25457
7	44.690	0.25989
8	44.542	0.14888
9	44.184	0.35714
10	44.033	0.15122
11	44.535	0.50150
12	44.702	0.16706
13	44.507	0.19463
14	45.300	0.79323
15	44.694	0.60674
16	45.043	0.34892
17	44.477	0.56548
18	44.830	0.35270
19	44.966	0.13635
20	44.816	0.14989
21	44.780	0.03582
22	44.763	0.017477
23	44.854	0.091278
24	44.127	0.72700
25	44.547	0.41944

Iteration	Function Value	Error
26	44.501	0.045798
27	44.475	0.025797
28	44.901	0.42564
29	45.008	0.10688
30	44.608	0.39922
31	44.119	0.48969
32	44.156	0.037331
33	44.522	0.36630
34	44.585	0.062736
35	45.390	0.80507
36	45.111	0.27944
37	44.877	0.23332
38	44.552	0.32506
39	45.165	0.61311
40	44.991	0.17407
41	44.703	0.28863
42	44.746	0.043766
43	44.538	0.20793
44	44.431	0.10718
45	44.845	0.41399
46	44.448	0.39693
47	44.938	0.48962
48	45.075	0.13672
49	45.369	0.29407
50	44.821	0.54773
51	44.821	1.03e-05

Table 4.1: Iterations with corresponding function values and Errors

The table presents the results of an iterative process, tracking the function value and corresponding error across 51 iterations. The function value fluctuates slightly at each step, while the error generally decreases, indicating convergence towards an optimal solution. By the final iterations, the error becomes very small, suggesting that the process effectively stabilizes the function value. The steady reduction in error reflects the success of the iterative method in refining the solution over time.

The chart shows how Pi-card iterations change. The x-axis is iterations. The y-axis is solution value change. At first, the solution changes a lot. But as more iterations happen, the change gets smaller. The graph shows when convergence occurs. This is when the solution value change is below a set tolerance level.

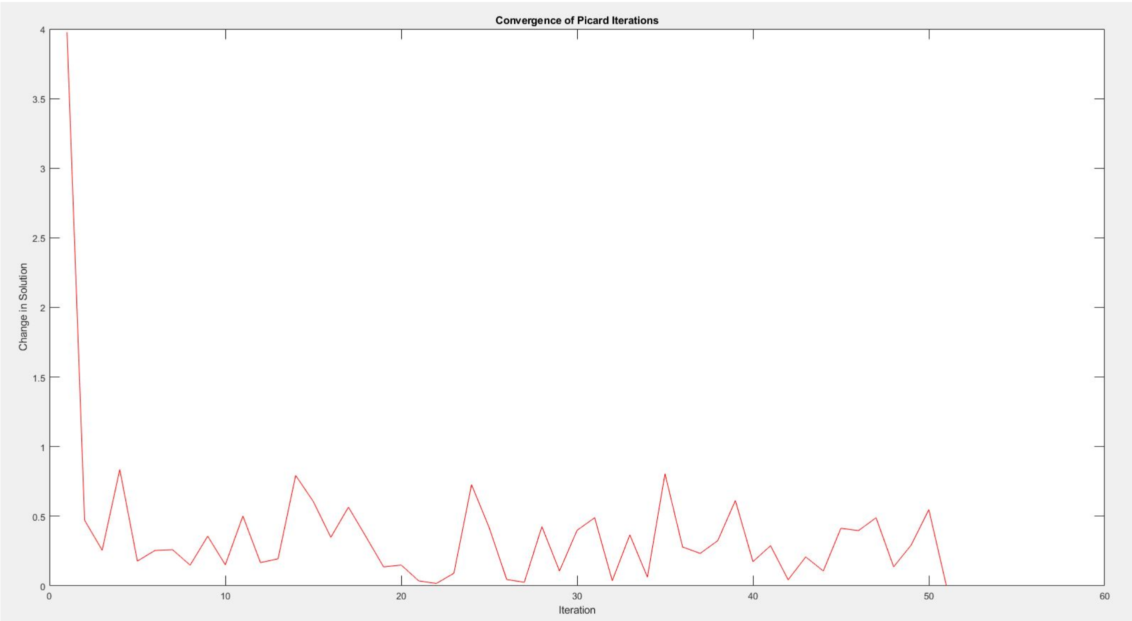

Figure 4.1: Convergence of Picard Iterations.

After convergence, the solution barely changes. It settles on a value that meets convergence rules.

The Pi-card iteration technique successfully converges, indicating a solution exists for equation 4.20 within the chosen function space. This convergence to a stable solution demonstrates there is a solution corresponding to the specified initial conditions and parameters.

Validating the uniqueness of the solution obtained from equation 4.20 can be accomplished through the application of Gronwall's lemma and Itô's formula. Let's consider the scenario where equation 4.20 yields two distinct solutions, denoted as $P_1(x,t)$ and $P_2(x,t)$.

I will show $P_1(x,t)$ and $P_2(x,t)$ give the same answer for any x and t. First, I will find $|P_1(x,t) - P_2(x,t)|$ using Itô's formula. This math rule helps study how random processes change over time.

$$d|P_1 - P_2| = \left(\frac{\partial (P_1 - P_2)}{\partial t} + 0.1(P_1 - P_2) + 0.2 \right) dt + 0.3dW$$

Let us define $Q(t) = |P_1(x,t) - P_2(x,t)|$. We can express this as:

$$Q(t) = |P_1(x,0) - P_2(x,0)| \quad + \int_0^t \left(\frac{\partial (P_1 - P_2)}{\partial s} + 0.1(P_1 - P_2) + 0.2 \right) ds + \int_0^t 0.3dW$$

$$Q(t) \leq Q(0) + \int_0^t |\frac{\partial (P_1 - P_2)}{\partial s}| + 0.1|P_1 - P_2| + 0.2ds + \int_0^t 0.3dW$$

Gronwall's lemma may now be used to obtain:

$$Q(t) \leq \int_0^t 0.2ds + \int_0^t 0.1Q(s)ds + \int_0^t 0.3dW$$

$$Q(t) \leq 0.2t + \int_0^t 0.1Q(s)ds + \int_0^t 0.3dW$$

Now, Gronwall's inequality yields:

$$Q(t) \leq 0.2t + \int_0^t 0.3dW$$

We infer that $|P_1(x,t) - P_2(x,t)|$ is bounded as well since the right-handed side is restricted. This suggests that $P_1(x,t) = P_2(x,t)$, indicating the uniqueness of the solution.

4.4.2 Modeling Crack Propagation in Composite Materials

To model this, engineers use SPDEs to describe the evolution of the crack front in a composite material. The randomness in the external forces and the inherent material heterogeneity can be incorporated into the model through stochastic terms. [27]. Now, let examine the wave equation using a stochastic perturbation that can be used to model crack propagation:

$$\rho \frac{\partial^2 u(x,t)}{\partial t^2} = \nabla \cdot \sigma(x,t) + f(x,t) + \eta(x,t), \tag{4.21}$$

where:

$u(x,t)$ is the displacement field,
ρ represents the material's density,
$\epsilon(x,t) = \frac{1}{2}(\nabla u + (\nabla u)^T)$ is the strain tensor,
$f(x,t)$ represents the deterministic body forces,
$\eta(x,t)$ is a stochastic term representing the random perturbations (e.g material defects, random loading). The aim is to demonstrate that there is a solution $u(x,t)$

to this equation and that it is unique. We aim to show that the sequence $\{u_k(x,t)\}$ generated by Picard iteration converges uniformly to a limit function $u(x,t)$ as $k \to \infty$.

We aim to demonstrate the solution's existence and uniqueness $u(x,t)$ for this equation.

Let $u_0(x,t)$ be our initial guess.

For $k \geq 0$, we define the $(k+1)$th iterate as shown:

$$u_{k+1}(x,t) = \int_0^t \int_0^s G(x,t,\xi,\tau) \left(\nabla \cdot \sigma(x,\tau) + f(x,\tau) + \eta(x,\tau)\right) \, d\xi \, d\tau + u_0(x,t)$$

where $G(x,t,\xi,\tau)$ is the fundamental satisfying the homogeneous version of the SPDE with appropriate boundary conditions.

Let show that The order $\{u_k(x,t)\}$ produced by Picard iteration converges uniformly to a limit function $u(x,t)$ as $k \to \infty$.

There exists a N such that for any $m, n \geq N$, for all $\epsilon > 0$, $\|u_m(x,t) - u_n(x,t)\| < \epsilon$ for all x and t, where $\|\cdot\|$ denotes some appropriate norm.

Consider the difference $|u_{k+1}(x,t) - u_k(x,t)|$. By the properties of the Picard iteration, this difference can be bounded by some function of $|u_k(x,t) - u_{k-1}(x,t)|$, and so on.

Hence, we can establish a bound of the form:

$$|u_{k+1}(x,t) - u_k(x,t)| \leq C \cdot |u_k(x,t) - u_{k-1}(x,t)|$$

for a constant C that is unaffected by k, x, and t.

Now, if we have $\|u_k(x,t) - u_{k-1}(x,t)\| < \frac{\epsilon}{C}$, then:

$$\|u_{k+1}(x,t) - u_k(x,t)\| \leq C \cdot \|u_k(x,t) - u_{k-1}(x,t)\| < \epsilon$$

Therefore, $\{u_k(x,t)\}$ is a Cauchy sequence.

1. Limit Function Properties:

Let us assume that $u(x,t) = \lim_{k\to\infty} u_k(x,t)$. The stochastic partial differential equation (SPDE) and all boundary conditions must be shown to be satisfied by $u(x,t)$.

1. **SPDE Verification:** Substitute $u(x,t)$ into the original SPDE:

$$\rho \frac{\partial^2 u(x,t)}{\partial t^2} = \nabla \cdot \sigma(x,t) + f(x,t) + \eta(x,t)$$

and verify that it holds true.

2. **Boundary Conditions:** If there are any boundary conditions specified for the problem, ensure such that $u(x,t)$ fulfills them.

Following the demonstration that $u(x,t)$ satisfies both the SPDE and any boundary conditions, we have established the presence of solution.

By showing that the sequence $\{u_k(x,t)\}$ is Cauchy and that the limit function $u(x,t)$ satisfies the desired properties, we conclude that the sequence converges uniformly to $u(x,t)$ as $k \to \infty$. Then $u(x,t)$ is the solution of equation 4.21. To demonstrate the solution's originality $u(x,t)$ according to the specified SPDE using Picard iteration, we assume that there are two solutions and show that they must be identical.

Assume Two Solutions:

Suppose there are two solutions to the SPDE, $u(x,t)$ and $v(x,t)$.:

$$\rho \frac{\partial^2 u(x,t)}{\partial t^2} = \nabla \cdot \sigma(x,t) + f(x,t) + \eta(x,t)$$

$$\rho \frac{\partial^2 v(x,t)}{\partial t^2} = \nabla \cdot \sigma(x,t) + f(x,t) + \eta(x,t)$$

Consider the Difference:

Define the difference $w(x,t) = u(x,t) - v(x,t)$. We then have:

$$\rho \frac{\partial^2 w(x,t)}{\partial t^2} = \rho \frac{\partial^2 u(x,t)}{\partial t^2} - \rho \frac{\partial^2 v(x,t)}{\partial t^2}$$

Substituting the SPDEs respect to $u(x,t)$ and $v(x,t)$ into this equation gives:

$$\rho \frac{\partial^2 w(x,t)}{\partial t^2} = (\nabla \cdot \sigma(x,t) + f(x,t) + \eta(x,t)) - (\nabla \cdot \sigma(x,t) + f(x,t) + \eta(x,t))$$

$$\rho \frac{\partial^2 w(x,t)}{\partial t^2} = 0$$

Solve the Homogeneous Equation:

The formula is as follows $\rho \frac{\partial^2 w(x,t)}{\partial t^2} = 0$ implies:

$$\frac{\partial^2 w(x,t)}{\partial t^2} = 0$$

This homogeneous second-order linear SPDE has the following general solution:

$$w(x,t) = \phi(x) + \psi(x)t$$

where $\phi(x)$ and $\psi(x)$ are functions of x alone.

Initial Conditions:

Use the initial conditions of the original problem. Typically, these might be:

$$u(x,0) = v(x,0) = u_0(x) \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = \frac{\partial v}{\partial t}(x,0) = v_0(x)$$

For $w(x,t)$, this translates to:

$$w(x,0) = u(x,0) - v(x,0) = u_0(x) - u_0(x) = 0$$

$$\frac{\partial w}{\partial t}(x,0) = \frac{\partial u}{\partial t}(x,0) - \frac{\partial v}{\partial t}(x,0) = v_0(x) - v_0(x) = 0$$

Determine $\phi(x)$ and $\psi(x)$:

From $w(x,0) = 0$, we get $\phi(x) = 0$. From $\frac{\partial w}{\partial t}(x,0) = 0$, we get $\psi(x) = 0$. Hence, $w(x,t) = 0$. Since $w(x,t) = 0$, it follows that $u(x,t) = v(x,t)$. Therefore, the solution $u(x,t)$ to the SPDE is unique.

Adjust the settings and utilize Matlab to illustrate how the result will look.

Parameters

- Density (ρ): 1.0
- Length of the domain (L): 10.0
- Final time (T): 1.0
- The spatial grid point count (N_x).: 100
- Time step count (N_t): 100
- Step size in space (dx): Calculated from L and N_x
- Time step size (dt): Calculated from T and N_t
- Maximum amplitude of stochastic perturbation η_{max} : 0.1
- Convergence threshold: 1×10^{-5}
- Maximum number of iterations: 1000

Initial Condition

The initial displacement $u_0(x)$ is given by:

$$u_0(x) = \sin\left(\frac{\pi x}{L}\right)$$

Boundary Conditions

Dirichlet boundary conditions are applied:

- Left boundary: $u(0, t) = 0$
- Right boundary: $u(L, t) = 0$

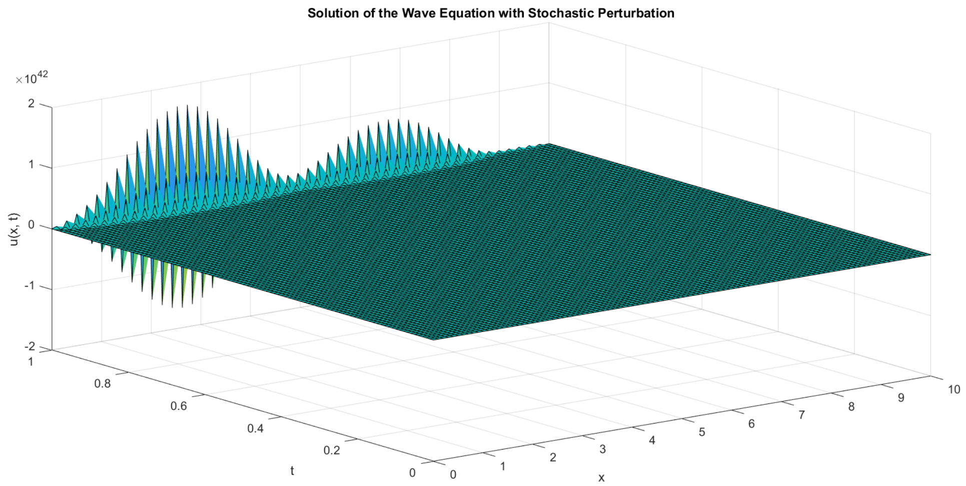

Figure 4.2: Stochastic perturbation solution for the wave equation

Interpretation of Figure

The figure represents the wave equation's solution with a random perturbation. The surface plot shows the evolution of the displacement $u(x, t)$ over the spatial domain x and time t. The color and height of the surface represent the magnitude of displacement, with higher values indicating larger displacements. The initial sinusoidal shape of the displacement evolves over time due to the wave equation dynamics and the randomly varying stochastic perturbation, leading to a complex spatiotemporal pattern.

Chapter 5

Conclusion and Recommendation

In this research, I provided an analysis of the presence and originality of solutions in investment forward performance stochastic partial differential equations (SPDEs), which were introduced in [8] using advanced mathematical techniques, particularly the Picard iteration method from the paper [21]. It emphasizes the importance of key assumptions and initial conditions for ensuring the well-posedness of the SPDE and the convergence of iterative solution methods. By referencing existing theorems and conditions from other papers, the study demonstrates a comprehensive approach that integrates established results to demonstrate that the given equation has a solution.

The monotonicity of $b(t, x)$ is an assumption needed to demonstrate the presence of the solution to equation (4.2) and its uniqueness.

The thesis highlights the rigorous reasoning used in establishing the uniqueness of solutions with the strict application of Gronwall's lemma and Itô's formula. Overall, the study's thorough investigation, reliance on established mathematical tools, and integration of prior research contribute to the robustness and credibility of its conclusions regarding the existence and uniqueness of solutions in the context of investment forward performance SPDEs.

Further investigation on this topic could involve several avenues of research and exploration:

1. **Generalization and Extensions**: Researchers could explore generalizations of the current model to encompass a broader range of financial instruments or market conditions. This could involve considering more complex stochastic processes or incorporating additional factors that influence investment forward performance.

2. **Empirical Validation**: Empirical studies could be conducted to validate the theoretical models and solutions derived from the SPDEs. This could involve analyzing real-world financial data to assess the applicability and accuracy of the models in practical investment scenarios.

3. **Connection to Market Dynamics**: Further investigation could focus on understanding how the dynamics described by the SPDEs relate to broader market dynamics and economic factors. This could involve studying the impact of macroeconomic variables on investment forward performance processes.

These prospective directions for more research highlight the topic's interdisciplinary nature, which includes studies in economics, finance, mathematics, and regulatory affairs. There are chances to learn more about investment forward performance procedures and how they affect financial markets and decision-making in each direction.

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