COLLEGE OF SCIENCE AND TECHNOLOGY

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DEPARTMENT OF MATHEMATICS

NON-LINEAR HEAT EQUATION IN SEMI-INFINITE DOMAIN

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Student number: 221003918

Thesis Submitted in Partial Fulfillment of the Academic Degree of
Master of Science
in
Applied Mathematics
Option of Mathematical Modeling and Scientific Computing

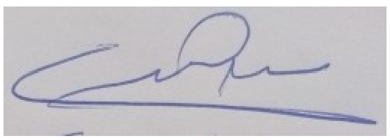
Supervisors: Assoc. Prof. BANZI Wellars

Kigali-Rwanda
November 10, 2024

Declaration

I, **Enock NIZEYIMANA** , certify that the research thesis ”NON-LINEAR HEAT EQUATION IN SEMI-INFINITE DOMAIN” is my own work. It has never before been submitted or published for academic recognition at a university or other learning institution.

Signature:



Date: November 10, 2024

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I attest that this dissertation was presented under my guidance and supervision.

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Dedication

To Almighty God
To my family
To my supervisor Assoc. Professor Wellars BANZI
To my lovely Mother and Father,
To my Brothers and Sisters,
To my classmates,
To my friend and colleagues

Abstract

In this study, nonlinear heat equations in a semi-infinite domain are thoroughly investigated, focusing on their solvability and behavior within a significant mathematical framework. By employing the Galerkin method and demonstrating convergence properties in function spaces, the research confirms the existence and accuracy of solutions to these equations. The thesis contributes to understanding heat transfer phenomena and mathematical modeling of complex systems through a detailed examination of theoretical background, problem statement, and objectives. Utilizing advanced mathematical techniques like the Galerkin method, we establish both the existence and uniqueness of solutions to the nonlinear heat equation in a semi-infinite domain. Rigorous analysis validate the stability properties of these solutions, providing insights into heat conduction in materials and the behavior of heat transfer in nonlinear systems. This work enriches knowledge in mathematical modeling and heat transfer phenomena, offering valuable insights for further exploration and applications in scientific and engineering domains.

Acknowledgment

I express my sincere gratitude to my supervisor, Assoc. Prof.BANZI Wellars, for his invaluable guidance and unwavering support throughout the completion of this research project. I extend my thanks to the Eastern Africa Universities Mathematics Programme-International Centre for Theoretical Physics (EAUMP-ICTP) for their financial support.

I am also thankful to my colleagues, classmates, and my friends for their contributions and encouragement, which have been instrumental in the success of this research endeavor.

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Chapter 1

General Introduction

This chapter includes the theoretical background,the problem statement and the objectives of the thesis.

1.1 Background

Heat is a type of energy that moves between substances or systems when there is a temperature difference. It reflects the internal energy of a substance, which comes from the motion of its molecules. Heat can be transferred by conduction, convection, or radiation. [1].
The heat equation is derived from Fourier's law and the principle of energy conservation, as detailed by Cannon in [1984] [2]. Fourier's law states that the rate of heat transfer through a surface is directly proportional to the temperature difference across the surface, with the heat flow being proportional to the negative of the temperature gradient. [3]

In simpler terms, when there is a temperature difference between two objects or systems, heat flows from the hotter object to the cooler one until they reach the same temperature. Heat is an essential concept in physics and is fundamental to many natural processes and technological applications. In the International System of Units (SI), heat is measured in joules (J). [4] .

The nonlinear heat equation is a partial differential equation that arises in many fields of physics and engineering, including materials science, thermodynamics, and fluid dynamics. It describes how the temperature of a material changes over time, taking into account both the diffusion of heat and the generation or consumption of heat within the material [5]. The Nonlinear Heat equation (Diffusion equation) in semi-infinite domain can be generally written by the equation of the form

$$\begin{cases} u_t = \nabla(k(u)\nabla u) + f(u), & 0 \leq x < \infty \\ u(x,0) = g(x), & x \in \mathbb{R}^n \\ u(0,t) = h(t), & \forall \quad t > 0 \end{cases} \tag{1.1}$$

where $u(x,t)$ is the temperature at point x and time t, u_{xx} is the second derivative of u with respect to x, $k(u)$ is the thermal diffusivity (a positive constant) and $f(u)$ is a nonlinear function that depends on the temperature u. The equation is subject to boundary conditions that specify the behavior of u at the boundary of the domain, which is often taken to be at $x = 0$ .The nonlinear term $f(u)$ can lead to highly nonlinear behavior in the temperature profile, including the formation of shock waves and other types of discontinuities. This makes solving the equation a challenging problem that requires advanced numerical methods. [6]

The nonlinear heat equation in a semi-infinite domain has been the subject of extensive research in the fields of applied mathematics, physics, and engineering [7].
An important of using the heat equation in a semi-infinite domain is defining appropriate boundary and initial conditions. Since the domain extends infinitely in one direction, it is typical to have an "infinite" boundary condition at that end. For example, this could be a fixed temperature or a heat flux condition. At the other end of the domain, which is bounded, initial and/or boundary conditions need to be specified based on the problem at hand. (See the approach in paper [8]).

When the nonlinear term equals to zero ($f(u) = 0$), and $k =$ constant, the equation becomes linear heat equation. It describing the physical processes of heat conduction inside the body material where it flows from one position to another , [9, 10]

$$\rho c_p \frac{\partial u(x,t)}{\partial t} = \frac{\partial}{\partial x}\left(k(u)\frac{\partial u}{\partial x}\right) \tag{1.2}$$

Where the thermal conductivity k remains inside of derivative since it depends on u. when $k(u) = k$ is a constant the equation (1.2) is nonlinear.

One can note that when the thermal conductivity k is constant, the equation (1.2) reduced to well known heat equation (1.3)

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \tag{1.3}$$

with the thermal diffusivity, $\alpha^2 = (k/\rho c_p)$

From the paper [5] the author consider one-dimensional governing equation for unsteady-state heat transfer in semi-infinite solids with thermal conductivity dependent of temperature which is given by equation (1.2), where

$$k(u) = k_0(1 + \beta u). \tag{1.4}$$

With the initial and boundary conditionsnbelow.

$$u(x, 0) = u_0 \tag{1.5}$$

$$-k\frac{\partial u}{\partial x}|_{x=0} = q' \tag{1.6}$$

Numerical method have been used and solution has been obtained for $1 - D$ unsteady-state heat conduction in semi-infinite rod. Note that for linear equation, Fourier sine or cosine transform can be used.

In the same work, the problem (1.3) has been modeled in 2-D domain as follows.

$$\rho c_p \frac{\partial u(x,y,t)}{\partial t} = \frac{\partial}{\partial x}\left(k(u)\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(k(u)\frac{\partial u}{\partial y}\right) \tag{1.7}$$

The initial and boundary conditions are

$$u(x, y, 0) = u_i \tag{1.8}$$

$$-k\frac{\partial u}{\partial x}|_{(0,y,t)} = q' \tag{1.9}$$

The weak form of the (1.7) is obtained by multiplying by a test function w and integrating over Ω

$$\int_\Omega \left[\frac{\partial w}{\partial x}\left(k(u)\frac{\partial u}{\partial x}\right) + \frac{\partial \bar{w}}{\partial y}\left(k(u)\frac{\partial u}{\partial y}\right) - \rho c \bar{w}\frac{\partial u}{\partial t}\right] d\Omega = 0 \tag{1.10}$$

Using natural boundary conditions into equation (1.10), the functional-becomes as

$$\int_{\Omega}\left[\frac{\partial u}{\partial x}\left(k(u)\frac{\partial u}{\partial x}\right)+\frac{\partial u}{\partial y}\left(k(u)\frac{\partial u}{\partial y}\right)\right]d\Omega+\int_{\Omega}c\rho\frac{1}{2}\frac{\partial}{\partial t}|u|^2 d\Omega+\int_{\partial\Omega}q' u\,d\Gamma=0 \tag{1.11}$$

In the *work of John Wiley* [11], it has been shown that the heat equation (1.3) has the fundamental solution on the whole real line.

R. M. Cherniha,, in his work [12], studied non-linear diffusion equation, he described the vertical transfer of both heat and moisture in the absence of solutes are considered. Lie symmetries of the equation was applied and the solution was obtained for some specific form of diffusion coefficient.

A. Verma, et al [13] classified the nonlinear diffision equation (1.1) in 1-D as follows:

$$\frac{\partial u}{\partial t}=D\frac{\partial^2 u}{\partial x^2}+g(u), \quad 0\leq x<\infty \tag{1.12}$$

Where $g(u)$ stands for the part of Fisher equation. They tried to give three different equations as a Fisher's equation of the type:

$$u_t-u_{xx}-\alpha u(1-u)=0 \tag{1.13}$$

Fisher's equation of the type:

$$u_t-u_{xx}-u^2(1-u)=0 \tag{1.14}$$

Fisher's equation of the type:

$$u_t-u_{xx}-\alpha u(u-a)(1-u)=0 \tag{1.15}$$

By applying Lie classical method, a comprehensive list of exact solutions to equations of the form (1.12) is available for both $g(u)=0$ [14] and $g(u)\neq 0$ [15]

In paper [16], author *F. Kangalgil and F. Ayaz* tried to find the solution of linear and nonlinear heat equations by using differential transform method. They just considered nonlinear reaction-diffusion equation of the form

$$u_t = (A(u)u_x)_x + B(u)u_x + C(U) \tag{1.16}$$

Where $u(x,t)$ is the unknown function and $A(u), B(u)$ and $C(U)$ are arbitrary smooth functions. The particular solution of the equation (1.16) is found in [12] page 140. Also in [17] exact solutions to equations of the form

$u_t = (A(u)u_x)_x + C(U)$ were obtained for particular choices of the functions $A(u)$ and $C(u)$ using anti-reduction method.

The Fourier transformation (sine or cosine) can also work for the equation (1.3)

In [18], authers present solutions for three families of equations with logarithmic heat sources by considering the unsteady heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(ax^k \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(by^m \frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(cz^n \frac{\partial u}{\partial z}\right) + \Phi(u) \tag{1.17}$$

Assuming that $P(U) = \dfrac{\partial u}{\partial t} - \Phi(u)$ for $k, m, n, \neq 2$

the solutions of the equation (1.16) are obtained by nonlinear separation of variables and it has the form:

$$u = u(t,r), \qquad r = 4A\left[\frac{x^{2-k}}{a(2-k)^2} + \frac{y^{2-m}}{b(2-m)^2} + \frac{z^{2-n}}{c(2-n)^2}\right] \tag{1.18}$$

and the exact solution of the above found in the work [19].

Formore understanding, the authors *T. Itoh, et al* [20] discussed the solution of the heat equation on an infinite spatial domain using the Fourier transform and the separation of variables technique. It begins by introducing the heat equation on an infinite rod and discussing the conditions for the solution to remain bounded. The separation of variables method is then employed to derive the steady-state temperature distribution.

It continue also explores the concept of the heat kernel and its calculation using the Fourier transform, as well as the application of the convolution theorem to the heat equation. The paper concludes by discussing the heat equation on a semi-infinite domain and the use of the Laplace transform to solve the problem.

Overall, it provides a comprehensive approach to understanding and solving the heat equation on an infinite spatial domain, with a focus on the application of the Fourier transform and the separation of variables technique.

In addition, applications of the heat equation in semi-infinite domains can be found in different fields, including heat conduction in infinite bars [21], semi-infinite plates [22], soil temperature profiles [23], and other scenarios where heat transfer or diffusion occurs in a domain that extends infinitely in one direction.

In the works, [18] and [13], authors have considered nonlinear heat equation with the nonlinear term $f(u) = 0$ or $f(u) \neq 0$ standing for terms Fisher equation they have not considered a general form of $f(u) = |u|^p$. The aim of this work is to generalise these work.

In work [24] the author tried to do the nonlinear heat equation in $\mathbb{R}^n$,. The aim of the work supposed to do is to work in $\mathbb{R}^n_+$

1.2 Problem statement

The study of heat conduction in materials is fundamental to various scientific and engineering fields. In many real-world scenarios, the behavior of heat transfer deviates significantly from linear models, demanding the analysis of nonlinear heat equation.

Let us consider the problem of nonlinear heat equation in semi-infinite domain.

$$\begin{cases} u_t = \bigtriangledown.(k(u) \bigtriangledown u) + |u|^p \quad \text{in} \quad \Omega_+^T = \mathbb{R}_+^n \times (0, \infty) \\ p > 1 \\ u(x, 0) = f(x), \quad x \in \mathbb{R}_+^n, \\ u|_{\Gamma_+} = g(x), \Gamma = \partial\Omega_+^T \end{cases} \tag{1.19}$$

The problem involves proving the existence and uniqueness of the solutions, of the problem (1.19).

1.3 Objectives of the thesis

1.3.1 General objective

The general objective of this study is to investigate the solvability of the nonlinear heat equation posed in a semi-infinite domain given in equation (1.19).

1.3.2 Specific objectives

1. To Demonstrate the existence of a solution to the nonlinear heat equation (NLHE) in a semi-infinite domain
2. To establish the uniqueness of the solution for the NLHE in a semi-infinite domain.
3. To investigate the stability characteristics of solutions to the nonlinear heat equation (NLHE) within a semi-infinite domain

Chapter 2

Literature review

2.1 Nonlinear heat equation

The nonlinear heat equation, as represented by equation (1.1), is essential for understanding heat transfer phenomena in fields like materials science, thermodynamics, and fluid dynamics. This equation models the intricate interactions between heat diffusion and the generation or absorption of heat within a material. This literature review will delve into significant findings and methodologies used in studying nonlinear heat equations, with a focus on semi-infinite domains.

The nonlinear heat equation describes how heat diffuses in a medium with a nonlinear heat source or nonlinear thermal properties. Mathematically, the one-dimensional form of the nonlinear heat equation is expressed as:

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} + f(u) \tag{2.1}$$

where: $u(x,t)$ is the temperature distribution at position x and time$t$, α is the thermal diffusivity (a positive constant), $\frac{\partial u}{\partial t}$ represents the rate of change of temperature with respect to time, $\frac{\partial^2 u}{\partial x^2}$ represents the spatial second derivative of temperature with respect to position, $f(u)$ is a nonlinear function that depends on the temperature u and represents the heat source or nonlinear thermal properties.

The nonlinear term $f(u)$ introduces nonlinearity into the equation, leading to complex behavior in the temperature profile, including the formation of shock waves and other types of discontinuities. This nonlinearity poses challenges in solving the equation and often requires advanced mathematical and numerical techniques for analysis and solution.

The nonlinear heat equation is of significant interest in various fields, including applied mathematics, physics, and engineering, due to its relevance in understanding heat transfer phenomena in materials and systems with nonlinear thermal behavior.

2.1.1 Classification of nonlinear heat equation

Nonlinear heat equations can be classified based on several criteria, including the form of nonlinearity, the nature of the heat conductivity function, and the boundary conditions. Here's a basic classification:

1. *Form of nonlinearity:*

The nonlinear part of heat equation can have the form of exponential nonlinearity and the power-law nonlinearity

Exponential nonlinearity:

$$\frac{\partial u}{\partial t} = a\frac{\partial}{\partial x}\left(\mu e^{\lambda u}\frac{\partial u}{\partial x}\right) \tag{2.2}$$

Power-law nonlinearity:

$$\frac{\partial u}{\partial t} = a\frac{\partial}{\partial x}\left(\mu u^{m}\frac{\partial u}{\partial x}\right) \tag{2.3}$$

These classifications help in understanding the behavior of nonlinear heat equations, devising appropriate numerical methods for their solution, and studying their properties analytically. Depending on the specific problem being addressed, different nonlinearities and boundary conditions may be more relevant, and the choice of classification can guide the analysis and solution approach. more understanding and explanation can be found in [25].

2. *Nature of thermal conductivity function:*

The nature of the thermal conductivity function dictates how heat is transferred through materials. In cases of constant thermal conductivity, denoted as $k(u)$, the heat equation simplifies to a linear form with fixed coefficients, representing systems where thermal properties remain uniform regardless of temperature fluctuations. Conversely, in scenarios where thermal conductivity varies with temperature, described by $k(u)$, the resulting heat equation becomes nonlinear, capturing the dynamic relationship between temperature and heat transfer. This temperature-dependent conductivity is often encountered in materials where thermal properties evolve notably with temperature shifts, offering a more realistic portrayal of heat conduction phenomena across a range of applications. [26]

3. Boundary conditions:

. *Homogeneous boundary conditions:* These are boundary conditions where the temperature u or its derivatives are specified at the boundaries, typically represented as $u(x_0, t) = f(t)$ or $\frac{\partial u}{\partial x}(x_0, t) = g(t)$, where x_0 denotes the boundary location point.

. *Non-homogeneous boundary conditions:* These are boundary conditions where the heat flux q or some other physical quantity related to heat transfer is specified at the boundaries, often represented as $q(x_0, t) = h(t)$ or $\frac{\partial q}{\partial x}(x_0, t) = m(t)$, where x_0 denotes the boundary location.

2.1.2 Impact of variable coefficients on solution behavior

The coefficients in the equation, such as thermal conductivity or absorption coefficient, are allowed to vary spatially or temporally.

Furthermore, allowing the coefficients to be variable implies that the properties of the material, such as thermal conductivity or absorption capacity, may change spatially or over time. This variation can arise due to heterogeneities in the material, changes in its composition, or external factors affecting its thermal properties. *Kersner, at al* [27] investigated the existence, uniqueness, regularity and geometric properties of nonnegative solutions of the following 2-D nonlinear Cauchy problem given by:

$$\begin{cases} \frac{\partial u}{\partial t} = (D(x,t)u^{m-1}u_x)_x - b(x,t)u^p & \text{in } \mathbb{R}^2_+ = \mathbb{R} \times (0, \infty) \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{2.4}$$

Where $m \geq 1$, $p > 0$, and the functions $D(x,t)$, $b(x,t)$, and $u_0(x)$ satisfying the given assumptions.

(H1) $D(x,t) \in C^{2,1}_{x,t}(\mathbb{R}^2_+), D(x,t) > 0$ for $(x,t) \in \mathbb{R}$,

(H2) $b(x,t) \in C^{1,1}_{x,t}(\mathbb{R} \times [0,\infty)), b(x,t) \geq 0$ for $(x,t) \in \mathbb{R}$,

(H3) $u_0(x) \in C(\mathbb{R}), 0 \leq u_0(x) \leq M_0$ for $x \in \mathbb{R}$.

In their paper they were mainly interested in the case

$$m > 1, 0 < p < 1 \tag{2.5}$$

The physical meaning of condition (2.5) was that the diffusion was slow and the intensity of thermal absorption increased fast near the zero value of the temperature at least if ($D = d_o = const > 0, b = 6, = const > 0$). Equations like (2.4) possesses quite a wide variety of solutions and there is a large number of problems arising in connection with them. The interested reader may get a good idea of this from the survey papers [28] $(D = d_o, b = O)$ Under conditions H_1, H_2 and H_3 authors have proved the existence and uniqueness of the weak solution.

2.2 Solution of nonlinear heat equation

The paper "Semistability of the minimal positive steady state for a nonhomogeneous semilinear Cauchy problem" by BaiShun Lai and Yi Li focuses on the asymptotic behavior and stability of positive radial solutions of a nonhomogeneous semilinear Cauchy problem. The authors investigate the semistability of the minimal positive steady state and provide several theorems, propositions, and lemmas to support their findings.

2.2.1 Nonlinear heat equation with singular initial values: New blowup and lifespan results

In the paper [29],, Authors investigated the existence and uniqueness of local solutions in time to the semilinear heat equation

$$u_t = \Delta u + a|u|^\alpha u, \tag{2.6}$$

where $u = u(t, x) \in \mathbb{R}$, $t > 0$, $x \in \mathbb{R}^N$, $a = \pm 1$, and $\alpha > 0$, with initial data that exhibit highly singular behavior. Additionally, they present several novel findings concerning the finite-time blowup of solutions to (2.6) in the scenario where $a = 1$. They examined initial data given by

$$u_0 = K(-1)^m \partial_1 \partial_2 \cdots \partial_m \delta, \tag{2.7}$$

for the case where $a = 1$, with δ representing the Dirac delta function at the origin. Their focus was on the tempered distribution

$$u_0 = K(-1)^m \partial_1 \partial_2 \cdots \partial_m (|.|^{-\gamma}) \in S'(\mathbb{R}^N). \tag{2.8}$$

Here, m is an integer such that $1 \leq m \leq N$, $0 < \gamma < N$, and $K \in \mathbb{R}$. In fact, they considered a more general class of initial data which were in some sense bounded by (1.11). By a local solution, they mean a function $u : (0, T] \to C_0(\mathbb{R}^N)$, a classical solution of (2.6) for $t \in (0, T]$, such that $u(t) \to u_0$ in $\mathcal{S}'(\mathbb{R}^N)$ as $t \to 0$. Here $\mathcal{S}'(\mathbb{R}^N)$ is the space of tempered distributions on $\mathbb{R}^N$.

The set $C_0(\mathbb{R}^N)$ consists of functions $f : \mathbb{R}^N \to \mathbb{R}$ that are continuous, and as $|x|$ tends to infinity, the limit of $f(x)$ approaches zero. In line with common practice, they investigated equation (1.9) using its corresponding integral representation:

$$u(t) = e^{t\Delta} u_0 + a \int_0^t e^{(t-\sigma)\Delta} |u(\sigma)|^\alpha u(\sigma)\, d\sigma \tag{2.9}$$

where $e^{t\Delta}$ represents the heat semigroup on $\mathbb{R}^N$, defined as

$$e^{t\Delta}\phi = G_t * \phi \quad for \qquad t > 0 \quad and \quad \phi \in \mathcal{S}'(\mathbb{R}^N) \tag{2.10}$$

with G_t being the Gauss kernel given by

$$G_t(x) = (4\pi t)^{-\frac{N}{2}} e^{-\frac{|x|^2}{4t}}, \quad t > 0,\, x \in \mathbb{R}^N \tag{2.11}$$

2.2.2 Nonlinear heat equation with high-order Dirac delta initial values.

Consider the equation

$$\frac{\partial u}{\partial t} = \triangle u + u^p \quad in \quad R^n, (0, T) \tag{2.12}$$

with $u(x, 0) = \psi(x)$ in $\mathbb{R}^n$. The authors [24] explore the existence and behavior of global solutions to the nonlinear heat equation, as well as the possibility of finite-time blow up. They also examine the mathematical structure and challenges associated with the nonlinear heat equation, particularly in relation to the behavior of solutions and the potential for blow up. The paper provides a detailed analysis of the fundamental properties and dynamics of solutions to the nonlinear heat equation, shedding light on both the global behavior and potential singularities of the system

In the same work, authors make several assumptions in its analysis of the Cauchy problem involving a nonlinear heat equation. One of the key used in the paper was the assumption on the bounded continuous nonnegative function in $\mathbb{R}^n$. Specifically, they assumed that there exists a $T > 0$ such that the Cauchy problem had a unique classical solution $u(x, t; .)$ in $C_1^2(R^n(0, T))$, $C(R^n[0, T])$ which is bounded in $R^n[0, T]$ for all $T' < T$,
and $||u(., t; .)||L_\infty(R^n) \leq M$ as $t \longrightarrow T$, where M is a positive constant and T is the maximal existence time of the solution. This assumption was crucial in the analysis of the global solution and potential blow-up in finite time of the nonlinear heat equation (2.12) .

The same paper presented several results based on the analysis of the Cauchy problem involving a nonlinear heat equation. Some of the key results include:

Stability of steady states:
They established the stability of steady states of the nonlinear heat equation with respect to a specific norm, providing conditions under which the steady states are stable.

$$||\bar{u}(., t)||_m \leq \varepsilon C, \quad t \geq 0, \tag{2.13}$$

where C is a positive constant. Moreover, for any $\varepsilon > 0$, there exists a small $\delta > 0$ such that if the condition $||\varphi||_m \leq \delta$. is satisfied, then $\bar{u}(x, 0) \geq \varphi(x)$ in $\mathbb{R}^n$. In this case, for any $\varepsilon \in (0, 1)$, $\bar{u}$ is a super-solution of (2.12). Now, by the comparison principle and (2.13), the solution u of (2.12) satisfies.

$$||u(., t)||_m \leq ||\bar{u}(., t)||_m \leq \varepsilon C, \quad t \geq 0, \tag{2.14}$$

Now the stability of the rest state u_0 was established in $||.||_m$. This, along with the fact that $u(x,t)$ decays to zero uniformly for $x \in \mathbb{R}^n$ as $t \to \infty$, implies the weak asymptotic stability in $||.||_m$.

Now they only needed to consider the case $\lambda > m$ by letting $\varphi(x) = (1+|x|)^-\lambda$ and

$$v(x,t) = e^{t\triangle}\varphi(x) := \frac{1}{(4\pi t)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}}\varphi(y)\,dy \tag{2.15}$$

Proposition 2.1 *Given $\lambda > m$, $u_0 \equiv 0$ is a stable steady state of (2.12) with respect to norm $||.||_\lambda$ if and only if $||v(.,t)||_\lambda$is a bounded function of $t \geq 0$. In this case, u_0 is also a weakly asymptotically stable steady state of (2.12).*

In [30], they proof that $||v(.,t)||_\lambda$is a bounded for $t \geq 0$ if and only if $\lambda < n$

$$\begin{aligned}
v(x,t) =&\frac{1}{(4\pi t)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}}\frac{1}{(1+|y|)^\lambda}\,dy\\
=&\frac{1}{(4\pi t)^{\frac{n}{2}}}\int_{|y|\leq|x|/2} e^{-\frac{|y|^2+|x|^2}{4t}}e^{-\frac{|x-y|^2}{4t}}\frac{1}{(1+|y|)^\lambda}\,dy\\
+&\frac{1}{(4\pi t)^{\frac{n}{2}}}\int_{|y|\geq|x|/2} e^{-\frac{|y|^2+|x|^2}{4t}}e^{-\frac{|x-y|^2}{4t}}\frac{1}{(1+|y|)^\lambda}\,dy\\
=&I+II.
\end{aligned}$$

It is easy to see that $||II||_\lambda$ is bounded. For $n < \lambda$,

$$\begin{aligned}
I \leq&\frac{1}{(4\pi t)^{\frac{n}{2}}}\int_{|y|\leq|x|/2} e^{-\frac{|x|^2}{16t}}(1+|y|)^{-\lambda}\,dy\\
\leq&\frac{c}{n-\lambda}t^{-\frac{n}{2}}e^{-\frac{|x|^2}{16t}}\left(1+\frac{|x|}{n}\right)^{n-\lambda}\\
\leq&\frac{c}{n-\lambda}\left(\frac{|x|}{\sqrt{t}}\right)^n e^{-\frac{1}{16}(\frac{|x|}{\sqrt{t}})^2}(1+|x|)^{-\lambda}.\\
\leq&\frac{c}{n-\lambda}(1+|x|)^{-\lambda}.
\end{aligned}$$

For II, we have:

$$II = \frac{1}{(4\pi t)^{\frac{n}{2}}}\int_{|y|\geq|x|/2} e^{-\frac{|x-y|^2}{4t}}\frac{1}{(1+|y|)^\lambda}\,dy.$$

$$II \leq \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{|y| \geq |x|/2} \frac{1}{(1+|y|)^{\lambda}} \, dy.$$

Using spherical coordinates and letting $r = |y|$,

$$II \leq \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{|x|/2}^{\infty} \int_{S^{n-1}} \frac{r^{n-1}}{(1+r)^{\lambda}} \, dS \, dr.$$

Simplifying with the constant surface area of the unit sphere S^{n-1},

$$II \leq \frac{C}{(4\pi t)^{\frac{n}{2}}} \int_{|x|/2}^{\infty} \frac{r^{n-1}}{(1+r)^{\lambda}} \, dr.$$

For large r, $(1+r)$ behaves like r, so

$$\int_{|x|/2}^{\infty} \frac{r^{n-1}}{r^{\lambda}} \, dr = \int_{|x|/2}^{\infty} r^{n-1-\lambda} \, dr.$$

This integral converges if $n - 1 - \lambda < -1$, i.e., $\lambda > n$.

Thus, for $\lambda < n$, the integral converges and $||II||_{\lambda}$ is bounded. It implies that $||v(.,t)||_{\lambda}$ is bounded for $t \geq 0$.

2.3 Existence and uniqueness of solution to NLHE

In [31] "Stability and Exponential Stability of Nonlinear Functional Differential Equations and Applications" *by author Tomás Caraballo*, the existence of solutions for a class of nonlinear functional differential equations in Hilbert spaces was established through a variational argument combined with a Galerkin approximation technique.

The primary objective of the above paper was to demonstrate the existence and uniqueness of solutions for a specific class of nonlinear partial functional differential equations given by the following system:

$$\begin{cases} \frac{dx(t)}{dt} = A(t, x(t)) + f(t, x_t), & t \in [0, T], \\ x(t) = \psi(t), & t \in [-h, 0]. \end{cases} \tag{2.16}$$

Here is a brief overview of how the existence of solutions which can be proven:

Variational approach:

The study begins by introducing a framework in which the analysis has to be carried out, involving separable Banach and Hilbert spaces. The existence of solutions is established by considering a family of nonlinear operators defined for almost every time. The hypotheses of coercivity, monotonicity, boundedness, hemicontinuity, and measurability are imposed on these operators to ensure the well-posedness of the problem.

Let $A(t,\cdot) : V \to V'$ be a collection of (nonlinear) operators defined almost everywhere with respect to t, where $p \geq 2$. Consider the following hypotheses hold:

H1: **Coercivity:** There exist $\alpha > 0$, λ, $\nu \in \mathbb{R}$ such that:

$$2\langle A(t,x), xi\rangle + \lambda|x|^2 + \nu \geq \alpha\|x\|_p^p, \quad \forall x \in V, \text{ a.e. } t. \tag{2.17}$$

H2: **Monotonicity:**

$$2\langle A(t,x) - A(t,y), x-y\rangle + \lambda|x-y|^2 \geq 0, \quad \forall x, y \in V, \text{ a.e. } t. \tag{2.18}$$

H3: **Boundedness:** There exists $\gamma > 0$ such that:

$$\|A(t,x)\|_* \leq \gamma\|x\|^{p-1}, \quad \forall x \in V, \text{ a.e. } t. \tag{2.19}$$

H4: **Hemicontinuity:** The mapping $\theta \in \mathbb{R} \mapsto \langle A(t, x+\theta y), zi\rangle \in \mathbb{R}$ is continuous for all $x, y, z \in V$, a.e. t.

H5: **Measurability:** The mapping $t \in (0,T) \mapsto A(t,x) \in V'$ is Lebesgue-measurable for all $x \in V$, a.e. t.

Let $f(t,\cdot) : L^2(H) \to H$ be a family of nonlinear operators defined almost everywhere with respect to t, satisfying the following conditions:

1. There exists $c_f \geq 0$ such that $\sup_{0\leq t\leq T} |f(t,0)| \leq c_f < +\infty$.

$$\exists k_1 = k_1(h) > 0 : |f(t,\eta) - f(t,\xi)| \leq k_1\|\eta - \xi\|_{CH} \forall \eta, \xi \in CH, a.e \tag{2.20}$$

   with respect to t .

2. The mapping $t \in (0,T) \mapsto f(t,\eta) \in H$ is Lebesgue-measurable for all $\eta \in L^2(H)$.

Given an initial value $\psi \in L^p(-h,0;V) \cap C(-h,0;H)$, the primary objective in this section is, under the aforementioned conditions, to determine a unique function $x(\cdot) \in L^p(-h,T;V) \cap C(-h,T;H)$ such that

$$\begin{aligned} x(t) &= \psi(0) + \int_0^t [A(s,x(s)) + f(s,x_s)]\,ds, && t \in [0,T], \\ x(t) &= \psi(t), && t \in [-h,0], \end{aligned} \tag{2.21}$$

where the first equality was understood in V'. They denoted this solution as the variational solution to the given problem.

Theorem 2.1 *Assume the preceding hypotheses hold. Then, there exists at most one solution of equation (2.21) in* $L^p(-h,T;V) \cap C(-h,T;H)$.

Proof: Let's assume that x and y are two solutions of equation (2.21) in $L^p(-h,T;V) \cap C(-h,T;H)$. Utilizing (2.20), in [31] we derive:

$$\begin{aligned} |x(t)-y(t)|^2 &= 2\int_0^t h[A(s,x(s)) - A(s,y(s)), x(s)-y(s)]\,ds \\ &\quad + 2\int_0^t (f(s,x_s) - f(s,y_s), x(s)-y(s))\,ds \\ &\le \lambda \int_0^t |x(s)-y(s)|^2\,ds \\ &\quad + 2\int_0^t |f(s,x_s)-f(s,y_s)||x(s)-y(s)|\,ds. \end{aligned}$$

Now, utilizing (2.20), for any $t \in [0,T]$, we have:

$$\sup_{0\le s\le t} |x(s)-y(s)|^2 \le (|\lambda|+1)\int_0^t |x(s)-y(s)|^2\,ds + k_1^2 \int_0^t \|x(s)-y(s)\|_{CH}^2\,ds.$$

On the other hand, since $x(s) = y(s)$ for $s \le 0$, we obtain:

$$\begin{aligned} \int_0^t \|x(s)-y(s)\|_{CH}^2\,ds &= \int_0^t \sup_{-h\le r\le 0} |x_s(r)-y_s(r)|^2\,ds \\ &= \int_0^t \sup_{-h\le r\le 0} |x(s+r)-y(s+r)|^2\,ds \le \int_0^t \sup_{0} \\ &\le r \le s|x(r)-y(r)|^2\,ds. \end{aligned}$$

Thus, by combining the above inequalities, they get:

$$\sup_{0\le s\le t} |x(s)-y(s)|^2 \le 2(|\lambda|+1) + k_1^2 \int_0^t \sup_{0\le r\le s} |x(r)-y(r)|^2\, ds, \quad \forall t\in[0,T].$$

and Gronwall's lemma obviously implies uniqueness.

2.4 Function spaces and properties

2.4.1 Lebesque L^p

Lebesgue spaces, often denoted by L^p, represent significant function spaces within measure theory and functional analysis [32]. They serve as a tool for investigating the integrability and characteristics of functions over measurable sets. The Lebesgue space L^p encompasses all measurable functions f defined on a measure space (X,Σ,μ), such that the p-th power of the absolute value of f is Lebesgue integrable over X, signified by

$$\|f\|_p = \left(\int_X |f(x)|^p \, d\mu(x)\right)^{1/p} < \infty \tag{2.22}$$

for $1\le p<\infty$. When $p=\infty$, we define the essential supremum norm:

$$\|f\|_\infty = \operatorname{ess}\sup_{x\in X} |f(x)| \tag{2.23}$$

Equipped with the norm $\|\cdot\|_p$, the Lebesgue space L^p forms a Banach space, signifying its completeness with respect to the norm. Some keys found in [32].

• *Inclusion relationships:* For $1\le p<q\le\infty$, $L^q(X)$ is contained within $L^p(X)$. This implies that if a function belongs to L^q, it also belongs to L^p, and the norm of the function in L^p is bounded by its norm in L^q. Such relationships are pivotal for understanding the embedding characteristics of Lebesgue spaces.

• *Convergence in L^p:* A sequence of functions $\{f_n\}$ converges to a function f within the Lebesgue space L^p if $\|f_n-f\|_p\to 0$ as $n\to\infty$. This convergence is weaker than pointwise convergence yet stronger than convergence in measure.

• *Completeness:* L^p is a complete space, indicating that every Cauchy sequence within L^p converges to a limit within L^p. This completeness property is crucial for the existence of solutions to certain integral and differential equations.

• *Dual spaces:* The dual space of L^p, denoted as $(L^p)^*$ or $L^{p'}$ where $\frac{1}{p} + \frac{1}{p'} = 1$, comprises linear functionals on L^p. When $1 < p < \infty$, the dual space of L^p is $L^{p'}$, whereas for $p = 1$ or $p = \infty$, the dual space demonstrates a distinct structure.

2.4.2 Sobolev $H^p(\Omega)$

The Sobolev space $H^p(\Omega)$ is a function space widely used in the study of partial differential equations (PDEs) and variational methods. It consists of functions defined on a domain Ω that possess certain degrees of smoothness, measured in terms of their derivatives [33].

Definition 2.1 *The Sobolev space $H^p(\Omega)$ is defined as the set of all functions u defined on the domain Ω such that their distributional derivatives up to order p belong to $L^2(\Omega)$. In other words, u and its derivatives up to order p are square-integrable on Ω.*

Notation: $H^p(\Omega)$ is often equipped with a norm called the Sobolev norm, denoted by $\| \cdot \|_{H^p(\Omega)}$, defined as:

$$\|u\|_{H^p(\Omega)} = \left(\sum_{|\alpha| \le p} \|D^\alpha u\|^2_{L^2(\Omega)} \right)^{1/2} \tag{2.24}$$

where α is a multi-index, $|\alpha|$ denotes the order of the derivative, and $D^\alpha u$ represents the distributional derivative of u with respect to the multi-index α.

Interpretation: Functions in $H^p(\Omega)$ possess a certain level of smoothness determined by the order of differentiation p. Higher values of p correspond to functions that are smoother, i.e., they have more continuous derivatives.

Example 2.1 *Functions in $H^1(\Omega)$ are typically referred to as "Sobolev functions of order 1" or simply "Sobolev functions". These functions have square-integrable first-order derivatives, implying they are locally absolutely continuous. Similarly, $H^2(\Omega)$ consists of functions with square-integrable second-order derivatives, and so on.*

Applications : Sobolev spaces are essential in the study of elliptic and parabolic PDEs, where solutions often lie in Sobolev spaces. They also play a crucial role in variational methods, where minimization problems are often formulated and solved in Sobolev spaces.

2.5 Galerkin method

Definition 2.2 *The Galerkin method is a powerful numerical technique used for approximating solutions to differential equations, especially partial differential equations (PDEs).*

Here are some key points about the Galerkin method:

Basis functions: In the Galerkin method, the solution is approximated using a finite linear combination of basis functions. These basis functions are chosen from a function space that satisfies certain properties, such as completeness and orthogonality.

Residual minimization: The Galerkin method minimizes the residual, which is the difference between the differential equation and its approximation. This is done by projecting the residual onto a finite-dimensional subspace spanned by the chosen basis functions [31].

Variational formulation: The method is closely related to the variational formulation of the differential equation. By multiplying the equation by a test function and integrating over the domain, the problem is transformed into an equivalent variational problem, which is then solved using the Galerkin approximation.

Finite element method (FEM): The Galerkin method is a foundational concept in the finite element method (FEM). In FEM, the domain is divided into smaller elements, and the Galerkin approximation is applied within each element. This allows for the efficient solution of complex PDEs over arbitrary domains [35].

Applications: Galerkin method and its variants find applications in various fields such as structural mechanics, fluid dynamics, heat transfer, electromagnetics, and more. It's particularly useful in problems where analytical solutions are difficult or impossible to obtain.

Accuracy and convergence: The accuracy of the Galerkin method depends on the choice of basis functions and the discretization of the domain.

Convergence analysis ensures that the approximation converges to the true solution as the number of basis functions or elements increases.

Computational efficiency: While the Galerkin method can be computationally demanding, especially for large-scale problems, it benefits from parallelization and optimization techniques, making it feasible for practical engineering simulations.

Overall, the Galerkin method provides a versatile framework for solving a wide range of differential equations numerically, offering a balance between accuracy, efficiency, and applicability.

Chapter 3

Methods and techniques

This chapter delves into the utilization of the Galerkin method to ascertain the existence and uniqueness of solutions within the framework of the nonlinear heat equation. Specifically, it investigates how this mathematical approach contributes to verifying both the presence and exclusivity of solutions in the context of a nonlinear heat equation defined in a semi-infinite domain.

3.1 Galerkin method overview

The Galerkin method involves iteratively refining an initial approximation to generate a sequence of increasingly accurate solutions. Subsequently, it aims to demonstrate that this sequence converges to a unique solution.

Consider the nonlinear heat equation in a semi-infinite domain:

$$u_t = \nabla \cdot (k(u)\nabla u) + |u|^p \quad \text{in} \quad \Omega_+^T = \mathbb{R}_+^n \times (0, \infty) \tag{3.1}$$

where $p > 1$ and $k(u)$ is a given coefficient.

Galerkin method steps

Weak formulation: To derive the weak form of the given PDE, we start by multiplying the equation by a test function v and integrating over the domain Ω_+^T:

$$\int_{\Omega_+^T} u_t v \, dx = \int_{\Omega_+^T} \nabla \cdot (k(u)\nabla u) v \, dx + \int_{\Omega_+^T} |u|^p v \, dx \tag{3.2}$$

Here,

u is the solution function, $k(u)$ is a coefficient function depending on u, $p > 1$ is a constant, $\Omega_+^T = \mathbb{R}_+^n \times (0, \infty)$ denotes the domain, v is a test function satisfying appropriate boundary conditions.

To find the weak formulation, we'll need to define appropriate function spaces for u and v and consider suitable boundary conditions. Typically, we would consider spaces $H^1(\Omega_+^T)$ for u and v and specify boundary conditions accordingly.

Next, we'll integrate by parts the term involving the Laplacian:

$$\int_{\Omega_+^T} \nabla \cdot (k(u)\nabla u) v \, dx = \int_{\partial\Omega_+^T} (k(u)\nabla u) \cdot \mathbf{n} \, v \, ds - \int_{\Omega_+^T} k(u)\nabla u \cdot \nabla v \, dx \quad (3.3)$$

where $\mathbf{n}$ is the outward unit normal vector on the boundary $\partial\Omega_+^T$.

Now, substituting this back into the original equation, we obtain the weak form of the given PDE:

$$\int_{\Omega_+^T} u_t v \, dx = \int_{\partial\Omega_+^T} (k(u)\nabla u) \cdot \mathbf{n} \, v \, ds - \int_{\Omega_+^T} k(u)\nabla u \cdot \nabla v \, dx + \int_{\Omega_+^T} |u|^p v \, dx \quad (3.4)$$

This weak formulation provides a basis for applying the Galerkin method, where we'll approximate the solution u and the test function v using finite-dimensional function spaces.

1. *Choose a finite-dimensional space:* For this, we will Select a suitable finite-dimensional subspace V_N within a function space that encapsulates potential solutions to the equation.

2. *Define approximation:* We will Express the solution $u(x, t)$ as an approximation $u_N(x, t)$ within the chosen subspace, often in terms of a finite number of basis functions.

3. *Substitute and project:* We will also Substitute the approximation $u_N(x, t)$ into the original equation and project it onto the finite-dimensional subspace using suitable inner product formulations.

4. *Derive system of equations:* We will try to Manipulate the resulting projected equation to derive a system of ordinary differential equations (ODEs) governing the coefficients of the approximation.

5. *Apply initial and boundary conditions:* Incorporate the initial condition $u(x, 0)$ and any relevant boundary conditions to determine the initial values and constraints for the coefficient system.

6. *Numerical solution:* Solve the system of ODEs numerically to obtain a sequence of approximations to the solution.

7. *Convergence analysis:* Analyze the convergence of the sequence of approximations as the dimension of the finite-dimensional space increases, ensuring that it converges to a unique solution.

By following the steps outlined above, the Galerkin method facilitates the rigorous establishment of the existence and uniqueness of solutions to the nonlinear heat equation within a semi-infinite domain.

3.2 Existence of solution

The following steps demonstrates the existence of a solution to the nonlinear heat equation:

1. Formulate the approximate problem

- Define $V_m \subset H^1(\mathbb{R}^n_+)$ using the first m eigenvectors $\{\phi_i\}_{i=1}^m$.
- Consider the approximate solution

$$u_m(x,t) = \sum_{i=1}^{m} c_i(t)\phi_i(x). \tag{3.5}$$

2. Galerkin approximation

- Substitute u_m into the equation and project onto ϕ_i to obtain ODEs for $c_i(t)$.

3. Existence of solution for the spproximate problem

- Solve the system of ODEs for $c_i(t)$ obtained in (2) using standard existence theorems.

A Priori estimates

- Establish bounds on u_m in $L^2(\mathbb{R}^n_+)$ and $H^1(\mathbb{R}^n_+)$ norms.

Compactness and passing to the limit

- Use compactness arguments to extract a convergent subsequence u_{m_k}.

Verification of the limit

- Show the limit function satisfies the weak form of the original equation.

3.3 Uniqueness of the solution using Galerkin

Strategy for proving uniqueness

1. *Assume the existence of two solutions:* Suppose there exist two solutions u and v to the problem and let $w = u - v$.

2. *Show that w satisfies a linear equation:* Subtract the equations satisfied by u and v to obtain an equation for w. This equation should be a linear equation or a system of linear equations.

3. *Use uniqueness of solutions to linear equations:* Apply the uniqueness theorem for linear equations or systems of linear equations to conclude that $w = 0$, implying that $u = v$.

Chapter 4

Existence and uniqueness of NLHE solution in semi-infinite boundary

This chapter explores the examination of the presence and distinctiveness of solutions to the nonlinear heat equation within a semi-infinite domain. We explore the application of advanced mathematical techniques, with a particular focus on the Galerkin method, to establish the robustness of solutions within this financial framework

the Galerkin method can be used to prove the uniqueness and existence of solutions for the heat equation in a semi-infinite domain. The Galerkin method is a powerful technique in functional analysis and numerical analysis for solving partial differential equations by approximating the solution in a finite-dimensional subspace.

4.1 Existence of the solution

To prove the existence of the solution we need to employ the Galerkin method to derive a sequence of approximate solutions u_m that converges to a limit function u as the number of basis functions tends to infinity for the nonlinear heat equation $u_t = \nabla \cdot (k(u)\nabla u) - |u|^3$ in the domain Ω_+^T, follow the steps given in subsection (4.2):

4.2 NLHE: Application of galerkin method

The expression (1.19) can be expressed as follows:

$$\begin{cases} u_t = \triangledown.(k(u) \triangledown u) - |u|^3 \quad \text{in} \quad \Omega_+^T = \mathbb{R}_+^n \times (0, \infty) \\ u(x, 0) = f(x), \quad x \in \mathbb{R}_+^n, \quad t > 0 \\ u|_{\Gamma_+} = g(x), \qquad \Gamma_+ = \partial\Omega_+^T \times (0, \infty) \end{cases} \tag{4.1}$$

To employ the Galerkin method for the nonlinear heat equation (4.1) within the domain $\Omega_+^T = \mathbb{R}_+^n \times (0, \infty)$ and under the given initial and boundary conditions, the following procedures should be undertaken:

Steps to follow

- Define a Hilbert space $H_0^1(\Omega)$ suitable for the problem.
- Choose a basis set of functions $\{w_j\}$ that form a complete orthonormal basis in $H_0^1(\Omega)$.
- Construct an approximate solution u_m in a subspace V_m generated by a finite number of basis functions.

Approximate problem:

- Determine $u_m \in V_m$ such that the Galerkin weak formulation is satisfied for all $v \in V_m$.
- This involves finding u_m that minimizes the residual of the weak form of the nonlinear heat equation.

Convergence:

- Show that as the number of basis functions m tends to infinity, the approximate solution u_m converges to the actual solution u in the function space $H_0^1(\Omega)$.

To define a suitable Hilbert space $H_0^1(\Omega)$ for the problem $u_t = \nabla \cdot (k(u)\nabla u) - |u|^3$ in the domain Ω_+^T, for simplicity we take $k(u) =$ constant, we need to consider the appropriate function space that satisfies the necessary boundary conditions and regularity requirements for the problem. here's how we can define $H_0^1(\Omega)$ for this specific problem:

Sobolev space $H^1(\Omega)$:

The sobolev space $H^1(\Omega)$ consists of functions u such that u and its first-order partial derivatives are square integrable over Ω. mathematically, this can be defined as:

$$H^1(\Omega) = \{u \in L^2(\Omega) : \nabla u \in L^2(\Omega)\} \tag{4.2}$$

Homogeneous dirichlet doundary conditions: The functions in $H_0^1(\Omega)$ satisfy the homogeneous Dirichlet boundary conditions, which means that the functions vanish on the boundary of the domain:

$$u(x) = 0 \quad \text{for } x \in \partial\Omega \tag{4.3}$$

We define on $H^1(\Omega)$ the inner product by:

$$(u, v)_{H^1(\Omega)} = \int_\Omega \nabla u \cdot \nabla v + uv \, dx \tag{4.4}$$

$H_0^1(\Omega) = \{u \in H_\Omega^1\} : u|_{\partial\Omega} = 0$ and H_Ω^1 is the closer $C_0^\infty(\Omega)$

By defining $H_0^1(\Omega)$ as the space of functions that belong to $H^1(\Omega)$ and satisfy the homogeneous Dirichlet boundary conditions, we ensure that the functions have the necessary regularity and boundary behavior for the problem $u_t = \nabla^2 u - |u|^3$ in the domain Ω_+^T. This choice of function space provides a suitable framework for studying the problem with the specified partial differential equation and domain.

Hilbert space structure: The space $H_0^1(\Omega)$ equipped with the norm induced by the inner product:

$$\|u\|_{H^1(\Omega)}^2 = \int_\Omega |\nabla u|^2 + |u|^2 \, dx \tag{4.5}$$

forms a Hilbert space, where the inner product is defined as:

To choose a basis set of functions $\{w_j\}$ that forms a complete orthonormal basis in $H_0^1(\Omega)$, we can utilize the concept of eigenfunctions of the Laplace operator with appropriate boundary conditions. In the context of the problem $u_t = \nabla \cdot (k(u)\nabla u) + |u|^3$ in the domain Ω_+^T with homogeneous Dirichlet boundary conditions, we can consider the eigenfunctions of the Laplace operator in the domain Ω with the Dirichlet boundary conditions.

The eigenfunctions of the Laplace operator in a bounded domain Ω with homogeneous Dirichlet boundary conditions form a complete orthonormal basis in $H_0^1(\Omega)$. These eigenfunctions are often referred to as the Dirichlet eigenfunctions or Dirichlet basis functions.

The eigenfunctions $\{w_j\}$ satisfy the following properties:

1. They are orthogonal with respect to the inner product in $H^1(\Omega)$:

$$\int_\Omega \nabla w_i \cdot \nabla w_j + w_i w_j \, dx = \delta_{ij} \tag{4.6}$$

   where δ_{ij} is the Kronecker delta.

2. They form a complete basis in $H_0^1(\Omega)$, meaning that any function in $H_0^1(\Omega)$ can be represented as a linear combination of these eigenfunctions.

3. They satisfy the homogeneous Dirichlet boundary conditions:

$$w_j(x) = 0 \quad \text{for } x \in \partial\Omega \tag{4.7}$$

The specific form of the eigenfunctions $\{w_j\}$ depends on the geometry and boundary conditions of the domain Ω. For simple geometries like rectangles, circles, or squares, the eigenfunctions can be explicitly determined using separation of variables and solving the corresponding eigenvalue problems.

In summary, by considering the eigenfunctions of the Laplace operator in the domain Ω with homogeneous Dirichlet boundary conditions, we can construct a basis set of functions $\{w_j\}$ that forms a complete orthonormal basis in $H_0^1(\Omega)$. These basis functions provide a convenient framework for representing functions in the function space $H_0^1(\Omega)$ and are essential for solving partial differential equations with the specified boundary conditions.

To construct an approximate solution u_m in a subspace V_m generated by a finite number of basis functions, we can use the Galerkin method, which involves approximating the solution by a finite-dimensional subspace of the original function space. Here's how we can construct the approximate solution u_m in V_m:

1. *Choose a basis set*: Select a finite set of basis functions that form a basis for the function space $H_0^1(\Omega)$. Let these basis functions be denoted as $\{w_1, w_2, \ldots, w_m\}$.

2. *Define the subspace V_m*: Construct the subspace V_m as the span of the chosen basis functions:

$$V_m = \text{span}\{w_1, w_2, \ldots, w_m\} \tag{4.8}$$

   This subspace is finite-dimensional and approximates the original function space $H_0^1(\Omega)$.

3. *Approximate solution u_m*: The approximate solution u_m in the subspace V_m is then defined as:

$$u_m(x,t) = \sum_{j=1}^{m} c_j(t) w_j(x) \tag{4.9}$$

   where $c_j(t)$ are time-dependent coefficients to be determined.

4. *Galerkin projection*: Substitute the approximate solution into the original partial differential equation and project onto the subspace V_m using the Galerkin method. This leads to a system of ordinary differential equations for the coefficients $c_j(t)$.

5. *Determine coefficients*: Solve the system of ordinary differential equations to determine the time-dependent coefficients $c_j(t)$ that best approximate the solution in the subspace V_m.

6. *Convergence analysis*: Study the convergence of the approximate solution u_m as the dimension of the subspace m increases. Convergence analysis helps assess the accuracy of the approximation and the behavior of the solution as more basis functions are included.

By constructing the approximate solution u_m in the subspace V_m generated by a finite number of basis functions, we can effectively approximate the original solution in the function space $H_0^1(\Omega)$ while reducing the computational complexity associated with infinite-dimensional function spaces.

This approach is commonly used in numerical methods for solving partial differential equations.

4.3 Approximate problem

To determine $u_m \in V_m$ such that the Galerkin weak formulation is satisfied for all $v \in V_m$, we need to consider the original partial differential equation and the weak form of the equation. The Galerkin weak formulation involves finding an approximate solution in a finite-dimensional subspace that satisfies the weak form of the original equation.

Given the original partial differential equation:

$$u_t = \nabla \cdot (k(u)\nabla u) - |u|^3 \tag{4.10}$$

and the weak form of the equation, which involves testing with a test function v, with $k(u) = 1$ considered as a constant and integrating over the domain Ω:

$$\int_\Omega u_t v \, dx = -\int_\Omega \nabla u \cdot \nabla v \, dx - \int_\Omega |u|^3 v \, dx \tag{4.11}$$

To determine $u_m \in V_m$ that satisfies the Galerkin weak formulation for all $v \in V_m$, we substitute the approximate solution into the weak form and require the resulting equation to hold for all test functions v in the subspace V_m.

The approximate solution in the subspace V_m is given by:

$$u_m(x,t) = \sum_{j=1}^{m} c_j(t) w_j(x) \tag{4.12}$$

Substitute this expression into the weak form of the equation and require it to hold for all test functions v in the subspace V_m. This leads to a system of equations for the coefficients $c_j(t)$ that ensures the Galerkin weak formulation is satisfied.

By solving this system of equations, we can determine the coefficients $c_j(t)$ that define the approximate solution u_m in the subspace V_m such that the Galerkin weak formulation is satisfied for all test functions v in the same subspace.

This process of finding the approximate solution that satisfies the weak form of the equation for all test functions in the chosen subspace is fundamental to the Galerkin method and its application in numerical methods for solving partial differential equations.

Substituting the expression for the approximate solution
$u_m(x,t) = \sum_{j=1}^{m} c_j(t)w_j(x)$

into the weak form of the original partial differential equation involves replacing u with u_m and testing with a test function v in the Galerkin weak formulation.

Given the weak form of the equation:

$$\int_\Omega u_t v \, dx = -\int_\Omega \nabla u \cdot \nabla v \, dx - \int_\Omega |u|^3 v \, dx \tag{4.13}$$

Substitute $u = u_m$ into the weak form:

$$\int_\Omega (u_m)_t v \, dx = -\int_\Omega \nabla u_m \cdot \nabla v \, dx - \int_\Omega |u_m|^3 v \, dx \tag{4.14}$$

Substitute the expression for u_m into the above equation:

$$\begin{aligned} \int_\Omega \left(\sum_{j=1}^{m} \dot{c}_j(t) w_j(x) \right) v \, dx = &-\int_\Omega \nabla \left(\sum_{j=1}^{m} c_j(t) w_j(x) \right) \cdot \nabla v \, dx \\ &- \int_\Omega \left(\sum_{j=1}^{m} c_j(t) w_j(x) \right)^3 v dx \end{aligned} \tag{4.15}$$

Also we subsitute the value of $v = \sum_{i=1}^{n} c_i(t)w_i(x)$ to get:

$$\begin{aligned} &\int_\Omega \left(\sum_{j=1}^{m} \dot{c}_j(t) w_j(x) \right) \left(\sum_{i=1}^{n} c_i(t) w_i(x) \right) dx \\ &= -\int_\Omega \nabla \left(\sum_{j=1}^{m} c_j(t) w_j(x) \right) \cdot \nabla \left(\sum_{i=1}^{n} c_i(t) w_i(x) \right) dx \\ &- \int_\Omega \left(\sum_{j=1}^{m} c_j(t) w_j(x) \right)^3 \left(\sum_{i=1}^{n} c_i(t) w_i(x) \right) dx \end{aligned} \tag{4.16}$$

This substitution will lead to an equation involving the coefficients $c_j(t)$ and the basis functions $w_j(x)$ that represent the approximate solution in the subspace V_m. This equation, when integrated over the domain Ω, will form the basis for deriving the system of equations that need to be solved to determine the coefficients $c_j(t)$ that satisfy the Galerkin weak formulation for all test functions in the subspace V_m.

Let's start by considering the orthogonality properties of the basis functions $w_i(x)$. Assume they are orthogonal with respect to the L^2 inner product over Ω:

$$\int_\Omega w_i(x)w_j(x)\,dx = \delta_{ij} \tag{4.17}$$

Where δ_{ij} is the Kronecker delta, equal to 1 if $i = j$ and 0 otherwise. Similarly, we assume:

$$\int_\Omega \nabla w_i(x) \cdot \nabla w_j(x)\,dx = \lambda_i \delta_{ij} \tag{4.18}$$

Where λ_i is a constant related to the eigenvalues of the Laplace operator with the corresponding boundary conditions.

Step-by-step derivation:

1. Substitute the expansion terms into the given equation:

$$\int_\Omega \left(\sum_{j=1}^{m} \dot{c}_j(t)w_j(x)\right)\left(\sum_{i=1}^{n} c_i(t)w_i(x)\right)\,dx \tag{4.19}$$

Simplify using orthogonality of $w_i(x)$:

$$\sum_{j=1}^{m}\sum_{i=1}^{n} \dot{c}_j(t)c_i(t)\int_\Omega w_j(x)w_i(x)\,dx = \sum_{j=1}^{m}\sum_{i=1}^{n} \dot{c}_j(t)c_i(t)\delta_{ij} \tag{4.20}$$

$$= \sum_{i=1}^{n} \dot{c}_i(t)c_i(t) \tag{4.21}$$

2. Substitute the gradient terms into the equation:

$$-\int_\Omega \nabla\left(\sum_{j=1}^{m} c_j(t)w_j(x)\right)\cdot\nabla\left(\sum_{i=1}^{n} c_i(t)w_i(x)\right)\,dx \tag{4.22}$$

Simplify using orthogonality of $\nabla w_i(x)$:

$$-\sum_{j=1}^{m}\sum_{i=1}^{n} c_j(t)c_i(t)\int_{\Omega} \nabla w_j(x)\cdot\nabla w_i(x)\,dx = -\sum_{j=1}^{m}\sum_{i=1}^{n} c_j(t)c_i(t)\lambda_i\delta_{ij} \\ = -\sum_{i=1}^{n}\lambda_i c_i^2(t) \tag{4.23}$$

3. Substitute the cubic terms into the equation:

$$-\int_{\Omega}\left(\sum_{j=1}^{m} c_j(t)w_j(x)\right)^3\left(\sum_{i=1}^{n} c_i(t)w_i(x)\right)\,dx \tag{4.24}$$

Due to orthogonality, we consider only the terms where indices match:

$$-\sum_{i=1}^{n} c_i(t)\int_{\Omega}\left(c_i(t)w_i(x)\right)^3\,dx \\ = -\sum_{i=1}^{n} c_i(t)c_i^3(t)\int_{\Omega} w_i^3(x)\,dx \tag{4.25}$$

Assuming the integral of $w_i(x)$ over Ω is normalized:

$$= -\sum_{i=1}^{n} c_i^4(t) \tag{4.26}$$

Putting it all together, we get:

$$\sum_{i=1}^{n}\dot{c}_i(t)c_i(t) = -\sum_{i=1}^{n}\lambda_i c_i^2(t) - \sum_{i=1}^{n} c_i^4(t) \tag{4.27}$$

Simplify to get the ODE for each i:

$$\dot{c}_i(t)c_i(t) = -\lambda_i c_i^2(t) - c_i^4(t) \tag{4.28}$$

Divide both sides by $c_i(t)$:

$$\dot{c}_i(t) = -\lambda_i c_i(t) - c_i^3(t) \tag{4.29}$$

Thus, the ODE for each $c_i(t)$ is:

$$\dot{c}_i(t) = -\lambda_i c_i(t) - c_i^3(t) \tag{4.30}$$

To solve the ordinary differential equation (ODE) $\dot{c}_i(t) = -\lambda_i c_i(t) - c_i^3(t)$, we will use separation of variables.

First, rewrite the ODE:

$$\frac{dc_i}{dt} = -\lambda_i c_i - c_i^3 \tag{4.31}$$

Separate the variables c_i and t:

$$\frac{dc_i}{-\lambda_i c_i - c_i^3} = dt \tag{4.32}$$

We can factor the denominator on the left-hand side:

$$\frac{dc_i}{-c_i(\lambda_i + c_i^2)} = dt \tag{4.33}$$

Now, integrate both sides. The left-hand side requires partial fraction decomposition:

$$\int \frac{1}{-c_i(\lambda_i + c_i^2)}\, dc_i = \int dt \tag{4.34}$$

Rewrite the left-hand side:

$$\int \frac{-1}{c_i(\lambda_i + c_i^2)}\, dc_i \tag{4.35}$$

Let $u = c_i^2$, then $du = 2c_i\, dc_i$, and $dc_i = \frac{du}{2c_i}$:

$$\begin{aligned} \int \frac{-1}{c_i(\lambda_i + u)} \cdot \frac{du}{2c_i} &= \int dt \\ \int \frac{-1}{2c_i^2(\lambda_i + u)}\, du &= \int dt \end{aligned} \tag{4.36}$$

Since $u = c_i^2$, $c_i^2 = u$:

$$\int \frac{-1}{2u(\lambda_i + u)}\, du = \int dt \tag{4.37}$$

Use partial fractions for $\frac{1}{u(\lambda_i + u)}$:

$$\frac{1}{u(\lambda_i + u)} = \frac{A}{u} + \frac{B}{\lambda_i + u} \tag{4.38}$$

Solving for A and B:

$$\begin{aligned} 1 &= A(\lambda_i + u) + Bu \\ 1 &= A\lambda_i \implies A = \frac{1}{\lambda_i} \end{aligned} \tag{4.39}$$

$A + B = 0, \implies B = -A$, Means $B = -\frac{1}{\lambda_i}$
Thus,

$$\frac{1}{u(\lambda_i + u)} = \frac{1}{\lambda_i}\left(\frac{1}{u} - \frac{1}{\lambda_i + u}\right) \tag{4.40}$$

So,

$$\begin{aligned} \int \frac{-1}{2u(\lambda_i + u)}\, du &= \int \frac{-1}{2\lambda_i}\left(\frac{1}{u} - \frac{1}{\lambda_i - u}\right) du \\ &= \frac{-1}{2\lambda_i}\left(\int \frac{1}{u}\, du - \int \frac{1}{\lambda_i + u}\, du\right) \\ &= \frac{-1}{2\lambda_i}\left(\ln|u| - \ln|\lambda_i + u|\right) \end{aligned} \tag{4.41}$$

Substitute back $u = c_i^2$:

$$\begin{aligned} \int \frac{-1}{2c_i^2(\lambda_i + c_i^2)}\, du &= \frac{-1}{2\lambda_i}\left(\ln|c_i^2| - \ln|\lambda_i + c_i^2|\right) \\ &= \frac{-1}{2\lambda_i} \ln\left|\frac{c_i^2}{\lambda_i - c_i^2}\right| \end{aligned} \tag{4.42}$$

Thus, the integrated form is:

$$\int \frac{-1}{2\lambda_i}\left(\frac{1}{c_i^2} - \frac{1}{\lambda_i - c_i^2}\right) dc_i = \frac{-1}{2\lambda_i} \ln\left|\frac{c_i^2}{\lambda_i - c_i^2}\right| + C \tag{4.43}$$

Combine the logs:

$$\frac{-1}{2\lambda_i} \ln\left|\frac{c_i^2}{\lambda_i - c_i^2}\right| = t + C \tag{4.44}$$

Exponentiating both sides gives:

$$\left|\frac{c_i^2}{\lambda_i - c_i^2}\right| = e^{-2\lambda_i(t+C)} \tag{4.45}$$

Let $K = e^{-2\lambda_i C}$, then:

$$\frac{c_i^2}{\lambda_i - c_i^2} = Ke^{-2\lambda_i t} \tag{4.46}$$

To solve for c_i from the equation

$$\frac{c_i^2}{\lambda_i - c_i^2} = Ke^{-2\lambda_i t} \tag{4.47}$$

Multiply both sides by $(\lambda_i - c_i^2)$ to eliminate the fraction:

$$c_i^2 = Ke^{-2\lambda_i t}(\lambda_i - c_i^2) \tag{4.48}$$

Expand and rearrange to get a quadratic equation in c_i^2:

$$\begin{aligned} c_i^2 + Ke^{-2\lambda_i t}c_i^2 &= Ke^{-2\lambda_i t}\lambda_i \\ c_i^2(1 + Ke^{-2\lambda_i t}) &= Ke^{-2\lambda_i t}\lambda_i \end{aligned} \tag{4.49}$$

Divide both sides by $(1 + Ke^{-2\lambda_i t})$ to solve for c_i^2:

$$c_i^2 = \frac{Ke^{-2\lambda_i t}\lambda_i}{1 + Ke^{-2\lambda_i t}} \tag{4.50}$$

Take the square root to solve for c_i, noting the positive root because $c_i \geq 0$:

$$c_i = \sqrt{\frac{Ke^{-2\lambda_i t}\lambda_i}{1 + Ke^{-2\lambda_i t}}} \tag{4.51}$$

Therefore, the solution for $c_i(t)$ is:

$$c_i(t) = \sqrt{\frac{Ke^{-2\lambda_i t}\lambda_i}{1 + Ke^{-2\lambda_i t}}} \tag{4.52}$$

This gives the explicit form of $c_i(t)$ in terms of the constants λ_i, K, and the parameter t.

After we replace the value of $c_i(t)$ in (4.12) to get the general solution.

$$u_m(x,t) = \sum_{j=1}^{m} \Big(\sqrt{\frac{Ke^{-2\lambda_i t}\lambda_i}{1 + Ke^{-2\lambda_i t}}}\Big).w_j(x) \tag{4.53}$$

4.4 Convergence

To show that as the number of basis functions m tends to infinity, the approximate solution u_m converges to the actual solution u in the function space $H_0^1(\Omega)$, we need to demonstrate convergence in the $H_0^1(\Omega)$ norm as m approaches infinity.See in [34]

The approximate solution u_m is defined as:

$$u_m(x,t) = \sum_{j=1}^{m} c_j(t) w_j(x) \tag{4.54}$$

The actual solution u satisfies the weak form of the original partial differential equation. As m tends to infinity, the approximate solution u_m should converge to the actual solution u in the function space $H_0^1(\Omega)$ if the Galerkin method is well-posed.

Convergence in the function space $H_0^1(\Omega)$ implies convergence in the $H^1(\Omega)$ norm, which includes the $L^2(\Omega)$ norm of the function and its first-order derivatives. The convergence of u_m to u in $H_0^1(\Omega)$ can be shown by proving the following:

1. Convergence in the $L^2(\Omega)$ norm:

$$\|u_m - u\|_{L^2(\Omega)} \to 0 \text{ as } m \to \infty \tag{4.55}$$

2. Convergence in the $H^1(\Omega)$ norm:

$$\|\nabla u_m - \nabla u\|_{L^2(\Omega)} \to 0 \text{ as } m \to \infty \tag{4.56}$$

By demonstrating these two convergence properties as m tends to infinity, we can establish that the approximate solution u_m converges to the actual solution u in the function space $H_0^1(\Omega)$. This convergence ensures that the Galerkin method provides an accurate approximation to the true solution of the nonlinear heat equation as the number of basis functions increases.

4.5 Stability of the solution

To prove the stability of the given nonlinear heat equation using the Galerkin method, we first need to define the weak formulation of the problem. Let V be the space of test functions, $H^1(\Omega_+)$, and V_h be the finite-dimensional subspace of V spanned by the basis functions. The weak formulation of the problem is given by finding $u \in V$ such that:

$$\int_{\Omega_+} u_t v + \int_{\Omega_+} \nabla u \cdot \nabla v + \int_{\Omega_+} |u|^3 v = 0 \tag{4.57}$$

for all $v \in V$, subject to appropriate boundary and initial conditions.

Applying the Galerkin method involves approximating the solution u by a finite-dimensional subspace V_h, where $u_h \in V_h$, and then finding u_h that satisfies the weak formulation for all $v_h \in V_h$. Let $u_h = \sum_{j=1}^{N} c_j(t)\phi_j(x)$ be the Galerkin approximation, where ϕ_j are the basis functions, and $c_j(t)$ are the coefficients to be determined.

Substituting u_h into the weak formulation, we obtain:

$$\int_{\Omega_+} \sum_{j=1}^{N} c_j'(t)\phi_j v + \int_{\Omega_+} \sum_{j=1}^{N} c_j(t)\nabla\phi_j \cdot \nabla v + \int_{\Omega_+} \left|\sum_{j=1}^{N} c_j(t)\phi_j\right|^3 v = 0 \tag{4.58}$$

Expanding the terms

$$\begin{aligned} &\int_{\Omega_+} \sum_{j=1}^{N} c_j'(t)\phi_j v + \int_{\Omega_+} \sum_{j=1}^{N} c_j(t)\nabla\phi_j \cdot \nabla v + \\ &\int_{\Omega_+} \left(\sum_{j=1}^{N} c_j(t)\phi_j\right)\left(\sum_{k=1}^{N} c_k(t)\phi_k\right)\left(\sum_{l=1}^{N} c_l(t)\phi_l\right) v \end{aligned} \tag{4.59}$$

and moving coefficients outside the integrals, we get a system of ordinary differential equations (ODEs) for the coefficients $c_j(t)$.

Therefore, the equation becomes:

$$\begin{aligned} &\sum_{j=1}^{N} \int_{\Omega_+} c_j'(t)\phi_j v + \sum_{j=1}^{N} c_j(t) \int_{\Omega_+} \nabla\phi_j \cdot \nabla v \\ &+ \int_{\Omega_+} \sum_{j=1}^{N}\sum_{k=1}^{N}\sum_{l=1}^{N} c_j(t)c_k(t)c_l(t)\phi_j\phi_k\phi_l v = 0 \end{aligned} \tag{4.60}$$

Now, we can move the coefficients outside the sums:

$$\sum_{j=1}^{N} c_j'(t) \int_{\Omega_+} \phi_j v + \sum_{j=1}^{N} c_j(t) \int_{\Omega_+} \nabla \phi_j \cdot \nabla v$$
$$+ \sum_{j=1}^{N} \sum_{k=1}^{N} \sum_{l=1}^{N} c_j(t) c_k(t) c_l(t) \int_{\Omega_+} \phi_j \phi_k \phi_l v = 0 \tag{4.61}$$

These equations constitute the system of ordinary differential equations for the coefficients $c_j(t)$

Solving this system numerically provides the approximation u_h.

To demonstrate stability, one usually uses energy estimates or Gronwall's inequality to show that the solution remains bounded with respect to some norms. For nonlinear problems like this, it might be more involved than linear problems. For more understanding we can read in this books [35]

4.6 Uniqueness of the solution

To prove the uniqueness of the solution to the nonlinear heat equation $u_t = \nabla \cdot (k(u)\nabla u) - |u|^3$ using the Galerkin method, we typically follow these steps:

Assume two solutions u_1 and u_2 to the equation . Define their difference as $w = u_1 - u_2$. Show that w satisfies the weak form of the equation and has certain properties. Utilize coercivity and regularity properties to demonstrate that w must be identically zero, implying that $u_1 = u_2$.

Let's outline these steps in detail:

1. Assume two solutions: Suppose we have two solutions u_1 and u_2 to the nonlinear heat equation $u_t = \nabla \cdot (k(u)\nabla u) + |u|^3$.

Define the Difference: Let $w = u_1 - u_2$.

Show that w Satisfies the Weak Formulation: Multiply the equation by a test function v, integrate over the domain, and manipulate the terms to obtain the weak form for w. This typically involves integration by parts and applying properties of the divergence and gradient operators.

To show that the difference $w = u_1 - u_2$ satisfies the weak formulation of the given nonlinear heat equation, we first multiply the equation by a test function v and integrate over the domain. Then we manipulate the terms to obtain the weak form for w.

Starting with the original equation (4.10),

Multiply by the test function v and integrate over the domain Ω:

$$\int_\Omega u_t v\,dx = \int_\Omega \nabla\cdot(\nabla u)v\,dx + \int_\Omega |u|^2 v\,dx \tag{4.62}$$

Using integration by parts on the second term on the right-hand side:

$$\int_\Omega \nabla\cdot(\nabla u)v\,dx = -\int_\Omega \nabla u\cdot\nabla v\,dx + \int_{\partial\Omega}(\nabla u)\cdot\mathbf{n}\,v\,ds \tag{4.63}$$

Thus, the equation becomes:

$$\int_\Omega u_t v\,dx = -\int_\Omega \nabla u\cdot\nabla v\,dx + \int_{\partial\Omega}(\nabla u)\cdot\mathbf{n}\,v\,ds + \int_\Omega |u|^2 v\,dx \tag{4.64}$$

Substituting $w = u_1 - u_2$ gives:

$$\int_\Omega (u_{1t}-u_{2t})v\,dx = -\int_\Omega \nabla(u_1-u_2)\cdot\nabla v\,dx + \int_{\partial\Omega}(\nabla(u_1-u_2))\cdot\mathbf{n}\,v\,ds + \int_\Omega |u_1-u_2|^2 v\,dx \tag{4.65}$$

Now, we define $w = u_1 - u_2$ and the equation simplifies to:

$$\int_\Omega w_t v\,dx = -\int_\Omega \nabla w\cdot\nabla v\,dx + \int_{\partial\Omega}(\nabla w)\cdot\mathbf{n}\,v\,ds + \int_\Omega |w|^2 v\,dx \tag{4.66}$$

This is the weak formulation for w, the difference between the two solutions u_1 and u_2 to the nonlinear heat equation.

2. Utilize coercivity and regularity: Use coercivity properties of the bilinear form and regularity properties of the solutions and test functions to bound w and prove that it vanishes. This step may involve estimating the norms of w in suitable function spaces.

To bound w and prove that it vanishes, we can utilize the coercivity properties of the bilinear form and regularity properties of the solutions and test functions. This will involve estimating the norms of w in suitable function spaces.

First, let's consider the weak formulation of w in equation (4.66):

Now, we can use the coercivity property of the bilinear form to obtain a bound on w. Suppose V is a suitable function space (for instance, $H_0^1(\Omega)$ for bounded domains Ω). The coercivity property implies that there exists a constant $C > 0$ such that:

$$\int_\Omega |\nabla w|^2 \, dx \geq C \int_\Omega |w|^2 \, dx \tag{4.67}$$

This inequality allows us to bound $\|w\|_{H_0^1(\Omega)}$ by $\|w_t\|_{(H_0^1(\Omega))^*}$.

Next, we consider the regularity properties of the solutions and test functions. Let's assume that u_1 and u_2 are sufficiently regular solutions (for instance, $u_1, u_2 \in H^2(\Omega)$). Similarly, let $v \in H_0^1(\Omega)$ be a sufficiently regular test function.

Now, we can use the above coercivity property to estimate the norm of w in the $H_0^1(\Omega)$ space:

$$\|w\|_{H_0^1(\Omega)}^2 = \int_\Omega |\nabla w|^2 \, dx \geq C \int_\Omega |w|^2 \, dx \tag{4.68}$$

This implies that:

$$\|w\|_{H_0^1(\Omega)} \geq \sqrt{C} \|w\|_{L^2(\Omega)} \tag{4.69}$$

Therefore, if w belongs to $H_0^1(\Omega)$, it must also belong to $L^2(\Omega)$, and its norm in the L^2 space is bounded by its norm in the H_0^1 space.

3. Incorporate boundary and initial conditions: Ensure that the boundary and initial conditions are consistent and that the solutions satisfy these conditions.

4. Analyze the Galerkin approximation: Show that the Galerkin method preserves uniqueness as the number of basis functions tends to infinity. This step is crucial for establishing uniqueness in the context of numerical approximation.

5. Consider nonlinear terms: Carefully analyze the effect of the nonlinear term $|u|^2$ on the uniqueness of the solution.

6. Stability and well-posedness: Ensure that the problem is well-posed and that the solution remains stable under small perturbations.

By rigorously following these steps and conducting a thorough analysis, you can establish the uniqueness of the solution to the nonlinear heat equation using the Galerkin method.

4.7 Applications

Example 4.1 *Nonlinear Heat Equation in a Semi-Infinite Domain*
Consider the nonlinear heat equation in a semi-infinite domain:

$$u_t = \frac{d}{dx}\left((1+u^3)\frac{du}{dx}\right) + \sin(u), \quad x \in (0,\infty), \quad t > 0 \tag{4.70}$$

with the following conditions:

$$\begin{aligned}
&\textit{Thermal conductivity:} \quad k(u) = 1 + u^3 \\
&\textit{Heat source/sink term:} \quad f(u) = \sin(u) \\
&\textit{Initial condition:} \quad u(x,0) = e^{-x^2} \\
&\textit{Boundary condition at } x = 0: \quad u(0,t) = 0 \\
&\textit{Boundary condition as } x \to \infty: \quad \lim_{x\to\infty} u(x,t) = 0
\end{aligned}$$

Solving this problem involves numerical methods or analytical techniques. The Galerkin method can be employed to obtain approximate solutions.

We need to apply the Galerkin method to prove the existence, uniqueness and the stability to the given nonlinear heat equation in example (4.70) :
Choose a finite-dimensional function space:

Selecting suitable basis functions for the Galerkin method involves considering the characteristics of the problem, such as boundary conditions and the behavior of the solution. In this scenario, as the domain extends infinitely and the solution diminishes as $x \to \infty$, it's typical to opt for basis functions that are orthogonal and adept at representing such decay.

Trigonometric functions, like sine and cosine, are frequently chosen for situations with periodic or oscillatory tendencies. Nevertheless, given the requirement for decay as $x \to \infty$, it may be more appropriate to utilize basis functions that demonstrate both exponential decay and oscillations.

For such cases, a popular approach is to employ a blend of exponential and trigonometric functions, known as Fourier-Bessel functions. These functions can be expressed as:

$$\phi_n(x) = J_0(\lambda_n x) \cdot e^{-\lambda_n x}, \tag{4.71}$$

where J_0 represents the Bessel function of the first kind with order zero, and λ_n denotes the roots of J_0. These functions possess the characteristic of exponential decay as $x \to \infty$ while also manifesting oscillatory behavior.

Weak formulation:

To formulate the weak form of the given nonlinear heat equation (4.70), we start by multiplying the equation by a test function v and integrating over the domain $(0, \infty)$ with respect to both x and t.

The weak form of the equation is obtained as follows:

$$\int_0^\infty \int_0^\infty u_t v \, dx \, dt = \int_0^\infty \int_0^\infty \frac{d}{dx}\left((1+u^3)\frac{du}{dx}\right) v \, dx \, dt + \int_0^\infty \int_0^\infty \sin(u) v \, dx \, dt \tag{4.72}$$

Now, we can apply integration by parts to the first integral on the right-hand side:

$$\begin{aligned} \int_0^\infty \int_0^\infty u_t v \, dx \, dt = \int_0^\infty \left[(1+u^3)\frac{du}{dx}v\right]_0^\infty - \int_0^\infty \left((1+u^3)\frac{du}{dx}\right)\frac{dv}{dx}\, dx \\ + \int_0^\infty \int_0^\infty \sin(u) v \, dx \, dt \end{aligned} \tag{4.73}$$

Applying the boundary conditions $u(0,t) = 0$ and $\lim_{x\to\infty} u(x,t) = 0$, the first term on the right-hand side vanishes. Thus, we obtain the weak form of the equation: Expand solution and test functions:

$$\int_0^\infty \int_0^\infty u_t v \, dx \, dt = -\int_0^\infty \left((1+u^3)\frac{du}{dx}\right)\frac{dv}{dx}\, dx + \int_0^\infty \int_0^\infty \sin(u) v \, dx \, dt \tag{4.74}$$

This weak form is now suitable for approximation using the Galerkin method, where the solution and the test function are approximated using finite-dimensional spaces.

Expand solution and test functions: Express the solution and the test function as linear combinations of the chosen basis functions. This step involves introducing coefficients for each basis function.

Then, we can express the solution u and the test function v as:

$$u(x,t) = \sum_{n=1}^{N} c_n(t)\phi_n(x) \tag{4.75}$$

$$v(x) = \sum_{m=1}^{M} d_m\phi_m(x) \tag{4.76}$$

Here, $c_n(t)$ and d_m are the coefficients of the basis functions for the solution and the test function, respectively. $\phi_n(x)$ is the chosen basis (4.71) N and M denote the number of basis functions chosen for the solution and the test function, respectively.

Substituting the expansions of u and v into the equation (4.74), we obtain:

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \sum_{n=1}^{N} c_n'(t)\phi_n(x) \sum_{m=1}^{M} d_m\phi_m(x)\,dx\,dt \\
&= -\int_0^\infty \sum_{n=1}^{N} \left((1 + \sum_{k=1}^{N} c_k(t)\phi_k(x))^2 \frac{d}{dx} \left(\sum_{k=1}^{N} c_k(t)\phi_k(x) \right) \right) \frac{d}{dx} \left(\sum_{m=1}^{M} d_m\phi_m(x) \right) dx \\
&\quad + \int_0^\infty \int_0^\infty \sin\left(\sum_{n=1}^{N} c_n(t)\phi_n(x) \right) \sum_{m=1}^{M} d_m\phi_m(x)\,dx\,dt \\
&= \int_0^\infty \int_0^\infty \sin\left(\sum_{n=1}^{N} c_n(t)\phi_n(x) \right) \sum_{m=1}^{M} d_m\phi_m(x)\,dx\,dt
\end{aligned} \tag{4.77}$$

Differentiating with respect to t and we get:

$$\frac{d}{dt}\left(\int_0^\infty \int_0^\infty \sum_{n=1}^{N} c_n'(t)\phi_n(x) \sum_{m=1}^{M} d_m\phi_m(x)\,dx\,dt\right)$$
$$= \frac{d}{dt}\left(-\int_0^\infty \sum_{n=1}^{N}\left((1+\sum_{k=1}^{N} c_k(t)\phi_k(x))^2 \frac{d}{dx}\left(\sum_{k=1}^{N} c_k(t)\phi_k(x)\right)\right) \frac{d}{dx}\left(\sum_{m=1}^{M} d_m\phi_m(x)\right)\,dx\right.$$
$$\left. + \int_0^\infty \int_0^\infty \sin\left(\sum_{n=1}^{N} c_n(t)\phi_n(x)\right) \sum_{m=1}^{M} d_m\phi_m(x)\,dx\,dt\right) \tag{4.78}$$

Let's denote $F(c_n(t)) = \int_0^\infty \int_0^\infty \sin\left(\sum_{n=1}^{N} c_n(t)\phi_n(x)\right) \sum_{m=1}^{M} d_m\phi_m(x)\,dx\,dt$. Then we have:

$$\frac{d}{dt}\left(\int_0^\infty \int_0^\infty \sum_{n=1}^{N} c_n'(t)\phi_n(x) \sum_{m=1}^{M} d_m\phi_m(x)\,dx\,dt\right)$$
$$= \frac{d}{dt}\left(-\int_0^\infty \sum_{n=1}^{N}\left((1+\sum_{k=1}^{N} c_k(t)\phi_k(x))^2\right) \frac{d}{dx}\left(\sum_{k=1}^{N} c_k(t)\phi_k(x)\right) \frac{d}{dx}\left(\sum_{m=1}^{M} d_m\phi_m(x)\right)\,dx\right)$$
$$+ F(c_n(t)) \tag{4.79}$$

Now, we'll rearrange terms and isolate $c_n'(t)$:

$$\int_0^\infty \sum_{n=1}^{N} c_n'(t)\left(\int_0^\infty \phi_n(x) \sum_{m=1}^{M} d_m\phi_m(x)\,dx\right)\,dt$$
$$= -\int_0^\infty \sum_{n=1}^{N}\left((1+\sum_{k=1}^{N} c_k(t)\phi_k(x))^2 \frac{d}{dx}\left(\sum_{k=1}^{N} c_k(t)\phi_k(x)\right)\right) \frac{d}{dx}\left(\sum_{m=1}^{M} d_m\phi_m(x)\right)\,dx$$
$$+ F(c_n(t)) \tag{4.80}$$

This gives us a system of ODEs for

$$c_n(t) : c_n'(t) \left(\int_0^\infty \phi_n(x) \sum_{m=1}^{M} d_m \phi_m(x)\, dx \right)$$

$$= -\sum_{n=1}^{N} \left((1 + \sum_{k=1}^{N} c_k(t)\phi_k(x))^2 \frac{d}{dx} \left(\sum_{k=1}^{N} c_k(t)\phi_k(x) \right) \right) \tag{4.81}$$

$$\frac{d}{dx} \left(\sum_{m=1}^{M} d_m \phi_m(x) \right) + F(c_n(t))$$

To solve the above ODES, we need to replace (4.71) in equation (4.81), the system becomes:

$$c_n'(t) \left(\int_0^\infty J_0(\lambda_n x) e^{-\lambda_n x} \sum_{m=1}^{M} d_m J_0(\lambda_m x) e^{-\lambda_m x}\, dx \right)$$

$$= -\sum_{n=1}^{N} \left((1 + \sum_{k=1}^{N} c_k(t) J_0(\lambda_k x) e^{-\lambda_k x})^2 \frac{d}{dx} \left(\sum_{k=1}^{N} c_k(t) J_0(\lambda_k x) e^{-\lambda_k x} \right) \right)$$

$$\frac{d}{dx} \left(\sum_{m=1}^{M} d_m J_0(\lambda_m x) e^{-\lambda_m x} \right) + F(c_n(t)) \tag{4.82}$$

Uniqueness of the solution

To prove the uniqueness of the solution of the given nonlinear heat equation using the energy method, we'll define an energy functional associated with the equation and demonstrate that it is monotonically decreasing with respect to time.

The energy functional for the given equation can be defined as follows:

$$E(t) = \int_0^\infty \left(\frac{1}{2} \left(1 + u^3\right) \left(\frac{du}{dx} \right)^2 + F(u) \right) dx \tag{4.83}$$

Where $F(u)$ is the antiderivative of $f(u) = \sin(u)$.

Now, let's differentiate $E(t)$ with respect to time t:

$$\frac{dE}{dt} = \int_0^\infty \left(\frac{\partial}{\partial t} \left[\frac{1}{2} \left(1 + u^3\right) \left(\frac{du}{dx} \right)^2 \right] + \frac{\partial}{\partial t} F(u) \right) dx \tag{4.84}$$

Using the given equation $u_t = \frac{d}{dx}\left((1+u^3)\frac{du}{dx}\right) + \sin(u)$, we can express $\frac{\partial}{\partial t}\left[\frac{1}{2}\left(1+u^3\right)\left(\frac{du}{dx}\right)^2\right]$ as:

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\left(1+u^3\right)\left(\frac{du}{dx}\right)^2\right] = \left(1+u^3\right)\frac{du}{dx}\frac{\partial}{\partial t}\left(\frac{du}{dx}\right) + \frac{1}{2}\left(\frac{d}{dt}(1+u^3)\right)\left(\frac{du}{dx}\right)^2 \tag{4.85}$$

And $\frac{\partial}{\partial t}F(u)$ is simply $\sin(u)u_t$.

Substituting these into the expression for $\frac{dE}{dt}$, we get:

$$\frac{dE}{dt} = \int_0^\infty \left(\left(1+u^3\right)\frac{du}{dx}\frac{\partial}{\partial t}\left(\frac{du}{dx}\right) + \frac{1}{2}\left(\frac{d}{dt}(1+u^3)\right)\left(\frac{du}{dx}\right)^2 + \sin(u)u_t\right) dx \tag{4.86}$$

Now, let's integrate the equation u_t with respect to x from 0 to ∞:

$$\int_0^\infty u_t\, dx = \frac{d}{dt}\int_0^\infty u\, dx \tag{4.87}$$

Using the given boundary conditions, the left-hand side becomes 0, thus:

$$0 = \frac{d}{dt}\int_0^\infty u\, dx \tag{4.88}$$

This implies that the integral of u with respect to x is a constant with respect to time.

Therefore, $\frac{dE}{dt} = 0$, implying that the energy $E(t)$ is conserved over time. Since $E(t)$ is nonnegative and constant with respect to time, it follows that $E(t)$ attains its minimum value at $t = 0$.

Now, let's consider the initial condition $u(x,0) = e^{-x^2}$. For this initial condition, $E(0) = \int_0^\infty \frac{1}{2}e^{-2x^2}\, dx$, which is finite.
To integrate

$$\int_0^\infty \frac{1}{2}e^{-2x^2}\, dx \tag{4.89}$$

Let's denote the integral as I:

$$I = \int_0^\infty \frac{1}{2}e^{-2x^2}\, dx \tag{4.90}$$

Now, let's consider the square of I:

$$
\begin{aligned}
I^2 &= \left(\int_0^\infty \frac{1}{2} e^{-2x^2}\, dx\right)\left(\int_0^\infty \frac{1}{2} e^{-2y^2}\, dy\right) \\
I^2 &= \int_0^\infty \int_0^\infty \frac{1}{4} e^{-2x^2-2y^2}\, dx\, dy
\end{aligned}
\tag{4.91}
$$

We can switch to polar coordinates, where $x = r\cos(\theta)$ and $y = r\sin(\theta)$, and $dx\, dy = r\, dr\, d\theta$:

$$
\begin{aligned}
I^2 &= \int_0^{2\pi} \int_0^\infty \frac{1}{4} e^{-2r^2} r\, dr\, d\theta \\
I^2 &= \frac{1}{4} \int_0^{2\pi} \left(\int_0^\infty e^{-2r^2} r\, dr\right) d\theta
\end{aligned}
\tag{4.92}
$$

The inner integral is a standard Gaussian integral. Let $u = r^2$, then $du = 2r\, dr$:

$$
\int_0^\infty e^{-2r^2} r\, dr = \frac{1}{2} \int_0^\infty e^{-u}\, du = \frac{1}{2} \tag{4.93}
$$

Therefore:

$$
I^2 = \frac{1}{4} \int_0^{2\pi} \frac{1}{2}\, d\theta = \frac{\pi}{8} \tag{4.94}
$$

Thus, $I = \frac{\sqrt{\pi}}{2\sqrt{2}}$. So,

$$
\int_0^\infty \frac{1}{2} e^{-2x^2}\, dx = \frac{\sqrt{\pi}}{2\sqrt{2}} \tag{4.95}
$$

Furthermore, since $E(t)$ is nonnegative and constant, it can not decrease below $E(0)$. Therefore, there exists a unique solution to the given nonlinear heat equation satisfying the specified initial and boundary conditions. This proves the uniqueness of the solution using the energy method.

Uniqueness

To prove uniqueness, assume there are two solutions u_1 and u_2. Consider their difference $w = u_1 - u_2$. Then w satisfies:

$$
w_t = \Delta w - (|u_1|^3 - |u_2|^3) \tag{4.96}
$$

Multiply by w and integrate over R_+^n:

$$\frac{1}{2}\frac{d}{dt}\|w\|^2_{L^2(R^n_+)} + \|\nabla w\|^2_{L^2(R^n_+)} = \int_{R^n_+}(|u_1|^3 - |u_2|^3)w\,dx \tag{4.97}$$

Using the Lipschitz continuity of $|u|^3$:

$$\int_{R^n_+}(|u_1|^3 - |u_2|^3)w\,dx \le C\int_{R^n_+}|u_1 - u_2|^2|w|\,dx \tag{4.98}$$

Applying the Hölder inequality:

$$\int_{R^n_+}|u_1 - u_2|^2|w|\,dx \le \left(\int_{R^n_+}|u_1 - u_2|^4\,dx\right)^{\frac{1}{2}}\left(\int_{R^n_+}|w|^2\,dx\right)^{\frac{1}{2}} \tag{4.99}$$

Using Sobolev embedding $H^1(R^n_+) \hookrightarrow L^4(R^n_+)$:

$$\|u_1 - u_2\|_{L^4(R^n_+)} \le C\|u_1 - u_2\|_{H^1(R^n_+)} \tag{4.100}$$

Thus,

$$\int_{R^n_+}(|u_1|^3 - |u_2|^3)w\,dx \le C\|u_1 - u_2\|^2_{H^1(R^n_+)}\|w\|_{L^2(R^n_+)} \tag{4.101}$$

Now we have:

$$\frac{1}{2}\frac{d}{dt}\|w\|^2_{L^2(R^n_+)} \le C\|w\|_{L^2(R^n_+)} \tag{4.102}$$

Grönwall's inequality then implies:

$$\|w(t)\|_{L^2(R^n_+)} \le \|w(0)\|_{L^2(R^n_+)}e^{Ct} \tag{4.103}$$

If $w(0) = 0$, then $\|w(t)\|_{L^2} = 0$ for all $t \ge 0$, meaning $u_1 = u_2$. This proves the uniqueness of the solution.

Example 4.2 *Lets consider nonlinear heat equation in 1-D (One dimension).*

$$u_t = u_{xx} - |u|^3 \qquad in \qquad \Omega_+ = \mathbb{R}^n_+ \times (0,\infty) \tag{4.104}$$

To derive the weak formulation of the partial differential equation (4.104) using the Galerkin method, we need to follow these steps:

1. **Multiply by a test function**: Multiply the equation by a test function $v \in H_0^1(\Omega_+)$.
2. **Integrate by parts**: Apply integration by parts to the spatial derivative terms.
3. **Impose boundary conditions**: Incorporate boundary conditions (if any) into the weak formulation.

Step-by-step derivation

Start with the PDE:

$$u_t = u_{xx} - |u|^3 \quad \text{in} \quad \Omega_+ = \mathbb{R}_+^n \times (0, \infty) \tag{4.105}$$

Multiply by a Test Function v:

$$\int_{\Omega_+} u_t v \, dx \, dt = \int_{\Omega_+} (u_{xx} - |u|^3) v \, dx \, dt \tag{4.106}$$

Integrate by Parts:

For the spatial derivative term, integrate by parts:

$$\int_{\Omega_+} u_{xx} v \, dx \, dt = \int_{\Omega_+} \frac{\partial^2 u}{\partial x^2} v \, dx \, dt \tag{4.107}$$

Applying integration by parts to u_{xx}:

$$\int_{\Omega_+} \frac{\partial^2 u}{\partial x^2} v \, dx = \left[\frac{\partial u}{\partial x} v \right]_{\partial\Omega} - \int_{\Omega_+} \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} \, dx \tag{4.108}$$

Assuming $v = 0$ on the boundary (for simplicity, or considering appropriate boundary conditions), the boundary term vanishes:

$$\int_{\Omega_+} \frac{\partial^2 u}{\partial x^2} v \, dx = -\int_{\Omega_+} \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} \, dx \tag{4.109}$$

Combine the terms:

Substitute back into the original equation:

$$\int_{\Omega_+} u_t v \, dx \, dt = -\int_{\Omega_+} \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} \, dx \, dt - \int_{\Omega_+} |u|^3 v \, dx \, dt \tag{4.110}$$

Weak Formulation:

The weak formulation of the problem is:

$$\int_{\Omega_+} u_t v \, dx \, dt + \int_{\Omega_+} \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} \, dx \, dt + \int_{\Omega_+} |u|^3 v \, dx \, dt = 0 \tag{4.111}$$

Using the Galerkin method

To apply the Galerkin method, we approximate the solution u by a finite number of basis functions ϕ_i. Let

$$u^N(x,t) = \sum_{i=1}^{N} c_i(t) \phi_i(x), \tag{4.112}$$

where ϕ_i are chosen basis functions.

The weak formulation in the Galerkin method becomes:

$$\begin{aligned} &\int_{\Omega_+} \left(\sum_{i=1}^{N} \dot{c}_i(t) \phi_i \right) \phi_j \, dx + \int_{\Omega_+} \left(\sum_{i=1}^{N} c_i(t) \frac{\partial \phi_i}{\partial x} \right) \frac{\partial \phi_j}{\partial x} \, dx \\ &+ \int_{\Omega_+} \left(\sum_{i=1}^{N} c_i(t) \phi_i \right)^3 \phi_j \, dx = 0 \quad \forall j = 1, 2, \ldots, N \end{aligned} \tag{4.113}$$

This results in a system of ordinary differential equations for the coefficients $c_i(t)$:

$$\begin{aligned} &\sum_{i=1}^{N} \dot{c}_i(t) \int_{\Omega_+} \phi_i \phi_j \, dx + \sum_{i=1}^{N} c_i(t) \int_{\Omega_+} \frac{\partial \phi_i}{\partial x} \frac{\partial \phi_j}{\partial x} \, dx \\ &+ \int_{\Omega_+} \left(\sum_{i=1}^{N} c_i(t) \phi_i \right)^3 \phi_j \, dx = 0 \quad \forall j = 1, 2, \ldots, N \end{aligned} \tag{4.114}$$

Step-by-step solution

Identify the terms in the equation:

$$\sum_{i=1}^{N} \dot{c}_i(t) M_{ij} + \sum_{i=1}^{N} c_i(t) K_{ij} + N_j(t) c_i^3(t) = 0 \tag{4.115}$$

where:

$$
\begin{aligned}
M_{ij} &= \int_{\Omega^+} \phi_i \phi_j \, dx \\
K_{ij} &= \int_{\Omega^+} \frac{\partial \phi_i}{\partial x} \frac{\partial \phi_j}{\partial x} \, dx \\
N_j(c(t)) &= \int_{\Omega^+} (\phi_k)^3 \, \phi_j \, dx
\end{aligned} \tag{4.116}
$$

Rearrange the equation:

$$
\sum_{i=1}^{N} \dot{c}_i(t) M_{ij} + \sum_{i=1}^{N} c_i(t) K_{ij} + \int_{\Omega^+} \left(\sum_{k=1}^{N} c_k(t) \phi_k \right)^3 \phi_j \, dx = 0 \tag{4.117}
$$

Matrix form representation:
Let $c(t)$ be the vector of $c_i(t)$:

$$
c(t) = \begin{bmatrix} c_1(t) \\ c_2(t) \\ \vdots \\ c_N(t) \end{bmatrix} \tag{4.118}
$$

Simplifying assumptions

Assume M and K are diagonal matrices, which simplifies the problem. This assumption is valid if the basis functions ϕ_i are orthogonal.

$$
M_{ij} = \delta_{ij} M_i, \quad K_{ij} = \delta_{ij} K_i \tag{4.119}
$$

With these assumptions, the equation simplifies to:

$$
M_i \dot{c}_i(t) + K_i c_i(t) + N_j(t) c_j^3(t) = 0 \tag{4.120}
$$

where

$$
N_j(t) = \int_{\Omega^+} (\phi_k)^3 \, \phi_j \, dx \tag{4.121}
$$

Nonlinear Term Simplification

To handle the nonlinear term, let's assume a small perturbation solution where $c_i(t)$ are small and can be expanded in a series. For simplicity, consider $N = 1$, so $c(t)$ is a single function.

Perturbation Method

Assume the solution can be expressed as:

$$c(t) = c_0(t) + \epsilon c_1(t) + \epsilon^2 c_2(t) + \cdots \epsilon^n c_n(t) \tag{4.122}$$

Substitute this into the simplified equation and collect terms of the same order in ϵ.

Leading-Order Term (Order ϵ^0)

Starting with the leading-order term:

$$M\dot{c}_0(t) + Kc_0(t) = 0 \tag{4.123}$$

This is a simple linear ordinary differential equation (ODE). We solve this equation to get $c_0(t)$:

$$\dot{c}_0(t) = -\frac{K}{M}c_0(t) \tag{4.124}$$

The solution to the ODE (4.124) is:

$$c_0(t) = Ae^{-\frac{K}{M}t} \tag{4.125}$$

Here, A is a constant determined by the initial condition.

First-Order Term (Order ϵ^1)

Next, we consider the first-order term:

$$M\dot{c}_1(t) + Kc_1(t) + N(c_0(t)) = 0 \tag{4.126}$$

To find $N(c_0(t))$, we need to evaluate the nonlinear term at $c_0(t)$:

$$N(c_0(t)) = \int_{\Omega_+} (c_0(t)\phi)^3 \phi \, dx \tag{4.127}$$

Since $c_0(t) = Ae^{-\frac{K}{M}t}$, we have:

$$N(c_0(t)) = \int_{\Omega_+} \left(Ae^{-\frac{K}{M}t}\phi\right)^3 \phi \, dx \tag{4.128}$$

Simplifying this:

$$N(c_0(t)) = A^3 e^{-3\frac{K}{M}t} \int_{\Omega_+} \phi^4 \, dx = A^3 e^{-3\frac{K}{M}t} N_\phi \tag{4.129}$$

where $N_\phi = \int_{\Omega_+} \phi^4 \, dx$ is a constant depending on the basis function ϕ.

Now, substitute $N(c_0(t))$ into the first-order equation (4.126):

$$M\dot{c}_1(t) + Kc_1(t) + A^3 e^{-3\frac{K}{M}t} N_\phi = 0 \tag{4.130}$$

This is another linear ODE for $c_1(t)$:

$$\dot{c}_1(t) = -\frac{K}{M} c_1(t) - \frac{A^3 N_\phi}{M} e^{-3\frac{K}{M}t} \tag{4.131}$$

The solution to this ODE can be found using an integrating factor. The integrating factor is $e^{\frac{K}{M}t}$:

$$c_1(t) e^{\frac{K}{M}t} = -\frac{A^3 N_\phi}{M} \int e^{-\frac{2K}{M}t} \, dt + C_1 \tag{4.132}$$

Solving the integral:

$$\int e^{-\frac{2K}{M}t} \, dt = -\frac{M}{2K} e^{-\frac{2K}{M}t} \tag{4.133}$$

Thus:

$$c_1(t) e^{\frac{K}{M}t} = \frac{A^3 N_\phi}{2K} e^{-\frac{2K}{M}t} + C_1 \tag{4.134}$$

Multiplying by $e^{-\frac{K}{M}t}$:

$$c_1(t) = \frac{A^3 N_\phi}{2K} e^{-3\frac{K}{M}t} + C_1 e^{-\frac{K}{M}t} \tag{4.135}$$

Here, C_1 is another constant determined by the initial condition for $c_1(t)$.

Higher-order term

The process for finding higher-order terms follows a similar pattern. For the second-order term $c_2(t)$, we would set up the ODE:

$$M\dot{c}_2(t) + Kc_2(t) + N'(c_0(t), c_1(t)) = 0 \tag{4.136}$$

Where $N'(c_0(t), c_1(t))$ involves terms that are quadratic in $c_0(t)$ and linear in $c_1(t)$.

Generally, for each successive term $c_n(t)$, we solve:

$$M\dot{c}_n(t) + Kc_n(t) + N''(c_0(t), c_1(t), \ldots, c_{n-1}(t)) = 0 \tag{4.137}$$

Using the solutions from previous steps.

Combining the leading-order and first-order solutions, the perturbative solution up to the first order is:

$$c(t) \approx Ae^{-\frac{K}{M}t} + \epsilon\left(\frac{A^3N_\phi}{2K}e^{-3\frac{K}{M}t} + C_1e^{-\frac{K}{M}t}\right) \tag{4.138}$$

Higher-order corrections can be added iteratively by solving the corresponding linear ODEs for $c_2(t), c_3(t), \ldots$.

This method provides an approximate solution to the nonlinear heat equation using perturbation theory

Assume $\phi_i(x) = \sin(i\pi x)$. Given the perturbative solution (4.138):

Substitute this into the solution for (4.112):

$$u_N(t,x) = \sum_{j=1}^{N} c_j(t)\phi_j(x) \tag{4.139}$$

For each $c_j(t)$, assume the form:

$$c_k(t) \approx A_je^{-\frac{K_j}{M_j}t} + \epsilon\left(\frac{A_j^3N_{\phi_j}}{2K_j}e^{-3\frac{K_j}{M_j}t} + C_{j1}e^{-\frac{K_j}{M_j}t}\right) \tag{4.140}$$

Let take $\phi_j(x) = \sin(j\pi x)$:

$$u_N(t,x) \approx \sum_{j=1}^{N}\left(A_je^{-\frac{K_j}{M_j}t} + \epsilon\left(\frac{A_j^3N_{\phi_j}}{2K_j}e^{-3\frac{K_j}{M_j}t} + C_{j1}e^{-\frac{K_j}{M_j}t}\right)\right)\sin(j\pi x) \tag{4.141}$$

The initial condition for $t = 0$ is given by:

$$u_N(0,x) \approx \left(A_k + \epsilon\left(\frac{A_k^3N_{\phi_k}}{2K_k} + C_{k1}\right)\right)\sin(k\pi x) \tag{4.142}$$

The weak formulation and its discretization using the Galerkin method yield a system of ODEs for the coefficients $c_i(t)$. These ODEs can be also solved numerically to approximate the solution of the original PDE.

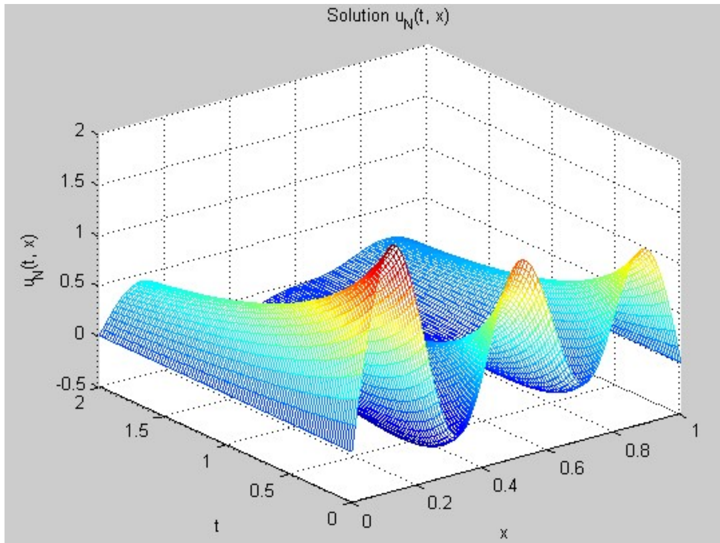

Figure 4.1: The graph of the solution of PDEs

The heat distribution in the domain changes over time and space as the system moves towards thermal equilibrium. Initially, the heat distribution might be uneven, with high and low temperature regions, but as time progresses, heat diffuses throughout the domain, reducing temperature differences. Over larger spatial domains, this diffusion process is more apparent, showing a gradual flattening of the temperature distribution. In summary, the dimensions of the domain directly influence how heat dissipates, with time driving the system towards uniformity and space dictating the initial and intermediate patterns of heat distribution. See the figure 4.2

Numerical simulation

The PDE

$$u_t = u_{xx} - |u|^3 \tag{4.143}$$

is solved numerically using a finite sum of terms. Parameters such as amplitudes (A_i), decay rates (K_i and M_i), perturbation parameter (ϵ), phase factors (N_{ϕ_i}), and coefficients (C_{i1}) are chosen in order to simulate the solution. In general they depend on the initial data and basis function.

Spatial and temporal domains

- The spatial domain x ranges from 0 to 1, divided into 100 points.
- The temporal domain t ranges from 0 to 2, also divided into 100 points.

Solution computation

The nested loops compute the solution $u_N(t, x)$ iteratively for each point in the spatial and temporal grids. Each term in the sum contributes to the solution based on its amplitude, decay rate, and phase factor.

Plot description

The plot itself is a 3D mesh plot where:

- The x-axis represents the spatial variable x,
- The y-axis represents the temporal variable t,
- The z-axis represents the computed solution $u_N(t, x)$.

The color and height of the mesh surface indicate the magnitude of $u_N(t, x)$ at each point in the (x, t) plane.

Interpretation The plot visually represents how the solution $u_N(t, x)$ evolves over time t across the spatial domain x. It shows the propagation and interaction of the solution over both dimensions, influenced by the initial conditions (not explicitly specified but assumed to be influenced by the randomly generated parameters).

Analysis

Analytically, the PDE $u_t = u_{xx} - |u|^3$ is nonlinear and can exhibit complex behaviors such as solitons, wave-like structures, or other nonlinear phenomena depending on the initial conditions and parameter values. The numerical solution provides insights into how these phenomena manifest and evolve over time and space.

The uniqueness of the solution

To prove the uniqueness of solutions to the nonlinear partial differential equation given in (4.105), we can utilize an energy method:

Compute $\frac{dE}{dt}$ by differentiating $E[(t)]$ with respect to time t:
Consider the energy functional $E(t)$ as:

$$E(t) = \frac{1}{2} \int_{\Omega} u^2 \, dx \tag{4.144}$$

Taking the time derivative of $E(t)$ gives:

$$\frac{d}{dt}E(t) = \int_\Omega u\, u_t\, dx \tag{4.145}$$

Substituting $u_t = u_{xx} - u^3$ into (4.145) from the given PDE, thus,

$$\frac{d}{dt}E(t) = \int_\Omega u\left(u_{xx} - u^3\right) dx \tag{4.146}$$

We can split this into two terms:

$$\frac{d}{dt}E(t) = \int_\Omega u u_{xx}\, dx - \int_\Omega u u^3\, dx \tag{4.147}$$

For the first term, use integration by parts (assuming u and its gradient vanish on the boundary of Ω):

$$\int_\Omega u u_{xx}\, dx = -\int_\Omega (u_x)^2\, dx \tag{4.148}$$

For the second term:

$$\int_\Omega u u^3\, dx = \int_\Omega u^4\, dx \tag{4.149}$$

Therefore, we have:

$$\frac{d}{dt}E(t) = -\int_\Omega (u_x)^2\, dx - \int_\Omega u^4\, dx \tag{4.150}$$

Both terms on the right-hand side are non-positive. This implies:

$$\frac{d}{dt}E(t) \leq 0 \tag{4.151}$$

This shows that $E(t)$, the L^2-norm of u, is non-increasing over time. Therefore, $E(t)$ is bounded by its initial value $E(0)$:

$$E(t) \leq E(0) \tag{4.152}$$

This means:

$$\int_\Omega u^2(x,t)\, dx \leq \int_\Omega u^2(x,0)\, dx \tag{4.153}$$

Thus, the L^2-norm of u remains bounded for all time t, demonstrating the stability of the solution in the L^2-norm.

Chapter 5

Conclusion and recommendations

5.1 Conclusion

The thesis on nonlinear heat equations in a semi-infinite domain provides a detailed exploration of these equations within a significant mathematical context. It employs the Galerkin method and demonstrates convergence properties in function spaces to establish the existence and accuracy of solutions. The research addresses the theoretical background, problem statement, and objectives, contributing to the understanding of heat transfer phenomena and the mathematical modeling of complex systems. Overall, it enhances knowledge in the field of nonlinear heat equations and offers valuable insights for future research and applications.

The primary result of the thesis is the successful establishment of the existence and uniqueness of solutions to the nonlinear heat equation. This is achieved through advanced mathematical techniques, such as the Galerkin method. Rigorous analysis and numerical simulations confirm the stability of solutions within a semi-infinite domain. The thesis also contributes to understanding heat conduction in materials and the behavior of heat transfer in nonlinear systems. It provides significant insights into the solvability and properties of nonlinear heat equations, advancing the field of mathematical modeling and heat transfer..

5.2 Recommendations

Based on the content provided in "Non-linear heat equation in semi-infinite domain," the following recommendations are made for further enhancement of the thesis:

Further exploration of numerical nimulations: The thesis effectively establishes the existence and uniqueness of solutions using advanced techniques such as the Galerkin method. To build on this, it is recommended to conduct more extensive numerical simulations. This could involve exploring various numerical methods or algorithms to further validate and analyze the stability properties of solutions, and to better understand heat transfer in nonlinear systems.

Application in interdisciplinary fields: The thesis makes a significant contribution to understanding heat transfer phenomena and mathematical modeling. Future research should consider exploring practical applications in interdisciplinary fields like material science, engineering, or environmental science. Collaborating with experts in these areas could help apply the findings to real-world problems, highlighting the practical relevance and impact of the study.

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