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Groups of special matchings and Bruhat graph automorphisms

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Abstract

Given any Weyl group W, the groups generated by special matchings on the Bruhat order of W have been introduced, and it turns out that those groups are not necessarily reflection groups.

In addition, the SageMath algorithm has been constructed, and from it, it was proven that for any butterfly in types B_2 up to B_9, D_4 up to D_6 and E_6 there must be an element that coves (or that is covered) by the maximal (or minimal) elements of that butterfly. This result have been used to prove Proposition 3.2.10 which is the second main result of this dissertation.

Declaration

I, Martin IRIBONEYE, hereby confirm that this thesis entitled **“Groups of special matchings and Bruhat graph automorphisms”** is my original work and has never been submitted or published anywhere by any person to any university or other institution of higher leaning for academic publication.

This thesis was conducted under the supervision of Dr.Vincent Umutabazi from the University of Rwanda, College of Science and Technology(UR-CST), School of Science, Department of Mathematics.

October, 2024

IRIBONEYE Martin

Certification

I, undersigned, certify that this thesis entitled **"Groups of special matchings and Bruhat graph automorphisms"** was conducted under my supervision and has been submitted under my approval.

Supervisor: Dr.Vincent UMUTABAZI

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Signature:

Date:

Dedication

To my family, friends, and supervisor Vincent Umutabazi.

Acknowledgements

Firstly, I want to thank the University of Rwanda, College of Science and Technology (UR-CST) under the School of Science, Department of Mathematics which helped me in my studies of masters in Mathematics.

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Chapter 1

Introduction

This chapter contains the background, problem statement, objectives, methodologies, and the structure of the thesis.

By a *perfect matching* on a (*undirected*) *Bruhat order* $\mathrm{Br}(u, v)$ of some Bruhat interval $[u, v]$ in a Coxeter group W, we mean a theoretical graph complete matching on $\mathrm{Br}(u, v)$. A special matching on a poset is a complete matching on its Hasse diagram with some extra conditions. The notion of special matching firstly was introduced in [3]. For Eulerian posets, du Cloux[8] introduced an independent concept of special matchings. For all those authors, their purposes were to find some algebraic properties that generalize the Bruhat order of a Coxeter group. For example, it is known that every non-trivial lower Bruhat interval on a Coxeter group has a special matching given by right multiplication by a simple generator of a Coxeter group.

1.1 Background

The Bruhat order on Coxeter groups plays an important role in the representation theory of Lie groups and the geometry of associated flag varieties and Schubert varieties. Coxeter groups appear in many branches of mathematics like algebraic combinatorics, etc,. A Bruhat order and a Bruhat graph of Coxeter group have several important properties in combinatorics. Among those properties, in this thesis, we focus on the special matchings of Bruhat orders and automorphisms of undirected Bruhat graphs.

Let (W, S) be a Coxeter system. Special matchings on Bruhat intervals in W possess interesting and central role in generalazing Bruhat orders, and in the study of Kazhdan-Lusztig polynomials. The Kazhdan-Lusztig polynomials have been introduced in [14]. More on Kazhdan–Lusztig–Vogan polynomials can be found in [15] and [18]. Special matchings have been studies extensively like in [6], and form posets called "zircons" which are generalisations of Bruhat orders, see [4], [16], and [12]. Special matchings have applications in studying Kazhdan-Lusztig polynomials; see for example [1]. In [11], special matchings on a Bruhat order of an interval in W turn to be automorphisms of undirected Bruhat graphs of the same interval in W. In [5], groups generated by two special matchings have been introduced. These groups are in fact dihedral; and play an important role in the computation of Kazhdan–Lusztig and R-polynomials, and in the study of Hecke algebra associated to the special matchings.

Let P be a partially ordered set and $x \in P$. Denote the set of special matchings of P by SM_P. Let also $P_{\leq x}\colon = \{y \in P : y \leq x\}$ be a subposet of P let SM_x be set of all special matchings of $P_{\leq x}$. We shall refer SM_x as the set of all special matchings of x. If M and N are elements of SM_x, then the set $\langle M, N\rangle$ is the dihedral subgroup of the symmetric group generated by all elements of SM_x. For any $z \in P$, the set $\langle M, N\rangle(z)$ is the orbit of z under the action of the dihedral group $\langle M, N\rangle$.

The motivation in this dissertation is to tackle on groups generated by more than two special matchings, on Bruhat intervals in W, and to extend the work in [11] especially Theorem 3.2.5 for some Weyl groups of types B, D, and E.

1.2 Problem statement

Let W be a Coxeter group of type A, or be a right-angled Coxeter group. Let also $\Gamma(u, v)$ be the undirected Bruhat graph of the Bruhat interval $[u, v]$ in W. It is known that if W is of type A, or a right-angled Coxeter group, then for any $u, v \in W$ with $u \leq v$, we have that a perfect matching of the interval $[u, v]$ is a special matching if and only if it is an automorphism on $\Gamma(u, v)$. However, it is not yet known if the same result holds for Coxeter groups of types B, D, H, F, and exceptional types. In this dissertation, we prove if the same result holds for Coxeter groups of types B, D and E. We also study some groups generated by some special matchings in W.

1.3 Objectives

1.3.1 Main objective

The main objective is to describe some perfect matchings that are special matchings in Coxeter groups of types B, D, and E; and to describe some groups generated by some special matchings on Bruhat intervals in Weyl groups.

1.3.2 Specific objectives

(i) The first specific objective is to determine some groups generated by some special matchings in the Bruhat intervals of a Weyl group.

(ii) The second specific objective is to describe and analyze perfect matchings that can be both special matchings and automorphisms of the intervals in types B, D, and E.

1.4 Methodologies

(i) The first specific objective has been achieved by using the theories from Lemma 3.1.1 and Lemma 2.2.23.

(ii) The second specific objective which is Proposition 3.2.10 has been achieved by using Theorem 3.2.5, Proposition 3.2.8, together with Proposition 3.2.9. The verification of this result is done by using the SageMath Algorithm in Appendix A where the Weyl group W is replaced by B_2, up to B_7, or D_4, or E_6.

1.5 Structure of the thesis

This thesis is organized as follow.

Chapter 1 contains the background, problem statement, objectives, and methodologies of the thesis. Chapter 2 contains the notations of posets, automorphisms, Bruhat order and Bruhat graphs, Coxeter groups of types B and D. Chapter 3 contains the main results.

Chapter 2

Preliminaries

In this chapter, some definitions, properties and discussions on posets, and Coxeter groups are recalled. Special matchings, posets automorphisms, Bruhat order and Bruhat graphs are summarized in order to be used in the results of this thesis.

2.1 Partially ordered sets

In this section, some literatures on partially ordered sets that are used in this thesis are presented. For more about partially ordered sets, see [17].

Definition 2.1.1. *A* partially ordered set *or (*poset*) is a set P, together with a binary operation "$\leq$" such that for all $a, b, c \in P$:*

(1) $a \leq a$ (reflexive),

(2) If $a \leq b$ and $b \leq a$ then $a = b$ (antisymmetric),

(3) If $a \leq b$ and $b \leq c$ then $a \leq c$ (transitive).

Definition 2.1.2. *Let $a < b$ in P. If there is no $c \in P$ such that $a < c < b$, we say that a is* covered *by b, and write $a \lhd b$.*

Definition 2.1.3. *The* Hasse diagram *of a poset P is the directed graph having P as a set of vertices and the cover relation as edge set, drawn in such a way that if $a \lhd b$, then a is below b.*

Definition 2.1.4. *A* maximum *$\hat{1} \in P$ is a unique element that satisfies $a \leq \hat{1}$ for all $a \in P$. Also, a* minimum *$\hat{0} \in P$ is a unique element that satisfies $\hat{0} \leq a$ for all $a \in P$. By an* induced subposet *of a poset P, we mean an ordered subset $R \subseteq P$ so that for all $a, b \in R$, $a \leq b$ in R if and only if $a \leq b$ in P. The subposet of P induced by $[a, b] := \{t \in P | \, a \leq t \leq b\}$ is called a (*closed*)* interval.

An element $a \in P$ is *minimal* if there is no point $b \in P$ such that $b < a$. An element $c \in P$ is said to be a *maximal* if there is no point $d \in P$ with $d > c$.

Definition 2.1.5. *An induced subposet I of P satisfying the property that, for each $t \in I$, all elements a below t (i.e., $a \leq t$) are also in I, is called an* order ideal*. An order ideal(resp., an order filter) of P is a subposet $Q \subseteq P$ with the property that if $x, y \in Q$, and $y \leq x$ (resp.,$y > x$) then $y \in Q$.*

Definition 2.1.6. *Let P_1 and P_2 be posets. A function $\phi : P_1 \to P_2$ is called an* order-preserving map *if for all $a_1, a_2 \in P_1$ with $a_1 \leq a_2$ in P_1 it holds that $\phi(a_1) \leq \phi(a_2)$ in P_2.*

A bijective order-preserving map $\phi : P_1 \to P_2$ whose inverse $\phi^{-1} : P_2 \to P_1$ is order-preserving is called an *isomorphism* of posets. An *automorphism* of a poset P is an isomorphism from P to itself.

2.2 Coxeter groups

In this section, some definitions and properties in the theory of Coxeter groups are recalled. They are used in the main results of the thesis. For more on Coxeter groups, we recommend the reader to consult [2] and [13].

Definition 2.2.1. *By a* Coxeter system*, we mean a pair (W, S) where W is a group and S is a set of simple generators in W subject to the conditions that for all $s \in S$, $s^2 = e$; for any $s \neq s' \in S$, $(ss')^m(s, s') = (s's)^m(s, s') = e$ where*

$$m(s, s') = m(s', s) \in \{1, 2, \cdots \infty\}.$$

If there is no condition between s and s' in S, it means that $m(s, s') = \infty$. Every $w \in W$ is a product of simple generators $s_i \in S$, i.e., $w = s_1 s_2 \cdots s_k$. If k is minimal in all expressions for w, we call such k the *length* of w, and we denote it as $\ell(w) = k$.

Definition 2.2.2. *A* Weyl group *is a finite Coxeter group for which $m(s, s') \in \{2, 3, 4, 6\}$ for all $s, s' \in S$.*

Definition 2.2.3. *The Coxeter system (W, S) is said to be* simply laced *if $m(s, s') \leq 3$. for all $s, s' \in S$; otherwise it is said to be* multiply laced.

Example 2.2.4. *The symmetric group S_n on the set $[n] = \{1, 2, \cdots, n\}$ is a Coxeter group whose simple generators are the simple transpositions $s_i = (i, i + 1)$ for all $1 \leq i \leq n - 1$. This is a finite simply laced Coxeter group.*

Definition 2.2.5. *A Coxeter system (W, S) is* right angled *system if $m(s, s') = \infty$ or $m(s, s') = 2$ for all $s \neq s' \in S$.*

Let W be a finite Coxeter group. There is an element $w_0 \in W$, called *maximal or longest* element, such that $w_0^2 = e$ (i.e., w_0 is an involution), and satisfies $\ell(w) < \ell(w_0)$ for all $w \in W$. In fact, w_0 is a unique element in W for which $\ell(sw_0) < \ell(w_0)$ for all $s \in S$. For example, the longest element in S_n is the *reverse permutation.*

Throughout the thesis, when we write W, we mean a Coxeter group with its set of simple generators S.

Propositions 2.2.6 and 2.2.7 below, are fundamental properties in Coxter groups.

Proposition 2.2.6. *. (Deletion Property). If $s_1 \cdots s_k$ is a non-reduced expression for w. Then there are indices $1 \leq i < j \leq k$ such that $s_1 \cdots \hat{s_i} \cdots \hat{s_j} \cdots s_k = w$ where the hats denote omission.*

Let $[k]$ the set defined by $[k] := \{1, 2, \cdots, k\}$.

Proposition 2.2.7. *(Exchange Property). If $s_i \cdots s_k$ is some expression for w and $\ell(w) > \ell(ws)$ for some $s \in S$, then $ws = s_1 \cdots \hat{s_i} \cdots s_k$ for some $i \in [k]$.*

We now present the Coxeter groups by their *Coxeter diagrams* (or *Coxeter graphs*). The *Coxeter graph* of W is the simple graph whose vertex set is S and whose edges are unordered pairs $\{s, s'\}$ if $m(s, s') \geq 3$. If $m(s, s') \geq 4$, then the edge $\{s, s'\}$ is labeled by that number; and the edge $\{s, s'\}$ has no label if $m(s, s') = 3$. In fact if s and s' commute, that is $m(s, s') = 2$, there is no edge between s and s'. An *irreducible* Coxeter group is one for which its graph is connected. The classification of finite irreducible Coxeter group has been done; see for example in [2]. In that classification we have:

(1) Three classical families of types A_n $(n \geq 1)$, B_n $(n \geq 2)$, D_n $(n \geq 4)$,

(2) Six exceptional groups of types E_6, E_7, E_8, F_4, H_3 and H_4,

(3) One family of dihedral groups of type $I_2(m)$, $m \geq 3$.

The Coxeter graphs of finite irreducible Coxeter groups are recorded in Figure 2.1. Note that $I_2(3) = A_2$ and $I_2(4) = B_2$.

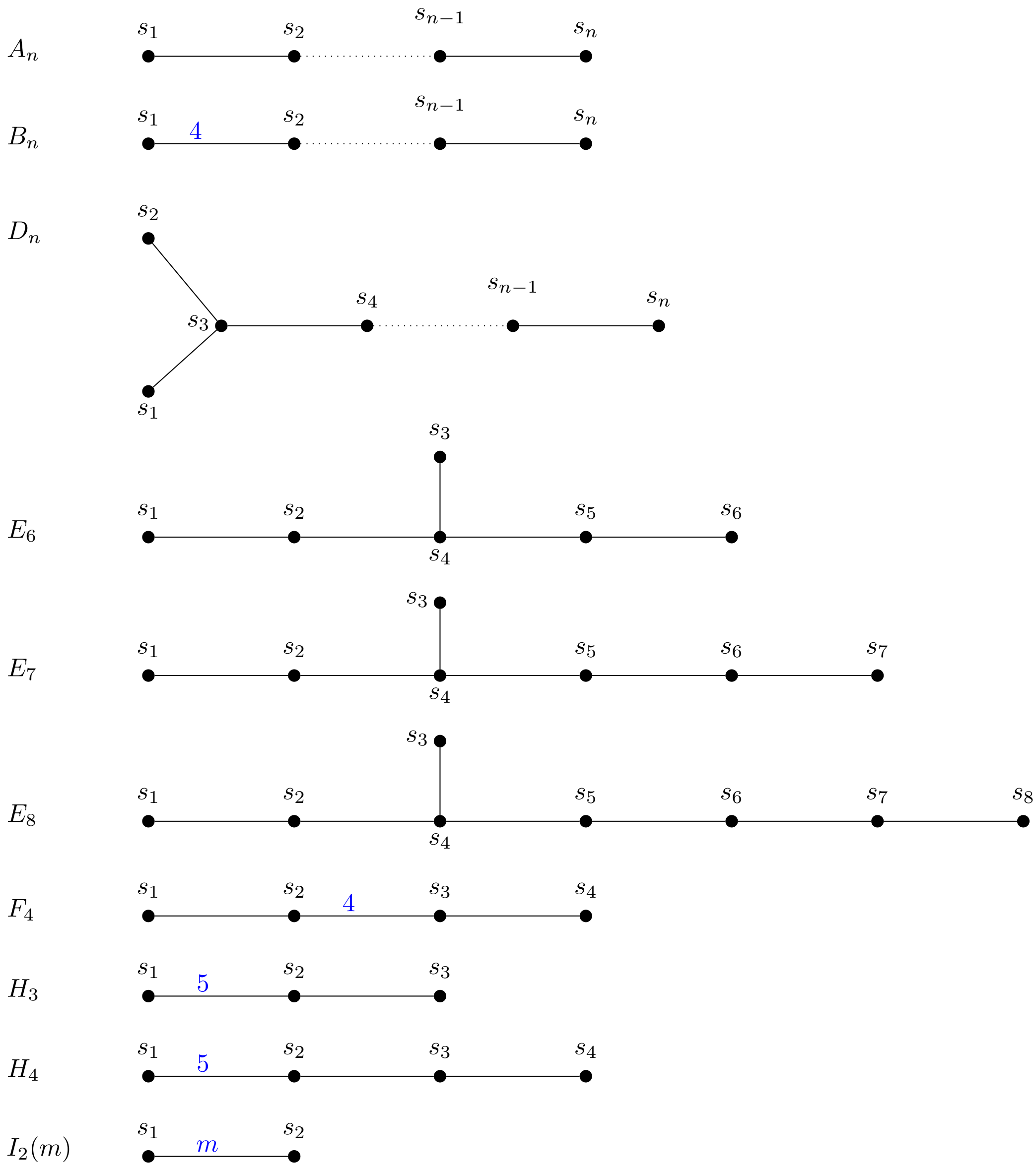

Figure 2.1: The finite, irreducible Coxeter groups.

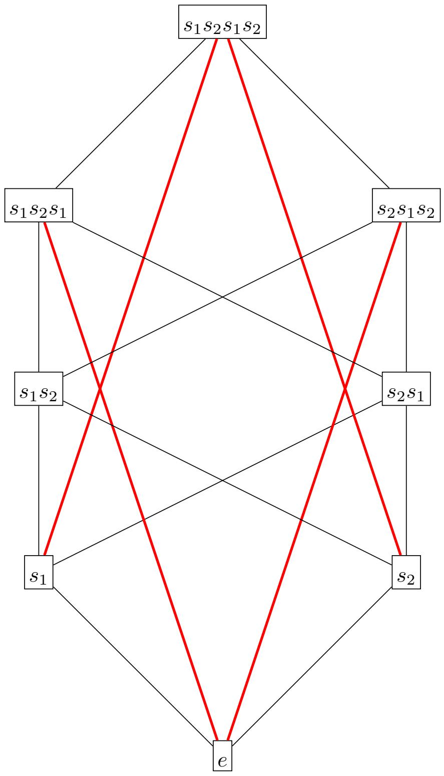

Figure 2.2: Bruhat order and Bruhat graph of B_3.

2.2.1 Bruhat graph and Bruhat order

In this subsection we recall definitions of the Bruhat order and Bruhat graph, and recall some of basic combinatorics properties of Coxeter groups which are useful in this thesis.

Let $T = \{wsw^{-1} : w \in W, s \in S\}$ be the set of *reflections* in W. For $v, w \in W$ define:

(i) $v \to w$ if $wb = bvt$ for some $t \in T$ with $\ell(v) < \ell(w)$.

(ii) $v \leq w$ if $v = v_0 \to v_1 \to \cdots \to v_m = w$ for some $v_i \in W$.

The *Bruhat graph* $\mathrm{Bg}_S(W)$ of (W, S) is the directed graph whose vertex set is W and whose edge set is $E_{\mathrm{Bg}WS} = \{(u, w) : u \to w\}$. The *Bruhat order* $\mathrm{Br}(W)$ is the partial order relation on W given by (ii).

The sets

$$D_L(w) := \{s \in S : \ell(sw) < \ell(w)\},$$

and

$$D_R(w) := \{s \in S : \ell(w)s < \ell(w)\}$$

are *left descent set* and *right descent set* of $w \in W$ respectively. For example in Figure 2.2, the Bruhat order of B_3 is represented by the poset with black edges while its Bruhat graph is the whole graph together with red edges.

Definition 2.2.8. *Let $u \leq w$ in* Br(W)*. Then $[u, w] := \{v \in W | u \leq v \leq w\}$ is called a* Bruhat interval*.*

Note that $\mathrm{Br}(W)$ is a poset for which the minimum is the identity element e. If W is finite, then $\mathrm{Br}(W)$ is a finite poset whose maximum and minimum are w_0 and e respectively.

Definition 2.2.9. *For any $w \in W$, the set $[e,w] := \{v \in W : e \leq v \leq w\}$ is called a* lower Bruhat interval.

Example 2.2.10. *For example, $[e, s_1s_3s_4] = \{e, s_1, s_3, s_4, s_1s_3, s_1s_4, s_3s_4, s_1s_3s_4\}$ is a lower Bruhat interval for $s_1s_3s_4 \in B_4$ where the Coxeter group B_4 is as in Figure 2.1.*

In Chapter 3, we consider undirected bruhat graphs (i.e., Bruhat graphs without arrows). For instance $\Gamma(u,v)$ denotes the undirected Bruhat graph of $\mathrm{Bg}_S(u,v)$ for an interval $[u,v]$ in W.

Example 2.2.11. *Take the symmetric group S_4. This group is the same as A_3. Here S_4 is generated by simple transpositions $s_i = (i, i+1)$ for all $1 \leq i \leq 3$. That is $s_1 = (1,2)$, $s_2 = (2,3)$, and $s_3 = (3,4)$ in cycle notations. In complete notation, $s_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}$, $s_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}$, and $s_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}$. Let us write s_1, s_2, and s_3 in one-line notations as $s_1 = 2134$, $s_2 = 1324$ and $s_3 = 1243$. Hence all reflections (i.e., all transpositions) are $T = \{(a,b) : 1 \leq a < b \leq 4\}$. That is $t_1 = (1,2)$, $t_2 = (2,3)$, $t_3 = (3,4)$, $t_4 = (1,3)$, $t_5 = (1,4)$, and $t_6 = (2,4)$. The Bruhat order and the Bruhat graph of S_4 are depicted in Figure 2.3 where the Bruhat order indicated by all vertices and the edges in black while the Bruhat graph is indicated by all vertices and edges.*

Theorem 2.2.12 is called *subword property.* It plays important roles in combinatorics and Coxeter groups like in the Bruhat order.

Theorem 2.2.12. *[2, 13] If (W,S) is a Coxeter system and $w = s_1s_2\cdots s_n$ is a reduced expression of $w \in W$, then $v \leq w$ if and only if $v = s_{j_1}s_{j_2}\cdots s_{i_k}$, $1 \leq j_1 < j_2 < \cdots \leq j_k \leq n$.*

The following lemma is called a *Lifting property.* It is very used in combinatorics for several purposes.

Lemma 2.2.13. *[7] Let (W,S) be a Coxeter system and $u < v$ in W. If $s \in D_R(v) \setminus D_R(u)$ then $u \leq uv$ and $us \leq v$.*

If W' is a subgroup W such that $W' = \langle W' \cap T \rangle$, then W' is called a *reflection* subgroup. A subgroup W' of W is called *dihedral* if $W' = \langle t, t' \rangle$ for some $t \neq t' \in T$.

Let $w \in W$ and define $N(w) := \{t \in T : \ell(wt) < \ell(w)$. In [9], any reflection subgroup W' of W has a generating set given by $\mathcal{R}(W') := \{t \in T : N(t) \cap W' = \{t\}\}$.

Proposition 2.2.14. *[10] If $I = [u,v]$ is an interval in W and $\ell(v) - \ell(u) = m$ for a positive integer m, then $W' = \langle xy^{-1} : x, y \in I \rangle$ is a reflection subgroup of W generated by $\mathcal{R}(W')$ where $|\mathcal{R}(W')| \leq m$. In addition, I is isomorphic to some Bruhat interval in W'.*

From Proposition 2.2.14, the following corollary follows.

Corollary 2.2.15. *Let W be finite. Then for every positive integer m there exist only finitely many isomorphism types of Bruhat intervals of fixed length m.*

From Corollary 2.2.15, it is clear that if W is a Weyl group, then W has finitely many intervals. Thus in section 3.2, we are calculating all intervals satisfying Conjecture 3.2.6 for some Weylgroups.

Proposition 2.2.16. *[10] If $t, t', t'', t''' \in T$ such that $tt' = t''t''' \neq e$, then $W' = \langle t, t', t'', t''' \rangle$ is a dihedral reflection subgroup of W.*

The above Proposition 2.2.16 has been generalised as Theorem 2.2.17 from [5] shows.

Theorem 2.2.17. *If W is a Coxeter group, and $t_1, \cdots, t_{2m} \in T$ are such that $t_1t_2 = t_3t_4 = \cdots = t_{2m-1}t_{2m} \neq e$, then*

$$W := \langle t_1, \cdots, t_{2m} \rangle$$

is a dihedral reflection subgroup.

Theorem 2.2.17 is used to show that any two elements in a Bruhat order of W can not cover (or be covered) by three elements or more.

Theorem 2.2.18. *[5] If (W,S) is a Coxeter system and $u, w \in W$ are such that $|\{x \in W : x \lhd u, x \lhd w\}| \geq 3$ or $|\{x \in W : u \lhd x, w \lhd x\}| \geq 3$ then $u = w$.*

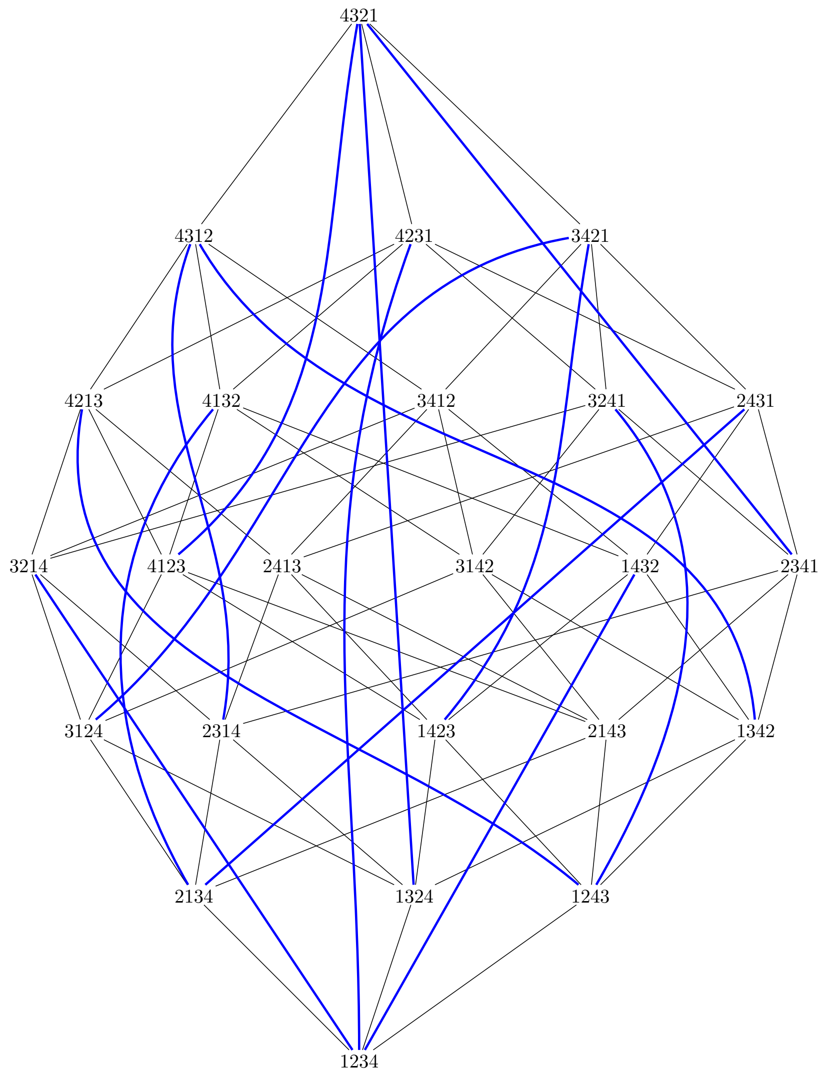

Figure 2.3: Undirected Bruhat order and Bruhat graph of S_4

2.2.2 Special matching and zircons

In this subsection, we recall the definitions of special matching of a poset and automorphisms of Bruhat graphs, and some of their properties that are employed in main results.

Definition 2.2.19. *A map M from a set Ω to itself is called* involution *if $M(M(x)) = x$ for all $x \in \Omega$.*

A *perfect matching* of a finite simple graph G (i.e, a finite graph without loops and multiple edges) with a set of vertices E_G, is an involution $M : G \to G$ such that for all vertices $x, y \in G$ where (x, y) is an edge and $M(x) = y$, the pair $\{x, M(x)\}$ is unique. This means that the set of edges $\{x, M(x)\}$ is unique for all vertices $x \in G$. For example the dotted edges in graph G depicted in Figure 2.4 form a perfect matching while the blue edges do not. The blue edges do not form a perfect matching because there is no blue edge between d and h (i.e, d and h are *unmatched*).

Definition 2.2.20. *Let M be a perfect matching of P. Then, M is called* special *if for all $a, b \in P$ with $a \lhd b$, either $M(a) = b$ or $M(a) < M(b)$.*

For $a \in P$, let $P_{\leq a} := \{q \in P | \, q \leq a\}$. The following proposition shows that every special matching of P restricts to $P_{\leq a}$ for a a none minimum in P.

Proposition 2.2.21 ([3]). *Let M be a special matching of a poset P, and $M(a) \lhd a$ for some $a \in P$. Then M restricts to a special matching of $P_{\leq a}$.*

If a poset is *Eulerian*, a special matching is equivalent to a *compression labelling* as was independently invented by du Cloux [8]. For the definition of Eulerian poset and its properties, see [17]. In particular, Bruhat orders are examples of Eulerian posets.

Definition 2.2.22. *A poset $P$is bounded if it has unique top and bottom elements, denoted $\hat{1}$ and $\hat{0}$ respectively.If P is bounded, then its proper part is $\bar{P} = P \backslash \{\hat{0}, \hat{1}\}$.*

Lemma 2.2.23 is called the *Lifting property for special matchings.* It appeared first in [3] where the author stated it by assuming that the poset is graded. There is a proof of it without this assumption that appears in [12].

If P is a poset in which every interval is finite, then P is called *locally finite.*

Lemma 2.2.23. *Let M be a special matching of a locally finite poset P. Let also $y, z \in P$ be such that $y < z$ and $M(z) < z$. The following conditions are satisfied.*

(1) $M(y) \leq z$,

(2) $M(y) < y \Rightarrow M(y) < M(z)$.

The following proposition from [2] makes a relationship between special matchings and Bruhat orders through Lemma 2.2.13.

Proposition 2.2.24. *For any Coxeter system (W, S), and $u, v \in W$ such that $s \in D_R(v) \setminus D_R(u)$. If $M(w) = ws$ for all $w \in [u, v]$ then M is a special matching of $[u, v]$.*

In particular, every lower Bruhat interval $[e, w]$ has a special matching $M : [e, w] \to [e, w]$ given by $M(x) = xs$ for every $x \in [e, w]$, and every simple generator $s \in S$ where also $s \in [e, w]$.

Note that there is a left version of Proposition 2.2.24 above.

Remark 2.2.25. *The converse of Proposition 2.2.24 is not true.*

Example 2.2.26. *From Figure 2.5, observe that the red edge set forms a special matching of* $\mathrm{Br}(A_2)$ *which is neither a right nor a left multiplication by a simple reflection.*

Note that special matchings have several important applications in combinatorics, like in the computation of Kazhdan-Lusztig polynomials and R-polynomials, but in this thesis, we are not going in the study of those polynomials.

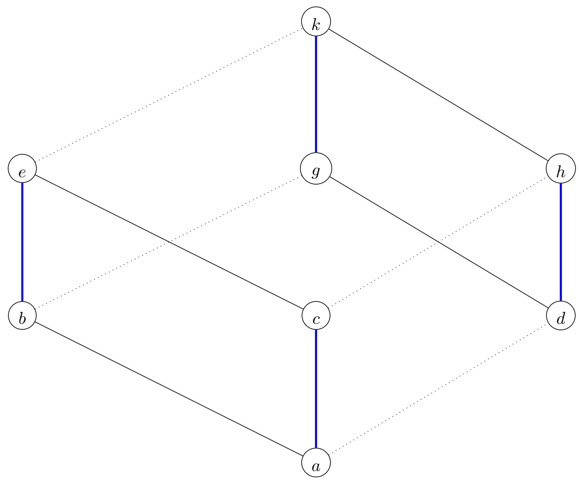

Figure 2.4: A perfect matching in a graph G.

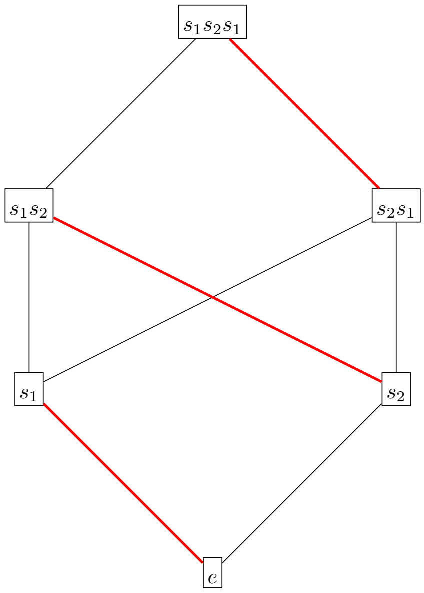

Figure 2.5: Bruhat order of A_2.

Definition 2.2.27. *A zircon is a poset P, such that for every non minimal element y the principal order ideal $p \leq y$ is finite and has a special matching.*

Example 2.2.28. *Consider Figure 3.2.11. It represents a zircon Z where the dotted lines indicate some special matching M. Observe in Z that for any non-minimal element $z \in Z$, Z_z is finite and admits a special matching.*

A *dual* of a poset P is a poset P' for which $x \leq y$ in P if and only if $y \leq x$ in P'. A poset P is *self dual* if P is isomophic to its dual.

Note that if Z is a zircon which is a self dual poset, then its dual Z' is clearly a zircon. However, the dual of a zircon need not be a zircon. To see it, take the affine Coxeter group $\tilde{A}_1$ generated by s_1 and s_2 where $m(s_1, s_2) = \infty$. Then $\mathrm{Br}(\tilde{A}_1)$ is a zircon. It has a minimum e, but no maximum. Let P be the dual of $\mathrm{Br}(\tilde{A}_1)$. Then P has a maximum e but no minimum. Take $s_1 \in P$. Then the principal order ideal $P_{\leq s_1}$ in P has a special matching but no minimum. Hence P is not a zircon.

It is known that if W is a finite Coxeter group, the translation by the longest element w_0 given by $w \to w_0 w w_0$ induces the automorphism of the Bruhat order. This translation is an inner automorphism of W, and induces an automorphism of a Coxeter graph.

The automorphisms of all Bruhat orders for irreducible Coxeter groups have been described. Theorem 2.2.29 counts those Bruhat order automorphisms in case W has a rank at least 3.

Theorem 2.2.29. *Let (W, S) be irreducibe and $|S| \geq 3$. Let also θ be an automorphism of* $\mathrm{Br}(W)$ *where $\theta(s) = s$ for all $s \in S$. Then either $\theta(w) = w^{-1}$ or $\theta(w) = w$ for every $w \in W$.*

From the above theorem, Corollary 2.2.30 follows.

Corollary 2.2.30. *Let (W, S) be irreducibe and $|S| \geq 3$. Then the automorphism group of* $\mathrm{Br}(W)$ *is generated by the map $w \to w^{-1}$ and the diagram automorphisms of W.*

Note that if the rank of W is 2 (i.e., if W is a dihedral group) then the automorphisms of $\mathrm{Br}(W)$ are not necessarily described in Theorem 2.2.29. For example, the dihedral group $I_2(m)$ has a Bruhat order whose group automorphism is isomorphic to $\mathbb{Z}_2^{m-1}$.

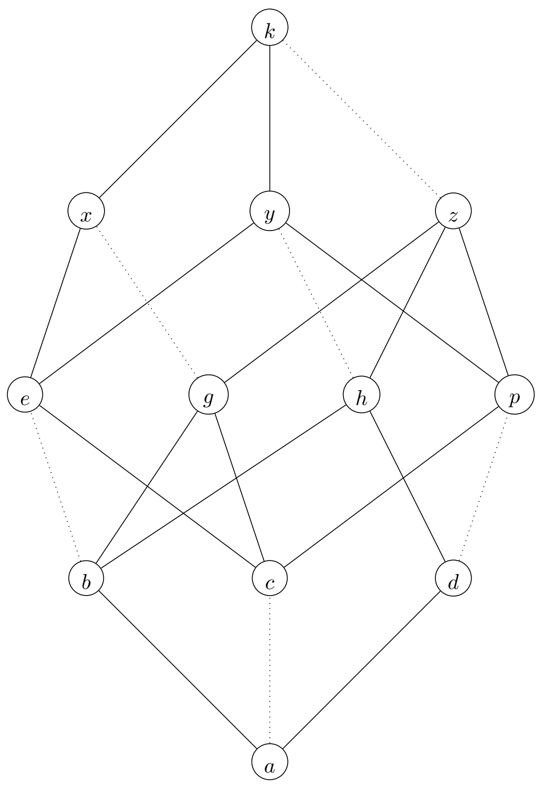

Figure 2.6: Zircon Z with a special matching

Chapter 3

Main results

In this chapter, we focus on the main results of the thesis. We construct groups that are generated by some special matching from Coxeter groups of type A. We also study and analyse perfect matchings that can be both special matchings and automorphisms of undirected intervals in types B, D, and E

3.1 Groups generated by some special matchings of Bruhat order in type A

Let P be a poset, and let M and M' be special matchings of P. By $\langle M, M' \rangle$, we mean the permutation group of P generated by M and N. For example, in the poset P whose Hasse diagram is Figure 3.1, and whose special matchings M and N are marked by red and blue colors respectively, that is $\langle M, N \rangle = \{id, M, N, MN = NM\}$. Here $MN = NM$ for all $x \in P$ because it is clear that $MN(x) = NM(x)$.

To achieve the specific objective (1) we extend $[Lemma 4.1, 6]$ which is stated below.

Lemma 3.1.1. *Let M and N be two special matchings of a finite graded poset P. Then $\langle M, N \rangle$ is a dihedral group.*

Definition 3.1.2. *A special matching M of* $\mathrm{Br}(W)$ *given by $M(x) = xs$ for all $x \in W$ and all $s \in S$ is called* initial special matching.

We state the following Corollary for a Weylgroup of type A and is valid for all finite Coxeter groups.

Corollary 3.1.3. *If (W, S) is a Coxeter system of type A, and M_1, M_2 are initial special matchings of* $\mathrm{Br}(W)$ *given by $M_1(x) = xs_1$ and $M_2(x) = xs_2$ respectively, then $\langle M_1, M_2 \rangle$ is a dihedral group.*

Proof. Let us first compute the group $\langle M_1, M_2 \rangle$. Since $M_1(x) = xs_1$, and $M_2(x) = xs_2$ then $M_1^2(x) = x = M_2^2(x)$. Thus $M_1^2 = M_2^2$. Also $M_1M_2(x) = xs_2s_1$, but $M_2M_1(x) = xs_1s_2$. Since $s_1s_2 \neq s_2s_1$, then $M_1M_2 \neq M_2M_1$. However, $M_1M_2M_1 = M_2M_1M_2$ because for all $x \in W$ we have $M_1M_2M_1(x) = xs_1s_2s_1 = xs_2s_1s_2 = M_2M_1M_2$ since $m(s_1, s_2) = m(s_2, s_1) = 3$. Hence $\langle M_1, M_2 \rangle = \{id, M_1, M_2, M_1M_2, M_2M_1, M_1M_2M_1\}$.

It remains to show that $\langle M_1, M_2 \rangle$ is a dihedral group. To see it, observe that $M_1^2 = M_2^2 = id$; and $(M_1M_2)^3(x) = (M_2M_1)^3(x) = x$ for all $x \in W$. Then $\langle M_1, M_2 \rangle$ has a representation of the form $\langle M_1, M_2 : M_1^2 = M_2^2 = id, (M_1M_2)^3 = (M_2M_1)^3 \rangle$. Thus $\langle M_1, M_2 \rangle$ is dihedral. □

In this section, we construct groups generated by initial special matchings in type A. With the established construction, one can do further construction for other types of irreducible finite Coxeter groups. The poset of the constructed group generated by initial special matchings turns to be isomorphic to the Bruhat order of a considered Coxeter group.

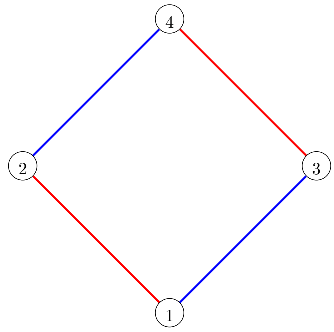

Figure 3.1: Poset P.

Proposition 3.1.4. *Let $G = \langle M_1, M_2, \cdots M_n \rangle$ where M_i are initial special matchings on the Bruhat order of A_n. Then G has the same order as A_n.*

In addition, G is not a reflection group.

Proof. Order the simple reflections $s_i \in S$ as $s_1 < s_2 < s_3, \cdots < s_n$ and let $M_i(x) = xs_i$ for all $x \in A_n$. The fact that G has the same order as A_n follows from Definition 3.1.2 and Theorem2.2.12. We now show that G is not a reflection group. Every initial special matching $M_i \in G$ can be identified as a permutation of $S_{(n+1)!}$ in the following way. Since $M_i(x) = xs_i$ for all $x \in A_n$ where A_n is identified as S_{n+1} and since S_{n+1} has $(n+1)!$ elements, then M_i can be defined as the permutation $M_i := \begin{pmatrix} x_1 & x_2 \cdots x_{(n+1)!} \\ x_1 s_i & x_2 s_i \cdots x_{(n+1)! s_i} \end{pmatrix}$.

But M_i is not a transposition (i.e; a reflection) because if $M_i(x_j) = x_j s_i$ and $M_i(x_j s_i) = x_j$ then other elements $x^* \neq x_j$ in $S_{(n+1)!}$ need not be fixed by M_i. Thus M_i is not a transposition. Since M_i was arbitrary, it follows that every M_i is not a transposition and hence not a reflection. Hence G is not generated by reflections. Thus G is not a reflection subgroup of $S_{(n+1)!}$; and hence not a reflection group. □

Example 3.1.5. *If W is of type A_3 and M_1, M_2, M_3 are initial special matchings of $Br(A_3)$ given by $M_1(x) = xs_1$, $M_2(x) = xs_2$, and $M_3(x) = xs_3$ for all $x \in A_3$ then by Proposition3.1.4 the group $\langle M_1, M_2, M_3 \rangle$ has the same cardnality as A_3. All elements of $\langle M_1, M_2, M_3 \rangle$ are id, M_1, M_2, M_3, M_2M_3, M_3M_2, M_1M_3, M_1M_2, M_2M_1, $M_1M_2M_3$, $M_2M_3M_2$, $M_2M_1M_3$, $M_3M_2M_1$, $M_1M_3M_2$, $M_1M_2M_1$, $M_1M_2M_3M_2$, $M_1M_2M_3M_1$, $M_2M_3M_1M_2$, $M_2M_3M_2M_1$, $M_3M_1M_2M_1$, $M_1M_2M_3M_1M_2$, $M_1M_2M_3M_2M_1$, $M_2M_3M_1M_2M_1$, $M_1M_2M_3M_1M_2M_1$.*

Corollary 3.1.6. *Let W be of type A_n and order the elements of S by $s_1 < s_2 < \cdots < s_n$. If every initial special matching of $Br(A_n)$ is $M_i(x) = xs_i$ for all $x \in A_n$, then the dihedral groups $\langle M_1, M_2 \rangle$, $\langle M_2, M_3 \rangle$, $\cdots$ $\langle M_i, M_{i+1} \rangle$, $\cdots$, $\langle M_{n-1}, M_n \rangle$ are pairwise isomorphic.*

Proof. The map $f : \langle M_{i-1}, M_i \rangle \to \langle M_i, M_{i+1} \rangle$ given by adding by 1 for every index of each M_j in the expression of a word in $\langle M_{i-1}, M_i \rangle$ for all $i - 1 \leq j \leq i$ is a well-defined map and a homomorphism. For example, $f(M_{i-1}M_iM_{i-1} = M_iM_{i+1}M_i)$. Observe that if $M_{j_1} = M_{j_2}$ belongs in $\langle M_{i-1}, M_i \rangle$ then $f(M_{j_1}) = M_{j_1+1} = M_{j_2+1} = f(M_{j_2})$. Also , if M_{j_1} and M_{j_2} are elements of $\langle M_{i-1}, M_i \rangle$ that are not necessarily equal, then $f(M_{j_1}M_{j_2}) = M_{j_1} + 1M_{j_2} + 1 = f(M_{j_1})f(M_{j_2})$. The homomorphism f is one-to-one since the kernel of f is $kerf = \{id_{\langle M_{i-1}, M_i \rangle}\}$. That is $id_{\langle M_{i-1}, M_i \rangle}$ is the only element such that $f(id_{\langle M_{i-1}, M_i \rangle}) = id_{\langle M_i, M_{i+1} \rangle}$. The homomorphism f is surjective since $\langle M_{i-1}, M_i \rangle$ and $\langle M_i, M_{i+1} \rangle$ have the same cardinality. Thus $\langle M_{i-1}, M_i \rangle$ is isomorphic to $\langle M_i, M_{i+1} \rangle$. □

Corollary 3.1.7. *The subgroup $\langle M_i, M_j \rangle$ of $\langle M_1, M_2, \cdots, M_n \rangle$ is given by $\{id, M_i, M_j, M_iM_j\}$ if and only if $|i - j| \geq 1$ for all $1 \leq i, j \leq n$.*

Proof. Since $\langle M_i, M_j\rangle$ has order 4 and $M_i \neq M_j$ then $M_iM_j = M_jM_i$. To see it, if M_iM_j would be different to M_jM_i, then $M_iM_jM_j \neq M_jM_iM_j$. But $M_jM_j = id$; then $M_i = M_iM_jM_j \neq M_jM_iM_j$ That is $M_jM_iM_j \neq M_i$. Similarly $M_jM_iM_j \neq M_j$. Since $M_jM_i \neq id$ for $j \neq i$. Then we have that $M_jM_iM_j(x) \neq x$ for all $x \in \langle M_i, M_j\rangle$. Hence $M_jM_iM_j$ is an extra element in $\langle M_i, M_j\rangle$ which is contradiction. Thus $M_iM_j = M_jM_i$, and this happens if $|M_i - M_j| > 1$. For the only if part, assume that $|i-j| > 1$. Then $s_is_j = s_js_i$ where $s_i, s_j \in S$. Hence $M_iM_j(y) = ys_js_i = ys_is_j = M_jM_i(y)$ for all $y \in A_n$. Hence $M_iM_j = M_jM_i$. It follows that $\langle M_i, M_j\rangle = \{id, M_i, M_j, M_iM_j\}$. □

We close this section by the following main result in Theorem 3.1.8.

Theorem 3.1.8. *Let (W, S) be a Coxeter system where W is a Weyl group with rank $|S| = n$. Let also M_i be an initial special matching given by $M_i(x) = xs_i$ where $s_1 < s_2 < \cdots < s_n \in S$. Then $\langle M_1, M_2, \cdots M_n\rangle$ has the same order as W. In addition $\langle M_1, \cdots M_n\rangle$ is isomorphic to a subgroup of $S_{|W|}$.*

Proof. For type A, Proposition 3.1.4 completes the proof. For other types, the theorem follows from the fact that if w_o is a longest element in W, then the element in $\langle M_1, M_2, \cdots M_n\rangle$ with the longest reduced expression is determined by the product of all initial special matchings indexed by same indexes as the indexes in w_o. □

As a remark, the group $\langle M_1 \cdots M_n\rangle$ possesses a poset which may produce an important class of Kazhdan -Lusztig polynomials. However in this dissertation, we are not studying the concept of the Kazdan-Lusztig polynomials.

3.2 Special matchings and automorphisms in types B, D, and E

Remember that for any set X, and any group Q, an orbit Q of $x \in X$ is a subset of X defined by $Q \cdot x = \{g \cdot x : g \in Q\}$. Let $\langle M, M'\rangle(x)$ be an orbit of $x \in P$ under the action of $\langle M, M'\rangle$.

Then, Definition 3.2.1 follows.

Definition 3.2.1. *Let $y \in P$ where P is a zircon, and let M, and M' be special matchings of P. Then M and M' are said to be* strictly coherent *if the order of $\langle M, M'\rangle(x)$ divides the order of $\langle M, M'\rangle(y)$ for all $x \leq y$ in P.*

Definition 3.2.2. *For a zircon P, we say that two special matchings M and M' are* coherent *if there exists a sequence of special matchings $M_0 = M_1, M_2, \cdots, M_j = M'$ such that every M_i and M_{i+1} are strictly coherent for all $i \in \{0, 1, \cdots, j-1\}$.*

Definition 3.2.3. *Let Z be a zircon. Then Z is called a* diamond *if for every non-minimal element $y \in Z$, any two special matchings M and M' of $Z_{\leq y}$ are coherent.*

Note that a zircon need not be a diamond; and a dual of a diamond need not be a diamond. Also the good examples of diamonds are all Bruhat orders of Coxeter groups. See many examples in [4] about diamonds and their properties.

Definition 3.2.4. *For any Coxeter W group and its associated Bruhat order $B_r(W)$, the elements $x, y, z, t \in W$ are said to form a* butterfly *if $x \lhd z$ and $y \lhd t$, then $x \lhd t$ and $y \lhd z$.*

For example the elements s_1s_2, s_2s_1, $s_1s_2s_1$, $s_2s_1s_2$ in B_2 form a butterfly, where B_3 is generated by simple generators s_1 and s_2 such that $m(s_1, s_2) = 4$. See Figure 3.2. Note that every butterfly cannot be embedded in a Bruhat order of rank two with four elements.

Theorem 3.2.5 is due to Gaetz and Gao in [11]. We extend it to Coxeter groups of types B, D, and E in order to achieve the first specific objective of this work. In [11], it is known that if $u \leq v$ and $u_1, u_2, y_1, y_2 \in [u, v]$ form a butterfly in symmetric group or a right-angled Coxeter group, then there is an element $y \in [u, v]$ such that $y_1, y_2 \lhd y$.

Theorem 3.2.5. *Let W be simply laced type A or right-angled Coxeter group. Let also $u \leq w$ in W. Then a perfect matching of the Hasse diagram of $[u, w]$ is a special matching if and only if it is an automorphism of a Bruhat graph of $[u, v]$.*

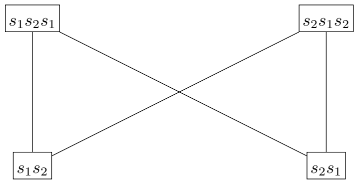

Figure 3.2: Butterfly in $B_r(B_3)$.

By *irregular interval* of a Bruhat interval I in a Coxeter group, we mean an interval whose undirected Bruhat graph is not regular. It is known that only type A_2, type G_2, and type B_2 butterflies exist in finite Weyl groups. In the same paper, it is also shown that in finite classical types only type B_2 and type A_2 butterflies exist. In this thesis, we have constructed an algorithm in SageMath that provides all Bruhat intervals together with Bruhat graphs in types B_2, B_3, B_4, B_5, B_6, D_4, D_5, and G_2 that are not isomorphic to a subgroup isomorphic to A_2, B_2, and G_2. All those intervals are irregular, and we have checked and have found the following proposition to be valid.

Conjecture 3.2.6. *[11]Let W be any Coxeter group, let $u \leq v \in W$, and suppose that the elements $x_1, x_2, y_1, y_2 \in [u, v]$ form a buttetrfly. Then there is an element $z \in [u, v]$ with $y_1, y_2 \lhd z$ or with $z \lhd x_1, x_2$.*

Let $I = [u, v]$ be an interval in a Coxeter group W; and suppose that the elements $y_1, y_2, z_1, z_2 \in I$ form a butterfly in I. Then , there exists $w \in I$ such that $w \lhd y_1, y_2$ or $z_1, z_2 \lhd w$

Though we do not have a complete proof of this conjecture for all Weyl groups; Proposition3.2.9 is a partial proof for $n \leq 10$. The butterfly presented in Example3.2.11 is a nice illustration of the above conjecture 3.2.6.

Definition 3.2.7. *If P is a poset with $\hat{0}, \hat{1} \in P$, every element $x \in P$ such that $\hat{0} \lhd x$ is called* atom*, while every element $y \in P$ such that $y \lhd \hat{1}$ is called* coatom*.*

Proposition 3.2.8. *Let $u < v$ in W, where W is of type B, or D or E. If $[u, v]$ is irregular and $\ell(v) - \ell(u) = 4$ such that $[u, v]$ contains a butterfly u_1, u_2, y_1, y_2 of type A or type B, then either u_1 and u_2 are atoms of u or y_1 and y_2 are coatoms of v.*

Proof. Since u_1, u_2, y_1, y_2 is a butterfly in $[u, v]$ and $\ell(v) - \ell(u) = 4$, then $u < u_1$, $u < u_2$, $y_1 < v$, and $y_2 < v$. If y_1 and y_2 are not coatoms of v, then $\ell(y_1) - \ell(u) < 3$ and $\ell(y_2) - \ell(u) < 3$. But since $u_1, u_2 \lhd y_1, y_2$ and $u < u_1$, $u < u_2$, it follows that u_1 and u_2 are atoms. Similarly if u_1 and u_2 are not atoms, it follows that y_1 and y_2 are coatoms. □

Since in the algorithm that we established all intervals $[u, v]$ containing butterflies satisfy $l(v) - l(u) \leq 4$,we now set Proposition 3.2.9 whose proof is a partial proof of Conjecture 6.7 in [11].

Proposition 3.2.9. *Let W be a Coxeter group of type B from B_2 up to B_7, or D_4, D_5, E_6. Let also $u \leq v$ in W. If $u_1, u_2, y_1, y_2 \in [u, v]$ form a butterfly in $[u, v]$. Then there is an element $w \in [u, v]$ such that either $y_1, y_2 \lhd w$ or $w \lhd u_1, u_2$.*

Proof. For types B, D, and E, See the algorithm in Appendix A together with Proposition 3.2.8. For type G_2, there are no irregular intervals of type A_2, A_4, and A_6. So G_2 satisfies Conjecture 6.7 in [11]. □

The analogue of the proof of Theorem 3.2.5, with Proposition 3.2.9, together with the proof of Proposition 6.4 in [11], we have the following proposition which is among the general results in this work.

Proposition 3.2.10. *Let W be a Coxeter group of type B from B_2 up to B_7, or D_4, D_5, E_6, or G_2. Let also $u \leq v$ in W. Then any perfect matching of the Hasse diagram of $[u, v]$ which is an automorphism of $\Gamma(u, v)$ is a special matching.*

Example 3.2.11. *The Figure 3.3 is a Bruhat order of interval $[s_3, s_3s_1s_2s_1s_3]$ in B_3. It contains a butterfly $s_2s_3, s_1s_3, s_1s_2s_3, s_2s_1s_3$ indicated by red edges. It is not hard to see that this interval is irregular since its Bruhat graph is not regular.*

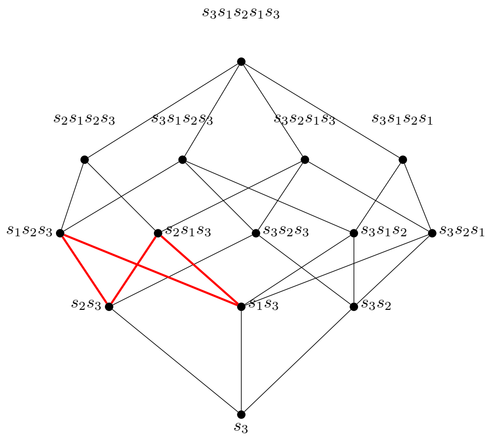

Figure 3.3: Irregular interval $[s_3, s_3s_1s_2s_1s_3]$ in B_3.

3.3 Conclusion, recommendation, and future work

3.3.1 Conclusion and recommendation

For the finite Weyl Coxeter group W of type A. We constructed a group which is generated by initial special martchings. That group has the same order as A_n and is not necessarily a reflection group. This result has been generalised to all finite irreducible Coxeter systems. The second result of this dissertation is based on the SageMath algorithm. That algorithm was constructed for providing irregular intervals that consist of butterflies, and irregular intervals that do not consist butterflies in types $B_n, n \leq 9$, $D_n, n \leq 6$, and E_6. As a recommendation, in future work, a leader can verify if Theorem 2.2.18 can be used to make a general proof of Proposition 3.2.10.

3.3.2 Discussion and future work

Since in the first main result we constructed groups that are only generated by initial special matchings, as a discussion, we hope that this work will be extended for groups generated by all special matchings in the Bruhat order of any Coxeter system. We even hope that the group constructed in that way will be used to study more on Kazhdan-Lusztig polynomials. For the second main result, we only showed that every automorphism of an undirected Bruhat graph of an interval in types $B_n, n \leq 9$, $D_n, n \leq 6$ and E_6 is a special matching of Bruhat order of that interval.

Question: Does the second main result hold true if W is E_7, or E_8? We need to increase and modify the algorithm to allow more calculations if the Coxeter group is to large.

In fiture work, we intend to increase our theory to every Weyl group.

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Appendices

In this appendix, we present an algorithm to complete the intervals with fixed length. Those interval are non-regular, By Proposition 2.2.14, Corolarly 2.2.15, and Proposition 3.2.8, those intervals are enough to prove Conjecture B for $B_n, 2 \leq n \leq 9, D_n, 4 \leq n \leq 6, E_6$.

Appendix A contains the algorithm that works for Weyl groups mentioned in the previous paragraph.

Appendix B contains some intervals that are irregular in types B and D.

Appendix A

Algorithm

```
W = WeylGroup(["put the type of  W", put the number indicating the rank of W], prefix="s")
[put the simple reflections] = W.simple_reflections()
ref = W.reflections().keys()
def elements_t_uv(u,v):
    bi = W.bruhat_interval(u,v)
    ret = []
    for r in ref:
        if all( r*x in bi for x in bi):
            ret.append(r)
    return ret
for v in W:
    for u in W.bruhat_interval(1,v):
        if (u!= v)&((v).length()-(u).length()==put the length of an interval either 2, or 3, or 6):
            if not (W.bruhat_graph(u,v).is_regular()):
                print([u,v])
```

Appendix B

Cases for B and D

B.1 Irregular interval of length 3

Here the interval such as $[s_2, s_2s_3s_1s_2]$ and other intervals are produced by the algorithm as $[s_2, s2*s3*s1*s2]$.

$[s_2, s2*s3*s1*s2]$, $[s1*s2*s3*s1, s2*s3*s1*s2*s3*s2*s1]$, $[s3*s1*s2*s3*s1, s3*s2*s3*s1*s2*s3*s2*s1]$, $[s3*s1*s2*s1, s3*s1*s2*s3*s1*s2*s1]$, $[s3*s1*s2, s3*s1*s2*s3*s1*s2]$, $[s3*s1, s3*s1*s2*s3*s1]$, $[s3*s1, s1*s2*s3*s2*s1]$, $[s2*s3*s1*s2, s2*s3*s1*s2*s3*s1*s2]$, $[s3*s2*s3*s2, s2*s3*s1*s2*s3*s1*s2]$, $[s2*s3*s1, s2*s3*s1*s2*s3*s1]$, $[s2*s3, s2*s3*s1*s2*s3]$, $[s3*s2, s3*s2*s3*s1*s2]$.

B.2 Irregular intervals of length 4

$[1, s2*s3*s1*s2]$, $[s2, s2*s3*s1*s2*s1]$, $[s3*s1*s2*s3*s1, s3*s2*s3*s1*s2*s3*s1*s2*s1]$, $[s3*s1*s2*s1, s2*s3*s1*s2*s3*s1*s2*s1]$, $[s2*s3*s1*s2, s2*s3*s1*s2*s3*s1*s2*s1]$, $[s1*s2*s3*s1, s2*s3*s1*s2*s3*s1*s2*s1]$, $[s3*s2*s3*s2, s2*s3*s1*s2*s3*s1*s2*s1]$, $[s2*s3*s1, s2*s3*s1*s2*s3*s2*s1]$, $[s1*s2*s1, s2*s3*s1*s2*s3*s2*s1]$, $[s1*s2*s3, s2*s3*s1*s2*s3*s2*s1]$, $[s2*s3, s2*s3*s1*s2*s3*s2]$, $[s3*s1*s2*s1, s3*s2*s3*s1*s2*s3*s2*s1]$, $[s1*s2*s3*s1, s3*s2*s3*s1*s2*s3*s2*s1]$, $[s3*s2*s3*s1, s3*s2*s3*s1*s2*s3*s2*s1]$, $[s3*s1*s2*s3, s3*s2*s3*s1*s2*s3*s2*s1]$, $[s1*s2*s1, s3*s1*s2*s3*s1*s2*s1]$, $[s3*s2*s1, s3*s1*s2*s3*s1*s2*s1]$, $[s3*s1*s2, s3*s1*s2*s3*s1*s2*s1]$, $[s3*s1, s3*s1*s2*s3*s2*s1]$, $[s3*s1*s2, s3*s2*s3*s1*s2*s3*s2]$, $[s1*s2, s3*s1*s2*s3*s1*s2]$, $[s3*s2, s3*s1*s2*s3*s1*s2]$, $[s3*s1, s3*s1*s2*s3*s1*s2]$, $[s1, s3*s1*s2*s3*s1]$, $[s3, s3*s1*s2*s3*s1]$, $[s1, s1*s2*s3*s2*s1]$, $[s3, s1*s2*s3*s2*s1]$, $[s2, s1*s2*s3*s1*s2]$, $[s3*s1, s1*s2*s3*s1*s2*s1]$, $[s2*s3*s1*s2, s3*s2*s3*s1*s2*s3*s1*s2]$, $[s3*s2*s3*s2, s3*s2*s3*s1*s2*s3*s1*s2]$, $[s3*s1*s2, s2*s3*s1*s2*s3*s1*s2]$, $[s1*s2*s1, s2*s3*s1*s2*s3*s1*s2]$, $[s2*s3*s2, s2*s3*s1*s2*s3*s1*s2]$, $[s2*s3*s1, s2*s3*s1*s2*s3*s1*s2]$, $[s3*s2*s3, s2*s3*s1*s2*s3*s1*s2]$, $[s3*s1, s2*s3*s1*s2*s3*s1]$, $[s2*s1, s2*s3*s1*s2*s3*s1]$, $[s2*s3, s2*s3*s1*s2*s3*s1]$, $[s3, s2*s3*s1*s2*s3]$, $[s2, s2*s3*s1*s2*s3]$, $[s2*s3*s1, s3*s2*s3*s1*s2*s3*s1]$, $[s3*s2, s3*s2*s3*s1*s2*s1]$, $[s2*s3, s3*s2*s3*s1*s2*s3]$, $[s3*s1, s3*s2*s3*s1*s2*s3]$, $[s3*s2, s3*s2*s3*s1*s2*s3]$, $[s2, s3*s2*s3*s1*s2]$, $[s3, s3*s2*s3*s1*s2]$.

B.3 Irregular intervals of length 3 in D_4

$[s4*s3*s1, s4*s2*s3*s1*s2*s4]$, $[s2*s4*s2, s4*s2*s3*s1*s2*s4]$, $[s2, s2*s4*s1*s2]$, $[s2, s2*s3*s1*s2]$, $[s4*s2*s3*s1*s2*s4*s2, s4*s2*s3*s1*s2*s4*s2*s3*s1*s2]$, $[s2, s2*s4*s3*s2]$, $[s1*s2*s4*s3*s1*s2*s1, s2*s3*s1*s2*s4*s2*s3*s1*s2*s1]$, $[s2*s4*s2*s3*s1, s4*s2*s3*s1*s2*s4*s2*s3]$, $[s1*s2*s4*s3*s1, s1*s2*s4*s1*s2*s3*s2*s1]$, $[s2*s4*s2*s3*s1, s4*s2*s3*s1*s2*s4*s2*s1]$, $[s2*s3*s2, s3*s2*s4*s3*s1*s2]$, $[s4*s3*s1, s3*s2*s4*s2*s3*s1]$, $[s2*s4*s2, s2*s4*s2*s3*s1*s2]$, $[s3*s1*s2*s4*s2, s4*s2*s3*s1*s2*s4*s3*s2]$, $[s4*s3*s1, s3*s1*s2*s4*s2*s3]$, $[s4*s3*s1, s4*s1*s2*s3*s2*s1]$, $[s4*s3*s1*s2*s1, s1*s2*s4*s2*s3*s1*s2*s1]$,

$[s3*s2*s4*s2*s3*s2*s1, s4*s2*s3*s1*s2*s4*s1*s2*s3*s1], [s1*s2*s1, s2*s4*s3*s1*s2*s1],$
$[s4*s1*s2*s3*s2, s3*s1*s2*s4*s1*s2*s3*s2], [s2*s4*s2, s2*s3*s1*s2*s4*s2],$
$[s2*s3*s1*s2*s4*s3*s1, s2*s3*s1*s2*s4*s1*s2*s3*s2*s1], [s4*s1*s2*s3*s2, s3*s2*$
$s4*s1*s2*s3*s1*s2], [s4*s3*s1, s1*s2*s4*s3*s2*s1], [s1*s2*s1, s1*s2*s4*s3*s2*s1],$
$[s4*s3*s1, s3*s2*s4*s1*s2*s3], [s2*s3*s2, s3*s2*s4*s1*s2*s3], [s3*s2*s4*s1*s2*s3*s2, s2*$
$s3*s1*s2*s4*s1*s2*s3*s1*s2], [s3*s1*s2*s4*s2*s3*s2, s4*s2*s3*s1*s2*s4*s1*s2*s3*s2],$
$[s3*s2*s4*s3*s1, s2*s3*s1*s2*s4*s1*s2*s3], [s3*s2*s4*s3*s1, s3*s2*s4*s1*s2*s3*s2*s1],$
$[s4*s1*s2*s3*s1*s2*s1, s3*s1*s2*s4*s1*s2*s3*s1*s2*s1], [s3*s1*s2*s4*s1*s2*$
$s1, s4*s2*s3*s1*s2*s4*s3*s1*s2*s1], [s1*s2*s4*s3*s1, s2*s3*s1*s2*s4*s3*s2*s1],$
$[s1*s2*s1, s1*s2*s4*s3*s1*s2], [s4*s3*s1, s1*s2*s4*s2*s3*s1], [s4*s3*s1, s3*s1*s2*s4*s3*s1],$
$[s4*s1*s2*s3*s2, s3*s1*s2*s4*s2*s3*s1*s2], [s4*s3*s1*s2*s1, s3*s1*s2*s4*s2*s3*s1*s2],$
$[s3*s1*s2*s4*s2, s3*s1*s2*s4*s2*s3*s1*s2], [s3*s1*s2*s4*s2, s4*s2*s3*s1*s2*s4*s1*s2],$
$[s4*s3*s1, s3*s1*s2*s4*s2*s1], [s1*s2*s4*s1*s2*s3*s1, s4*s2*s3*s1*s2*s4*s2*s3*s2*s1],$
$[s4*s3*s1*s2*s1, s3*s1*s2*s4*s3*s1*s2*s1], [s2*s3*s2, s2*s4*s1*s2*s3*s2],$
$[s2*s4*s2*s3*s1, s2*s3*s1*s2*s4*s2*s3*s1], [s1*s2*s4*s3*s1, s2*s3*s1*s2*s4*s2*s3*s1],$
$[s3*s2*s4*s3*s1, s2*s3*s1*s2*s4*s2*s3*s1], [s2*s3, s2*s4*s1*s2*s3], [s1*s2, s1*s2*s4*s3*s2],$
$[s2*s4*s2*s3*s1*s2, s2*s3*s1*s2*s4*s2*s3*s1*s2], [s1*s2*s4*s3*s1*s2, s2*s3*$
$s1*s2*s4*s2*s3*s1*s2], [s1*s2*s4*s1*s2*s1, s2*s3*s1*s2*s4*s2*s3*s1*s2],$
$[s3*s2*s4*s3*s1*s2, s2*s3*s1*s2*s4*s2*s3*s1*s2], [s3*s2*s4*s2*s3*s2, s2*s3*$
$s1*s2*s4*s2*s3*s1*s2], [s1*s2*s3*s1*s2*s1, s2*s3*s1*s2*s4*s2*s3*s1*s2],$
$[s2*s4*s1*s2*s3*s2, s2*s3*s1*s2*s4*s2*s3*s1*s2], [s2*s4*s3*s1*s2*s1, s2*s3*$
$s1*s2*s4*s2*s3*s1*s2], [s2*s3*s1*s2*s4*s2, s2*s3*s1*s2*s4*s2*s3*s1*s2],$
$[s2*s4*s2*s3, s2*s3*s1*s2*s4*s2*s3], [s3*s2*s4*s3, s2*s3*s1*s2*s4*s2*s3],$
$[s2*s4*s3*s1, s2*s3*s1*s2*s4*s2*s3], [s2*s3*s2*s1, s2*s4*s1*s2*s3*s2*s1],$
$[s1*s2*s3*s1, s2*s4*s1*s2*s3*s2*s1], [s2*s4*s3*s1, s2*s4*s1*s2*s3*s2*s1], [s3*s1*s2*$
$s4*s2*s1, s4*s2*s3*s1*s2*s4*s3*s2*s1], [s1*s2*s4*s1*s2*s1, s4*s2*s3*s1*s2*s4*s3*s2*s1],$
$[s1*s2*s4*s2*s3*s1, s4*s2*s3*s1*s2*s4*s3*s2*s1], [s4*s3*s1*s2, s3*s1*s2*s4*s3*s1*s2],$
$[s1*s2*s3*s2, s3*s1*s2*s4*s3*s1*s2], [s3*s1*s2*s1, s3*s1*s2*s4*s3*s1*s2],$
$[s4*s3*s1*s2, s1*s2*s4*s2*s3*s1*s2], [s4*s1*s2*s1, s1*s2*s4*s2*s3*s1*s2],$
$[s1*s2*s4*s2, s1*s2*s4*s2*s3*s1*s2], [s4*s1, s1*s2*s4*s2*s1], [s4*s3, s3*s2*s4*s2*s3],$
$[s3*s1, s1*s2*s3*s2*s1], [s3*s1*s2*s4*s2*s3*s2*s1, s4*s2*s3*s1*s2*s4*s1*s2*s3*s2*s1],$
$[s3*s1*s2*s4*s1*s2*s3*s1, s4*s2*s3*s1*s2*s4*s1*s2*s3*s2*s1], [s4*s2*s3*$
$s1*s2*s4*s3*s1, s4*s2*s3*s1*s2*s4*s1*s2*s3*s2*s1], [s3*s2, s3*s2*s4*s1*s2],$
$[s4*s2, s4*s2*s3*s1*s2], [s2*s1, s2*s4*s3*s2*s1], [s2*s4, s2*s3*s1*s2*s4], [s3*s1*s2*s4*$
$s2*s3, s4*s2*s3*s1*s2*s4*s1*s2*s3], [s3*s2*s4*s2*s3*s2, s4*s2*s3*s1*s2*s4*s1*s2*s3],$
$[s3*s2*s4*s2*s3*s1, s4*s2*s3*s1*s2*s4*s1*s2*s3], [s4*s1*s2*s3*s2*s1, s3*s1*$
$s2*s4*s1*s2*s3*s2*s1], [s1*s2*s3*s1*s2*s1, s3*s1*s2*s4*s1*s2*s3*s2*s1],$
$[s3*s1*s2*s4*s3*s1, s3*s1*s2*s4*s1*s2*s3*s2*s1], [s2*s4*s2*s1, s2*s3*s1*s2*s4*s2*s1],$
$[s1*s2*s4*s1, s2*s3*s1*s2*s4*s2*s1], [s2*s4*s3*s1, s2*s3*s1*s2*s4*s2*s1],$
$[s4*s3*s1*s2, s3*s2*s4*s2*s3*s1*s2], [s4*s2*s3*s2, s3*s2*s4*s2*s3*s1*s2],$
$[s3*s2*s4*s2, s3*s2*s4*s2*s3*s1*s2].$

B.4 Some irregular intervals of length 4 in D_4

$[s2*s4, s4*s2*s3*s1*s2*s4], [s4*s1, s4*s2*s3*s1*s2*s4], [s4*s3, s4*s2*s3*s1*s2*s4],$
$[s3*s1, s4*s2*s3*s1*s2*s4], [s4*s2, s4*s2*s3*s1*s2*s4], [s4*s1, s1*s2*s4*s1*s2*s1]$
$[s4*s3, s3*s2*s4*s2*s3*s2], [1, s2*s4*s1*s2], [s2*s1, s2*s4*s2*s3*s2*s1], [s4*s2, s4*s2*s3*s1*s2*s1],$
$[1, s2*s3*s1*s2], [s2*s4*s2*s3*s1*s2, s4*s2*s3*s1*s2*s4*s2*s3*s1*s2], [s1*s2*s4*s3*s1*$
$s2, s4*s2*s3*s1*s2*s4*s2*s3*s1*s2], [s1*s2*s4*s1*s2*s1, s4*s2*s3*s1*s2*s4*s2*s3*s1*s2],$
$[s3*s2*s4*s3*s1*s2, s4*s2*s3*s1*s2*s4*s2*s3*s1*s2], [s3*s2*s4*s2*s3*s2, s4*s2*$
$s3*s1*s2*s4*s2*s3*s1*s2], [s1*s2*s3*s1*s2*s1, s4*s2*s3*s1*s2*s4*s2*s3*s1*s2],$
$[s2*s4*s1*s2*s3*s2, s4*s2*s3*s1*s2*s4*s2*s3*s1*s2], [s2*s4*s3*s1*s2*s1, s4*s2*$
$s3*s1*s2*s4*s2*s3*s1*s2], [s2*s3*s1*s2*s4*s2, s4*s2*s3*s1*s2*s4*s2*s3*s1*s2],$

$[s4*s2*s3*s1*s2*s4, s4*s2*s3*s1*s2*s4*s2*s3*s1*s2]$, $[s2*s1, s3*s2*s4*s3*s2*s1]$, $[s2*s3, s1*s2*s4*s1*s2*s3]$, $[1, s2*s4*s3*s2]$, $[s2*s3, s2*s4*s1*s2*s3*s1$, $[s2*s4, s2*s3*s1*s2*s4*s1]$, $[s2*s4*s3*s1*s2*s1, s2*s3*s1*s2*s4*s2*s3*s1*s2*s1]$, $[s2*s4*s2*s3*s1*s2, s2*s3*s1*s2*s4*s2*s3*s1*s2*s1]$, $[s1*s2*s3*s1*s2*s1, s2*s3*s1*s2*s4*s2*s3*s1*s2*s1]$, $[s1*s2*s4*s1*s2*s1, s2*s3*s1*s2*s4*s2*s3*s1*s2*s1]$, $[s1*s2*s4*s3*s2*s1, s2*s3*s1*s2*s4*s2*s3*s1*s2*s1]$, $[s1*s2*s4*s3*s1*s2, s2*s3*s1*s2*s4*s2*s3*s1*s2*s1]$, $[s3*s2*s4*s3*s1*s2, s2*s3*s1*s2*s4*s2*s3*s1*s2*s1]$, $[s3*s2*s4*s2*s3*s2, s2*s3*s1*s2*s4*s2*s3*s1*s2*s1]$, $[s2*s4*s1*s2*s3*s2, s2*s3*s1*s2*s4*s2*s3*s1*s2*s1]$, $[s2*s3*s1*s2*s4*s2, s2*s3*s1*s2*s4*s2*s3*s1*s2*s1]$, $[s2*s4*s2*s3, s4*s2*s3*s1*s2*s4*s2*s3]$, $[s3*s2*s4*s3, s4*s2*s3*s1*s2*s4*s2*s3]$, $[s2*s4*s3*s1, s4*s2*s3*s1*s2*s4*s2*s3]$, $[s4*s2*s3*s1, s4*s2*s3*s1*s2*s4*s2*s3]$, $[s2*s4*s2*s1, s4*s2*s3*s1*s2*s4*s2*s3]$ $[s2*s4*s2*s3, s2*s3*s1*s2*s4*s2*s3*s2]$, $[s3*s2*s4*s3, s2*s3*s1*s2*s4*s2*s3*s2]$, $[s2*s4*s3*s1, s2*s3*s1*s2*s4*s2*s3*s2]$, $[s2*s3*s2*s1, s1*s2*s4*s1*s2*s3*s2*s1]$, $[s1*s2*s3*s1, s1*s2*s4*s1*s2*s3*s2*s1]$, $[s2*s4*s3*s1, s1*s2*s4*s1*s2*s3*s2*s1]$, $[s1*s2*s4*s1, s1*s2*s4*s1*s2*s3*s2*s1]$, $[s1*s2*s4*s3, s1*s2*s4*s1*s2*s3*s2*s1]$, $[s2*s3*s2*s1, s2*s4*s1*s2*s3*s1*s2*s1]$, $[s1*s2*s3*s1, s2*s4*s1*s2*s3*s1*s2*s1]$, $[s2*s4*s3*s1, s2*s4*s1*s2*s3*s1*s2*s1]$, $[s2*s4*s2*s1, s4*s2*s3*s1*s2*s4*s2*s1]$, $[s1*s2*s4*s1, s4*s2*s3*s1*s2*s4*s2*s1]$, $[s2*s4*s3*s1, s4*s2*s3*s1*s2*s4*s2*s1]$, $[s4*s2*s3*s1, s4*s2*s3*s1*s2*s4*s2*s1]$, $[s2*s4*s2*s3, s4*s2*s3*s1*s2*s4*s2*s1]$, $[s2*s4*s2*s1, s2*s3*s1*s2*s4*s1*s2*s1]$, $[s1*s2*s4*s1, s2*s3*s1*s2*s4*s1*s2*s1]$,