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Admissible transformations and the group classification of Schrödinger equations

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Dedication

To my family, Dancille Mukarugina,
Hope Benigne Ineza,
Dalton Bruce Hirwa,
For your patience during my studies.

Abstract

We study admissible transformations and solve group classification problems for various classes of linear and nonlinear Schrödinger equations with an arbitrary number n of space variables.

The aim of the thesis is twofold. The first is the construction of the new theory of uniform semi-normalized classes of differential equations and its application to solving group classification problems for these classes. Point transformations connecting two equations (source and target) from the class under study may have special properties of semi-normalization. This makes the group classification of that class using the algebraic method more involved. To extend this method we introduce the new notion of uniformly semi-normalized classes. Various types of uniform semi-normalization are studied: with respect to the corresponding equivalence group, with respect to a proper subgroup of the equivalence group as well as the corresponding types of weak uniform semi-normalization. An important kind of uniform semi-normalization is given by classes of homogeneous linear differential equations, which we call uniform semi-normalization with respect to linear superposition of solutions.

The class of linear Schrödinger equations with complex potentials is of this type and its group classification can be effectively carried out within the framework of the uniform semi-normalization. Computing the equivalence groupoid and the equivalence group of this class, we show that it is uniformly semi-normalized with respect to linear superposition of solutions. This allow us to apply the version of the algebraic method for uniformly semi-normalized classes and to reduce the group classification of this class to the classification of appropriate subalgebras of its equivalence algebra. To single out the classification cases, integers that are invariant under equivalence transformations are introduced. The complete group classification of linear Schrödinger equations is carried out for the cases $n = 1$ and $n = 2$.

The second aim is to study group classification problem for classes of generalized nonlinear Schrödinger equations which are not uniformly semi-normalized. We find their equivalence groupoids and their equivalence groups and then conclude whether these classes are normalized or not. The most appealing classes are the class of nonlinear Schrödinger equations with potentials and modular nonlinearities and the class of generalized Schrödinger equations with complex-valued and, in general, coefficients of Laplacian term. Both these classes are not normalized. The first is partitioned into an infinite number of disjoint normalized subclasses of three kinds: logarithmic nonlinearity, power nonlinearity and general modular nonlinearity. The properties of the Lie invariance algebras of equations from each subclass are studied for arbitrary space dimension n, and the complete group classification is carried out for each subclass in dimension (1+2). The second class is successively reduced into subclasses until we reach the subclass of (1+1)-dimensional linear Schrödinger equations with variable mass, which also turns out to be non-normalized. We prove that this class is mapped by a family of point transformations to the class of (1+1)-dimensional linear Schrödinger equations with unique constant mass.

Populärvetenskaplig sammanfattning

I denna avhandling studerar vi hur man kan klassificera både linjära och icke-linjära Schrödinger ekvationer med gruppsymmetri som klassificeringsprincip. Detta leder till introduktionen av en ny teori av så kallade uniformt halv-normaliserade klasser av differentialekvationer som kan tillämpas på de olika typer av ekvationsklasser. I synnerhet tillämpas metoden på linjära Schrödinger ekvationer med potential (både reell och komplex) och på så vis erhållas en fullständig klassificering av denna typ av Schrödinger ekvationer i dimension 1+1 och 1+2.

Även icke-linjära Schrödinger ekvationer, med vissa typer av icke-linjära termer, studeras och en fullständig klassificering erhållas för dessa icke-linjära termer i dimension 1+2. De klasser av icke-linjära Schrödinger ekvationer som studeras här innehåller en ekvationsklass med variabel massa. Vi visar bland annat att i dimension 1+1 varje Schrödinger ekvation med variabel massa kan avbildas på en Schrödinger ekvation med konstant massa.

Både linjära och icke-linjära Schrödinger ekvationer förekommer i fysik och teknik, och även linjära Schrödinger ekvationer med komplex potential har på senare år undersökts i kvantteori. Att klassificera sådana ekvationer enligt gruppsymmetriska egenskaper är att skapa ett slags ordning i en skenbar oordning, och det visar sig att det i grunden finns bara ett ändligt antal "grundtyper" av Schrödinger ekvationer som är sinsemellan inekvivalenta.

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Célestin Kurujyibwami

List of papers

The thesis consists of five papers, which will be referred to in the text by their numbers in Roman numerals.

I. Kurujyibwami C., Basarab-Horwath P. and Popovych R.O., Algebraic method for group classification of (1+1)-dimensional linear Schrödinger equations, submitted to *Acta Appl. Math.*, (2016) arXiv:1607.04118, 30 pp.

II. Kurujyibwami C., Basarab-Horwath P. and Popovych R.O., Group classification of multidimensional linear Schrödinger equations with the algebraic method, (in preparation) (2017).

III. Kurujyibwami C., Basarab-Horwath P. and Popovych R.O., Group classification of multidimensional nonlinear Schrödinger equations, (in preparation) (2017).

IV. Kurujyibwami C., Basarab-Horwath P. and Popovych R.O., Admissible transformations of (1+1)-dimensional Schrödinger equations with variable mass, (in preparation) (2017).

V. Kurujyibwami C., Equivalence groupoid for (1+2)-dimensional linear Schrödinger equations with complex potentials, *Journal of Physics: Conference Series*, **621** (2015),12008–12014.

Contribution
Authors's contribution to these papers is the following. I wrote Paper V myself. In the other four papers supervisors posed the problems to be studied and proposed the general plan of their solutions. The other piece of the work, which includes computing and formulating results and writing the papers, was done by me.

The results of the above papers were presented by me at four conferences:

- *Group classification of multidimensional Schrödinger equations with potentials and general modular nonlinearity*, The 3rd EAUMP conference "Advances in Mathematics and its applications" (October 26–28, 2016, Makerere University, Kampala, Uganda).

- *Group classification of multidimensional nonlinear Schrödinger equations*, Eighth International Workshop "Group Analysis of Differential Equations and Integrable Systems" (June 12–17, 2016, Larnaka, Cyprus).

- *Group classification of multidimensional linear Schrödinger equations with algebraic method*, 14th conference "Mathematics in Technical and Natural Sciences" (September 18–24, 2015, Koscielsco, Poland).

- *Algebraic method for group classification of* $(1+1)$*-dimensional linear Schrödinger equations*, Seventh International Workshop "Group Analysis of Differential Equations and Integrable Systems" (June 15–19, 2014, Larnaka, Cyprus).

These results were also presented in various seminars:

- *Group classification of linear Schrödinger equations by the algebraic method*, Licentiate seminar at the Department of Mathematics (February 24, 2016, Linköping University, Linköping, Sweden).
- *Group classification of (1+n)-multidimensional linear Schrödinger equations*, Seminar at the Department of Mathematics (January 21, 2016, University of Rwanda, Kigali, Rwanda).
- *Group classification of (1+1)-dimensional linear Schrödinger equations*, Mathematics colloquium (May 23, 2014, Linköping University, Linkoping, Sweden).
- *Group classification of Schrödinger equations*, First Network Meeting for SIDA and ISP funded PhD Students in Mathematics (March 7–8, 2017, Stockholm, Sweden).

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Theoretical background

1

Introduction

Lie symmetries of differential equations were first introduced by the Norwegian mathematician Sophus Lie in his efforts to create a theory for differential equations similar to the work of Galois on algebraic equations. In this approach, a symmetry of a differential equation (ordinary or partial) is an invertible transformation that maps the set of solutions of the equation to itself: if $L(u) = 0$ is a differential equation, where u stands for a solution of the equation, then we must have

$$L(\tilde{u}) = 0 \quad \text{whenever} \quad L(u) = 0,$$

where $\tilde{u}$ is the transform of u. It is easy to see that the set of all symmetries of a given equation form a group.

Lie's theory gave birth to the theory of continuous transformations group and then led to the concept of Lie groups: a Lie group is, in modern terminology, *a finite-dimensional analytic manifold whose underlying set is also a group so that the group operations of multiplication and taking the inverse,* $(g, h) \to gh$, $g \to g^{-1}$, *are analytic* (see [45]).

Lie's original work on symmetries of differential equations was done in terms of what are now called *Lie algebras and one-parameter groups.* These one-parameter groups are generated by vector fields (known also as infinitesimal symmetries) and the vector fields form a Lie algebra. The first step in a (Lie) symmetry analysis of a given differential equation is to find the vector fields that generate (local) one-parameter groups of symmetry transformations. To do this, Lie developed an algorithmic method of determining these vector fields. This was one of his many deep results, and it is still the basis of every investigation of the symmetry properties of differential equations. This is known as the *Lie infinitesimal method.* Having found the Lie algebra of infinitesimal symmetries of an equation, one can then reconstruct the local one-parameter groups that they generate, and from a given solution of the equation we can generate new ones by applying the local symmetry transformations.

In his approach, Lie posed two questions:

1. What symmetries does a given equation possess?

2. Given a group (in a given representation), what differential equations (of a given order and type) admit this group as a group of symmetries?

In this thesis, we look at a variant of the first question: given a class of differential equations, what symmetries can the equations in that class possess and how can we classify them according to the various symmetry properties that they have?

From 1899, the year in which Lie died, up to the 1950's, Lie's methods fell almost into disuse. However, two notable exceptions were Bateman's proof in 1909 that the free wave equation is invariant under the conformal group and Emmy Noether's work on the correspondence between the symmetries of a Lagrangian variational problem and the conservation laws of the Euleur-Lagrange equations [32].

In the late 1950's and early 1960's, Ovsiannikov began publishing his work on the symmetry analysis of differential equations ([34], [35]) and from the late 1960's to the present day there has been an exponential growth in the application of symmetry methods to differential equations (see for instancecite [3], [6], [33], [36], [37] and the references cited there).

Although Lie's symmetry algorithm based on the infinitesimal Lie method is fundamental in symmetry analysis, it is not always powerful enough on its own to give complete results. For classes of differential equations whose arbitrary elements depend on one variable only but not on any derivative, such as the equation

$$u_t = f(u)u_{xx} + f_u(u)u_x,$$

this approach is successful. However, the method is not powerful enough when confronted by equations of the type

$$u_t = f(u, u_x)u_{xx} + g(u, u_x).$$

The reason for this is quite simple: the infinitesimal method of Lie depends on being able to express the symmetry condition (on the coefficients of a symmetry vector field) as a polynomial in the derivatives. This is not possible when terms such as $f(u, u_x)$ and $g(u, u_x)$ are present. The way out of this problem is to supplement Lie's infinitesimal method with other techniques. In this thesis we use the algebraic method [4], which is based on the subgroup analysis of the equivalence group associated with a class of differential equations under study. This approach uses a synthesis of Lie's infinitesimal method, the technique of equivalence transformations [37] and the analysis of low-dimensional subalgebras as well as the *equivalence groupoid* of the equation (or class of equations) [40]

1.1 Lie symmetries of Schrödinger equations: known results

The study of Schrödinger equations from the symmetry point of view began in the early 1970's with the symmetry analysis of the linear Schrödinger equation ([7], [25], [27], [28], [29], [30]). In [27] Niederer found the usual Lie symmetry group of (1+3)-dimensional free linear Schrödinger equations using Lie's infinitesimal method. In subsequent papers ([28], [29], [30]) he obtained the maximal kinematical invariance group of the harmonic oscillator as well establishing an isomorphism between the invariance groups for both the harmonic and linear potential, and the free invariance particle. Niederer also obtained a symmetry classification of linear Schrödinger equations with arbitrary real potential $V = V(t,x)$ for space-dimension less or equal to three. Boyer ([7]) solved the problem of group classification of the class of linear Schrödinger equations with arbitrary real-valued time independent potential and found a generalization of Niederer's results. In the book [25], Miller gave a résumé of his important work on the symmetry properties and separation of variables for linear Schrödinger equations.

In the early 1990's, Fushchych and his collaborators in ([13], [14], [15], [16]) studied the symmetries and conditional symmetries and exact solutions of Schrödinger equations of the form

$$i\psi_t + \lambda\Delta\psi + F(|\psi|)\psi = 0 \quad \lambda \neq 0$$

(as well as considering other nonlinear equations). Earlier, in ([17], [18], [19] the group invariant solutions of (1+3)-dimensional generalized quintic nonlinear Schrödinger equations, with $F = a_0 + a_1|\psi|^2 + a_2|\psi|^4$ were constructed. Nonlinear Schrödinger equations of the type

$$i\psi_t + \psi_{xx} + F(t,x,\psi,\psi^*,\psi_x,\psi^*_x) = 0,$$

where F is an arbitrary smooth, complex-valued function, were studied in ([8]), in particular their invariance under subalgebras of the symmetry algebra of (1+1)-dimensional free Schrödinger equation. The symmetry classification for the general case of $F = F(\psi,\psi^*)$ was carried out in [31].

A complete classification of the above type of nonlinear Schrödinger equation with modular nonlinearities, supplementing Lie's infinitesimal approach with the notions of equivalence group and normalized classes of differential equations, was given in [40]. They also gave a complete description of all admissible transformations in the class $\mathcal{F}$ (that is, those that map equations of the class to equations in the same class).

In ([20]) the symmetry analysis of the class of (1+1)-dimensional cubic Schrödinger equations with variable coefficients of the form

$$i\psi_t + f(t,x)\psi_{xx} + g(t,x)|\psi|^2\psi + h(t,x)\psi = 0,$$

where f, g, h are complex-valued functions of (t,x) with $(\operatorname{Re} f)(\operatorname{Re} g) \neq 0$, was carried out.

Various other classes of nonlinear Schrödinger equations have over the years been studied using Lie's method supplemented with other techniques. Among these supplementary techniques is the exploitation of equivalence transformations, as developed systematically by Ovsiannikov (see [37]). This method has been enhanced and refined by Popovych and his collaborators (for details see [39], [40], [46]). In this thesis, we investigate linear and non-linear Schrödinger equations using the techniques developed by Popovych and his collaborators, and we enhance these methods with new results.

1.2 Motivation and relevance

We consider linear and nonlinear Schrödinger equations with complex potentials. Nonlinear Schrödinger equations have been used in nonlinear optics and plasma physics, as well as other fields. Real potentials have an obvious interpretation in quantum mechanics, but recently interest in complex-valued potentials has grown. These complex potentials still give a real-valued spectrum for the Hamiltonian (see for instance [1], [2], [12], [26], [42], [43]).

We are motivated in our treatment of linear and nonlinear Schrödinger equations for a variety of reasons.

First of all, we treat, in the first paper of this thesis, linear Schrödinger equations in (1+1)-time-space with a potential. Although this case was considered previously by Niederer ([27]) and Boyer ([7]), their results are not complete. Also, their approach to symmetries was based on an assumption of a linear representation of symmetries (as suggested by representation theory). Here, we use the Lie algorithm for finding the symmetries, and the form of the representation is then dictated by the equation. Our approach gives an exhaustive list of potentials, both real and complex, and we thus obtain completeness in our classification. As mentioned above, some complex potentials have been of interest in various applications, there are complex potentials which still allow the quantum mechanical Hamiltonian to have a real spectrum. In the second paper, we look at the case of linear Schrödinger equations with potential in $(1+n)$-time-space dimensions and then give a more detailed analysis for the (1+2)-time-space.

As well as completeness, using our method on this type of Schrödinger equation provides us with a laboratory for the method and the auxiliary concepts we develop. Indeed, we extend the methods described in ([4]). Further, in the third paper we extend the results of ([40]) to the case of several dimensions.

Since nonlinear Schrödinger equations have been used in many contexts, we use our methods to study nonlinear Schrödinger equations of the form

$$i\psi_t + G(t,x,\psi,\psi^*,\nabla\psi,\nabla\psi^*)\Delta\psi + F(t,x,\psi,\psi^*,\nabla\psi,\nabla\psi^*) = 0, \quad G \neq 0,$$

and then consider more specific cases of the nonlinearities F and G. This study is intended to help in the understanding of the various subclasses of nonlinear equations.

Finally, our study is intended to give a Lie symmetry classification of these Schrödinger equations and to use these Lie symmetries to construct exact solutions of the equations.

This thesis consists of two parts: in the first part we present a summary of the theoretical background needed to read the research papers and the second part consists of the research papers.

2

Basic notions of symmetry analysis

2.1 One-parameter groups and their infinitesimal generators.

Symmetry analysis of differential equations is based on the notion of Lie algebras of vector fields. These vector fields are the generators of (local) Lie groups of symmetry transformations. A symmetry transformation is, as noted in the introduction, simply a transformation which maps a solution of a given differential equation to another solution of the same equation. We begin by recalling some basic concepts and definitions which form the background of our approach. For simplicity, we work in $\mathbb{R}^n$ or open subsets of $\mathbb{R}$ as all our considerations in this thesis are of a local nature.

One of the fundamental ingredients in our approach is the concept of (local) one-parameter group. This is just a modification of the idea of a one-parameter group of transformations:

Definition 2.1. 1) A one-parameter group of smooth transformations of $\mathbb{R}^n$ (for some positive integer n) is a one-parameter family of smooth transformations $\Phi_t \colon \mathbb{R}^n \to \mathbb{R}^n$ with $t \in \mathbb{R}$, such that $(t, x) \to \Phi_t(x)$ is a smooth map $\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ with the properties that $\Phi_s \circ \Phi_t = \Phi_{s+t}$ for all $t \in \mathbb{R}$ and $\Phi_0 = \mathrm{id}_{\mathbb{R}^n}$.

2) A one-parameter *local* group of smooth transformations of an open subset $U \subset \mathbb{R}^n$ is a family of smooth transformations $\Phi_t \colon U \to U$ with $t \in I$ for some symmetric interval $I \subset \mathbb{R}$ about $0 \in \mathbb{R}^n$ such that $(t, x) \to \Phi_t(x)$ is smooth map $\mathbb{R} \times U \to U$ and such that $\Phi_t(x) \in U$ for all $t \in I$ and $x \in U$, $\Phi_0(x) = x$ for each $x \in U$ and $\Phi_s \circ \Phi_t(x) \in U$ for $x \in U$ and for all $s, t \in I$ such that $s + t \in I$.

Example 2.1

As examples of one-parameter groups of transformations we have

$$\begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix}, \quad \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}, \quad t \in \mathbb{R}.$$

Clearly, every one-parameter group of transformations is a local one-parameter group. The generator of a local one-parameter group is a vector field which we define as follows:

Definition 2.2. Suppose we are given a local one-parameter group of transformations $\Phi_t,\ t \in I$ on an open subset $U \subset \mathbb{R}^n$ as defined in Definition 2.1. Then a vector field X defined on U is the *generator* of the local one-parameter group on U if X_x is the tangent vector to the curve defined by $t \to \Phi_t(x)$ at $t = 0$. That is,

$$X_x = \frac{d}{dt}\Phi_t(x)\big|_{t=0}.$$

The generator of a local one-parameter group acts on (smooth) functions $f\colon U \to \mathbb{R}$ according to the rule

$$(Xf)(x) = \frac{d}{dt} f\left(\Phi_t(x)\right)\big|_{t=0}$$

and this defines a derivation on the multiplicative ring of functions: we have $X(f+g) = Xf + Xg$ and $X(fg) = (Xf)g + f(Xg)$, as is easily verified from the definition of the generator X.

2.2 Jet bundles

Jet bundles, or jet spaces, are the natural setting for discussing differential equations. A jet bundle over a manifold M is essentially a space whose local coordinates are the local coordinates $x = (x^1, \ldots, x^n)$ of M together with functions $u = (u^1, \ldots, u^m)$ on M as well as their derivatives. Thus, if the manifold is $\mathbb{R}^2$ and we have one function $u\colon \mathbb{R}^2 \to \mathbb{R}$, then the jet bundle of order two, denoted by $J^2(\mathbb{R}^2, \mathbb{R}$, is the space with local coordinates $(x, u_{(2)}) = (x^1, x^2, u, u_1, u_2, u_{11}, u_{12}, u_{22})$, where u_1 and u_2 denote the first-order derivatives $\partial_1 u$ and $\partial_2 u$ with ∂_1 and ∂_2 being the partial derivatives with respect to x^1 and x^2 respectively, and u_{11}, u_{12}, u_{22} denote the three second order partial derivatives of u: $u_{11} = \partial_1^2 u$, $u_{12} = \partial_{12}^2 u$, $u_{22} = \partial_2^2 u$. It is assumed that all mixed derivatives are equal. A rigorous definition of jet bundles and their properties can be found in [38], [41]. Our exposition below is informal and follows that of Olver in [33].

For a smooth real-valued function $u(x^1, \ldots, x^n)$ of n independent variables we have the pth order derivatives

$$u_\alpha = \frac{\partial^{|\alpha|} u}{\partial x^{\alpha_1} \partial x^{\alpha_2} \ldots \partial x^{\alpha_n}}$$

where α stands for the unordered n-tuple of integers $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and where the integers α_i, $i = 1, \ldots, n$ are such that $1 \leqslant \alpha_i \leqslant n$. α is referred to as a *multi-index* and its *length*, denoted by $|\alpha|$, is defined as $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$. The corresponding jet space is denoted by $J^p(\mathbb{R}^n, \mathbb{R})$ and has local coordinates $(x, u) = (x^1, \ldots, x^n, u)$ together with all the partial derivatives u_α, $1 \leqslant |\alpha| \leqslant p$. Thus, for instance, $J^3(\mathbb{R}^2, \mathbb{R})$ has local coordinates $(x, y, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, u_{xxxx}, u_{xxy}, u_{xyy}, u_{yyy})$. We define the "zeroth" jet bundle $J^0(\mathbb{R}^n, \mathbb{R})$ as being the space with local coordinates (x, u), the "zeroth" derivative of u being just the function itself.

If we now take a function $u\colon \mathbb{R}^n \to \mathbb{R}^m$ then the corresponding jet space of order p is denoted by $J^p(\mathbb{R}^n, \mathbb{R}^m)$ and its local coordinates are

$$(x, u) = (x^1, \ldots, x^n, u^1, \ldots, u^m)$$

together with all the partial derivatives u^a_α, $1 \leqslant |\alpha| \leqslant p$, $a = 1, \ldots, m$. The "zeroth" jet bundle $J^0(\mathbb{R}^n, \mathbb{R}^m)$ is then the space with local coordinates (x, u). We refer to the space $\mathbb{R}^n \times \mathbb{R}^m$ with local coordinates (x, u) as the *underlying space*. It is the graph space of the mapping $u\colon \mathbb{R}^n \to \mathbb{R}^m$. In the literature on symmetries of differential equations the $x = (x^1, \ldots, x^n)$ and $u = (u^1, \ldots, u^m)$ are called the *independent variables* and the *dependent variables*, respectively.

The jet bundles $J^p(\mathbb{R}^n, \mathbb{R}^m)$ can be given the structure of a differentiable manifold (see [41]) and they are also *fibred manifolds*: there is a map (the projection map) $\pi_p\colon J^p(\mathbb{R}^n, \mathbb{R}^m) \to \mathbb{R}^n \times \mathbb{R}^m$ which is surjective (it is a submersion) given by $\pi_p(x, u_{(p)}) = (x, u)$, where $u_{(p)}$ stands for the collection of all partial derivatives of u up to and including order p, with the "zeroth" derivatives being just the functions themselves. The *fibre* above (x, u) is just the inverse image $\pi_p^{-1}(x, u)$ (see [38], [41]). One also has the projections $\pi_{p,l}\colon J^p(\mathbb{R}^n, \mathbb{R}^m) \to J^q(\mathbb{R}^n, \mathbb{R}^m)$ for $p \geqslant q$ given by $\pi_{p,q}(x, u_{(p)}) = (x, u_{(q)})$ and it is easy to verify that $\pi_{j,p} \circ \pi_{p,q} = \pi_{j,q}$ whenever $j \geqslant p \geqslant q$. We also have $\pi_{p,p} = \mathrm{id}$ and $\pi_p = \pi_l \circ \pi_{p,q}$ for all $p \geqslant q$.

The structure of a fibred manifold allows us to define diffeomorphisms of $J^p(\mathbb{R}^n, \mathbb{R}^m)$. Note that any diffeomorphism $\Phi\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m$ of the underlying space $\mathbb{R}^n \times \mathbb{R}^m$ induces a transformation of the derivatives of all orders of $u = (u^1, \ldots, u^m)$. The induced transformation on the jet space $J^p(\mathbb{R}^n \times \mathbb{R}^m)$ obtained in this way (by the usual chain rule) is called the pth *prolongation of* Φ. It is clear that the pth prolongation of a diffeomorphism Φ will depend only on the local coordinates $(x, u_{(p)})$. We use the notation $\Phi|_{(x,u_{(p)})}$ to denote the pth prolongation of Φ.

The jet bundles described above are endowed with differential operators D_i, $i = 1, \ldots, n$ called the *total derivatives* with respect to x_i and they are defined as $\mathrm{D}_i = \partial_i + u^a_{\alpha+\delta_i}\partial_{u^a_\alpha}$, where δ_i is the multi-index whose ith entry equals 1 and whose other entries are zeroes. The variable u^a_α of the jet space $J^p(\mathbb{R}^n, \mathbb{R}^m) \to \mathbb{R}$ is given by $\partial^{|\alpha|}u^a/\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}$. Here and below we assume summation over repeated indices.

Thus for the functions $X\colon J^p(\mathbb{R}^n, \mathbb{R}^m) \to \mathbb{R}$ we have

$$D_i X = \frac{\partial X}{\partial x^i} + u^a_{\alpha+\delta_i}\frac{\partial X}{\partial u^a_\alpha},$$

where the summation over α is over all n-tuples $(\alpha_1, \ldots, \alpha_n)$ of all lengths $|\alpha| \geqslant 1$. The sum is well-defined since X is a function of only a finite number of arguments. Here $u^a_{\alpha,i}$ is defined for $\alpha = (\alpha_1, \ldots, \alpha_n)$ as

$$u^a_{\alpha,i} = \frac{\partial u^a_\alpha}{\partial x^i} = \frac{\partial^{|\alpha|+1} u^a}{\partial x^i \partial x^{\alpha_1} \ldots \partial x^{\alpha_n}}.$$

Example 2.2

The space on which second-order ordinary differential equations are defined is $J^2(\mathbb{R}, \mathbb{R})$. The underlying space is $\mathbb{R} \times \mathbb{R}$ with local coordinates (x, y). The local coordinates on $J^2(\mathbb{R}, \mathbb{R})$ are then (x, y, y', y'') and the total derivative with respect to x is

$$D_x = \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + y'' \frac{\partial}{\partial y'} + y''' \frac{\partial}{\partial y''}.$$

Example 2.3

For $J^2(\mathbb{R}^2, \mathbb{R})$ we have the underlying space $\mathbb{R}^2 \times \mathbb{R}$ with local coordinates (x, y, u) and the local coordinates on $J^2(\mathbb{R}^2, \mathbb{R})$ are then $(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy})$. We have two total derivatives

$$\begin{aligned} D_x =& \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{xy} \frac{\partial}{\partial u_y} + u_{xxx} \frac{\partial}{\partial u_{xx}} + u_{xxy} \frac{\partial}{\partial u_{xy}} \\ &+ u_{xyy} \frac{\partial}{\partial u_{yy}}, \end{aligned}$$

$$\begin{aligned} D_y =& \frac{\partial}{\partial y} + u_y \frac{\partial}{\partial u} + u_{xy} \frac{\partial}{\partial u_x} + u_{yy} \frac{\partial}{\partial u_y} + u_{yyy} \frac{\partial}{\partial u_{yy}} + u_{yyx} \frac{\partial}{\partial u_{xy}} \\ &+ u_{xxy} \frac{\partial}{\partial u_{xx}}. \end{aligned}$$

The total derivatives D_x, D_y mentioned in this example are examples of *generalized* vector fields rather than vector fields since the coefficient y''' and $u_{xxx}, u_{xxy}, u_{yyy}$ appear, and these are coordinates on the jet bundles $J^3(\mathbb{R}, \mathbb{R})$ and $J^3(\mathbb{R}^2, \mathbb{R})$. This problem can be circumvented by working on the infinite jet bundle $J^\infty(\mathbb{R}^n, \mathbb{R}^m)$ where the local coordinates include derivatives u^a_α of arbitrary order (that is with $|\alpha|$ takes on all possible positive integer values), and in this case the operators D_i mentioned above are vector fields. Functions on this infinite jet bundle are defined as being functions of only a finite number of non-zero arguments. A rigorous definition is given in [41], but we shall not need this construction. We do not use jet bundles other than in an informal way and the total derivatives are to be understood as generalized vector fields.

2.3 Lie symmetries of differential equations

In this Section we are interested in *point symmetries* of differential equations, that is, invertible point transformations acting on the space of independent and dependent variables which map solutions of differential equations to solutions of the same equations (see for instance [6], [33]).

Definition 2.3. A *symmetry* of differential equation is a smooth transformation that maps any solution of differential equation to another solution of the same equation. Then a *point symmetry group* of differential equation is a smooth, invertible point transformation which maps every solution of the differential equation to a solution of the same differential equation.

Note that according to this definition, a symmetry may be a transformation which depends on derivatives, whereas a *point* symmetry is a transformation on the underlying space of a differential equation (that is, the space of independent and dependent variables). In the following, we refer to point symmetries as just symmetries.

We begin with a definition of what we mean by a system of differential equations ([33]):

Definition 2.4. A system of l differential equations of order p for m unknown functions $(u^1, \dots, u^m)$ in n independent variables $(x_1, \dots, x_n)$ is a smooth function

$$\mathcal{L}\colon J^p(\mathbb{R}^n, \mathbb{R}^m) \to \mathbb{R}^l, \quad \text{such that} \quad \mathcal{L}(x, u_{(u_p)}) = 0,$$

where p, m, n and l are positive integers. This system is of *maximal rank* if the smooth function $\mathcal{L}\colon J^p(\mathbb{R}^n, \mathbb{R}^m) \to \mathbb{R}^l$ has rank l, that is, the differential $D\mathcal{L}$ mapping the tangent space at each point of the domain of definition of $\mathcal{L}$ to $\mathbb{R}^l$ has rank l.

Example 2.4

A system of two equations of order two of one function of independent variables (x_1, x_2) is a smooth map $\mathcal{L}\colon J^2(\mathbb{R}^2, \mathbb{R}) \to \mathbb{R}^2$ such that

$$\mathcal{L}(x, u_{(2)})\colon = \begin{cases} \mathcal{L}_1(x, u_{(2)}) = 0, \\ \mathcal{L}_2(x, u_{(2)}) = 0. \end{cases}$$

For illustration one can take $\mathcal{L}(x, u_{(2)})$ to be the system

$$\mathcal{L}_1\colon u_t - u_{xx} - 3u_x - u = 0, \quad \text{and} \quad \mathcal{L}_2\colon u_{tt} - u_x - 2u_{xx} = 0.$$

The differential $D\mathcal{L}$ is then easily computed to be

$$D\mathcal{L} = \left(\frac{\partial \mathcal{L}_\nu}{\partial x_i}, \frac{\partial \mathcal{L}_\nu}{\partial u_J^\alpha}\right) = \begin{pmatrix} 0 & 0 & -1 & 1 & -3 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & -2 \end{pmatrix},$$

which is of rank two whenever $\mathcal{L}(x, u_{(2)}) = 0$.

Definition 2.5. A symmetry transformation of a system of differential equations

$$\mathcal{L}\colon J^p(\mathbb{R}^n,\mathbb{R}^m)\to\mathbb{R}^l,\quad \text{such that}\quad \mathcal{L}(x,u_{(u_p)})=0,$$

is a smooth, invertible map

$$\Phi\colon \mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^n\times\mathbb{R}^m,\qquad \Phi(x,u)=(\tilde{x},\tilde{u}) \tag{2.1}$$

so that $\mathcal{L}(\tilde{x},\tilde{u}_{(p)})=0$ whenever $\mathcal{L}(x,u_{(p)})=0$.

Any symmetry Φ will induce an action on the derivatives of a function. This action is defined as follows:

Definition 2.6. The map $(x,u_{(p)})\to(\tilde{x},\tilde{u}_{(p)})$ induced by the map $\Phi\colon(x,u)\to(\tilde{x},\tilde{u})$ is called the *pth prolongation* of Φ, denoted by $\mathrm{pr}^{(p)}\Phi$.

It is an essentially hopeless task to find symmetries of differential equations with this definition. However, using this definition we can restrict our attention to (locally defined) one-parameter groups generated by vector fields of the form

$$X=\xi^i(x,u)\partial_{x_i}+\eta^a(x,u)\partial_{u^a}.$$

The action of this vector field can be extended to an action on functions on the jet bundle $P\colon J^p(\mathbb{R}^n,\mathbb{R}^m)\to\mathbb{R}$. This extended action is called the pth *prolongation* of the vector field X, and is denoted by $X_{(p)}$ and is defined by the formula

$$X_{(p)}=\xi^i(x,u)\partial_{x_i}+\eta^a(x,u)\partial_{u^a}+\sum_{a=1}^{m}\eta^a_\alpha\partial_{u^a_\alpha}.$$

The sum over α is over all n-tuples α with $|\alpha|\geqslant 1$. The functions η^a_α are defined as follows for each $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_n)$:

$$\eta^a_\alpha=D_\alpha\left(\eta^a-\xi^i u^a_i\right)+\xi^i u^a_{\alpha,i}$$

with $D_\alpha=D_{\alpha_1}D_{\alpha_2}\dots D_{\alpha_n}$. Alternatively, they may be given recursively as

$$\eta^a_i=D_i\eta^a-u^a_bD_i\xi^b,\quad \eta^a_{\alpha,i}=D_i\eta^a_\alpha-u^a_{\alpha,b}D_i\xi^b$$

for $|\alpha|\geqslant 1$.

Example 2.5

For the vector field

$$X=x\partial_x+2u\partial_u$$

defined on the underlying space $\mathbb{R}\times\mathbb{R}$ the first prolongation $X_{(1)}$ is

$$X_{(1)}=x\partial_x+2u\partial_u+\eta_x\partial_{u_x}=x\partial_x+2u\partial_u+u_x\partial_x$$

since $\eta_x=D_x(2u-xu_x)+xu_{xx}$.

Example 2.6

The vector fields $X = \partial_x,\ Y = x\partial_x,\ Z = x^2\partial_x$ form the Lie algebra $\mathrm{sl}(2,\mathbb{R})$ and the commutation relations are $[Y,X] = -X$, $[Y,Z] = Z$ and $[X,Z] = 2Y$. Their second prolongations for the underlying space $\mathbb{R}^2 \times \mathbb{R}$ are

$$\begin{aligned}
&X_{(2)} = \partial_x, \quad Y_{(2)} = x\partial_x - u_x\partial_{u_x} - 2u_{xx}\partial_{u_{xx}}\\
&Z_{(2)} = x^2\partial_x - 2u_x\partial_{u_x} - (2u_x + 4xu_{xx})\partial_{u_{xx}}.
\end{aligned}$$

For more details on these definitions and examples, we refer to [33].

Lemma 2.1. *Let $\Phi_X(t)$ denote the local flow of a vector field X on $\mathbb{R}\times\mathbb{R}^m$ and $\Phi_{X_{(p)}}(t)$ the local flow of $X_{(p)}$ on $J^p(\mathbb{R}\times\mathbb{R}^m)$, then*

$$\mathrm{pr}^{(p)}\Phi_X(t) = \Phi_{X_{(p)}}(t).$$

For a proof, see [33].

We can now define what we mean by an *infinitesimal symmetry operator* for a differential equation:

Definition 2.7. The vector field X defined by

$$X = \xi^i(x,u)\partial_{x_i} + \eta^a(x,u)\partial_{u^a}$$

is an *infinitesimal point symmetry* of the system of differential equations

$$\mathcal{L}(x,u,u_{(p)}) = 0, \quad \text{where} \quad \mathcal{L}\colon J^p(\mathbb{R}^n,\mathbb{R}^m) \to \mathbb{R}^l,$$

if the local diffeomorphism generated by the local flow $\Phi_X(t)$ of X is a symmetry of this system of differential equations.

The basic result on symmetries of differential equations is the following:

Theorem 2.1. *If the (smooth) vector field X defined on the underlying space $\mathbb{R}^n\times\mathbb{R}^m$ is a symmetry of the system of differential equations $\mathcal{L}(x,u_{(p)}) = 0$, where $\mathcal{L}\colon J^p(\mathbb{R}^n,\mathbb{R}^m)\to\mathbb{R}^l$ is of maximal rank, then*

$$X_{(p)}\cdot\mathcal{L} = 0 \quad \textit{whenever} \quad \mathcal{L}(x,u_{(p)}) = 0.$$

If X, Y are two infinitesimal point symmetries of the system of differential equations of maximal rank $\mathcal{L}(x,u_{(p)}) = 0$ then so is their commutator $[X,Y]$. Here $X_{(p)}\cdot\mathcal{L} = 0$ means $X_{(p)}\mathcal{L}_q = 0$ for all $q = 1,\ldots,l$.

Note that this result only says that *if* $\Phi_X(t)$ is a symmetry, then $X_{(p)}\cdot\mathcal{L} = 0$. It requires that, at least for $t\in I$ for some interval I of zero, $\Phi_X(t)u$ to be a solution of $\mathcal{L}(x,u,u_{(p)}) = 0$ whenever u is a solution. The converse statement requires the extra condition of *nondegeneracy*.

2.3.1 Nondegenerate systems of differential equations

Our work deals with *non-degenerate* systems of differential equations. We have the following definitions (see [33]):

Definition 2.8. A system of pth order differential equations $\mathcal{L}(x, u_{(p)}) = 0$ is said to be *locally solvable* at the point $(x, u_{(p)})$ for which $\mathcal{L}(x, u_{(p)}) = 0$ if there exists a smooth solution $u = f(x)$ of the system $\mathcal{L}(x, u_{(p)}) = 0$ which is defined in some neighborhood of x and satisfying $u_{(p)} = \mathrm{pr}^{(p)} f(x)$ where $\mathrm{pr}^{(p)} f(x)$ denotes the collection of all derivatives of $f(x)$ of order up to and including p.

It is *locally solvable* if it is locally solvable at every point of the set

$$\{(x, u_{(p)}) \colon \mathcal{L}(x, u_{(p)}) = 0\}.$$

Definition 2.9. A system of differential equations is said to be *non-degenerate* if it is locally solvable and of maximal rank at each point of the set

$$\{(x, u_{(p)}) \colon \mathcal{L}(x, u_{(p)}) = 0\}.$$

The importance of non-degeneracy is illustrated in the following theorem:

Theorem 2.2. *Suppose $\mathcal{L}(x, u_{(p)}) = 0$ is a non-degenerate system of differential equations. A vector field X on the underlying space $\mathbb{R}^n \times \mathbb{R}^m$ is an infinitesimal symmetry of the system if and only if*

$$X_{(p)} \cdot L = 0 \quad \textit{whenever} \quad \mathcal{L}(x, u_{(p)}) = 0.$$

For a proof, see [33].

3

Classes of differential equations and equivalence transformations

We give here a short exposition of the main technical concepts that are used in our approach to the group classification of differential equations: these are the idea of a class of differential equations and equivalence transformations and concepts related to them.

3.1 Classes of differential equations

To introduce the concept of a *class* of differential equations we look at the example of an evolution equation of second order:

$$u_t = F(t, x, u, u_x, u_{xx}),$$

where F is a smooth function of its arguments, and it belongs to a class of equations with the following conditions on F

$$F_{u_t} = F_{u_{tx}} = F_{u_{tt}} = 0, \quad F_{u_{xx}} \neq 0.$$

We call F an *arbitrary element* of the system

$$u_t = F(t, x, u, u_x, u_{xx}), \quad F_{u_t} = F_{u_{tx}} = 0, \quad F_{u_{xx}} \neq 0.$$

This can be symbolized by

$$L_\theta(t, x, u_{(2)}, \theta(t, x, u_{(2)})) = 0$$

where $\theta = F$ and belongs to the set of (locally) smooth functions satisfying

$$\theta_{u_t} = \theta_{u_{tx}} = \theta_{u_{tt}} = 0, \quad \theta_{u_{xx}} \neq 0.$$

The functions θ are called *arbitrary elements*. Note that in a system of differential equations, there will in general be more than one arbitrary element.

In our work we follow [33], and we consider systems of differential equations

$$\mathcal{L}_\theta\colon\, L(x, u_{(p)}, \theta_{(q)}(x, u_{(p)})) = 0,$$

parameterized by the tuple of arbitrary elements $\theta(x, u_{(p)})$, where θ denotes the tuple $\theta = (\theta^1, \dots, \theta^k)$ and $\theta_{(q)}$ is the set of derivatives of θ up to and including order q. This tuple of arbitrary elements θ satisfies an auxiliary system of equations $\mathcal{S}$ which consists of equations of the form $S(x, u_{(p)}, \theta_{(q)}(x, u_{(p)})) = 0$. This auxiliary system may also be augmented by some non-vanishing condition written representatively as $\Sigma(x, u_{(p)}, \theta_{(q)}(x, u_{(p)})) \neq 0$, which is taken to mean that no component of Σ vanishes.

In this way, we have the *class of systems of differential equations* $\mathcal{L}_\theta$ where the arbitrary elements θ run through $\mathcal{S}$. We denote this class by $\mathcal{L}|_\mathcal{S}$.

Example 3.1

In the above example of the class of second order evolution equation we have: $n = 2$, $m = 1$, $p = 2$, $l = 1$, $\theta = F$, the auxiliary system of the class is $\mathcal{S}$: $\theta_{u_t} = \theta_{u_{tx}} = \theta_{u_{tt}} = 0$ and the non-vanishing condition is $\theta_{u_{xx}} \neq 0$.

Example 3.2

Consider the class of second order nonlinear wave equations of the form

$$u_{tt} = f(x, u_x)u_{xx} + g(x, u_x),$$

where the arbitrary elements f and g depend on x and u_x. We have a single system with two arbitrary elements $\theta^1 = f$ and $\theta^2 = g$ depending on x and u_x. The auxiliary system associated to this class is formed by the equations:

$$\begin{aligned}
&\theta^1_t = \theta^1_u = \theta^1_{u_t} = \theta^1_{u_{tx}} = \theta^1_{u_{xx}} = \theta^1_{u_{tt}} = 0,\\
&\theta^2_t = \theta^2_u = \theta^2_{u_t} = \theta^2_{u_{tx}} = \theta^2_{u_{xx}} = \theta^2_{u_{tt}} = 0.
\end{aligned}$$

The nonvanishing conditions are $\theta^1 \neq 0$ and $(\theta^1_{u_x}, \theta^2_{u_x u_x}) \neq (0, 0)$.

Example 3.3

For the class of linear Schrödinger equations $i\psi_t + \psi_{xx} + V(t, x)\psi = 0$, we have a single arbitrary element $\theta = V$ with the auxiliary system of the form

$$\begin{aligned}
&\theta_\psi = \theta_{\psi^*} = \theta_{\psi_t} = \theta_{\psi^*_t} = \theta_{\psi_x} = \theta_{\psi^*_x} = 0,\\
&\theta_{\psi_{tx}} = \theta_{\psi^*_{tx}} = \theta_{\psi_{tt}} = \theta_{\psi^*_{tt}} = \theta_{\psi_{xx}} = \theta_{\psi^*_{xx}} = 0.
\end{aligned}$$

3.2 Equivalence transformations

Group classification of differential equations uses the notion of *equivalence transformations*. An equivalence transformation is essentially an invertible transformation that sends an equation of a particular class to another equation of that class. In this section we look at equivalence transformations of classes of differential equations and some of their properties. More details can be found in [40].

If we take two systems of equations $\mathcal{L}_\theta$ and $\mathcal{L}_{\tilde{\theta}}$ from a given class $\mathcal{L}|_\mathcal{S}$, then the elements θ, $\tilde{\theta}$ may or may not be equivalent under some invertible transformation. For instance, the evolution equations

$$u_t = u^2\left[\frac{u_3}{u_1^3} - \frac{3}{2}\frac{u_2^2}{u_1^4}\right], \qquad u_t = \left[\frac{u_3}{u_1^3} - \frac{3}{2}\frac{u_2^2}{u_1^4}\right]$$

cannot be mapped into one another by any equivalence transformation of the class of third-order evolution equations of the form $u_t = F(t, x, u, u_1, u_2)u_3 + G(t, x, u, u_1, u_2)$. Thus we are led to the concept of *admissible transformations*: that is, transformations that map θ to $\tilde{\theta}$.

Definition 3.1. If $\mathcal{L}_\theta$ and $\mathcal{L}_{\tilde{\theta}}$ are two systems belonging to the class $\mathcal{L}|_\mathcal{S}$ we say *an admissible transformation* from $\mathcal{L}_\theta$ to $\mathcal{L}_{\tilde{\theta}}$ is an invertible point transformation of the underlying space having local coordinates (x, u) which maps $\mathcal{L}_\theta$ to $\mathcal{L}_{\tilde{\theta}}$. We denote by $T(\theta, \tilde{\theta})$ the set of point transformations from $\mathcal{L}_\theta$ to $\mathcal{L}_{\tilde{\theta}}$.

Definition 3.2. We define the set of *admissible transformations in the class of differential equations* as

$$\mathcal{G}^\sim(\mathcal{L}|_\mathcal{S})\colon = \{(\theta, \tilde{\theta}, \varphi) | \theta, \tilde{\theta} \in \mathcal{S}, \varphi \in \mathrm{T}(\theta, \tilde{\theta})\}.$$

We note (see [23]) that the admissible transformations for the class $\mathcal{L}|_\mathcal{S}$ have the structure of a *groupoid*: the identity transformation is quite obviously an admissible transformation; if $\varphi \in \mathrm{T}(\theta_1, \theta_2)$ and $\psi \in \mathrm{T}(\theta_3, \theta_4)$ then the composition $\psi \circ \varphi$ is defined only if $\theta_2 = \theta_3$ and then $\psi \circ \varphi \in \mathrm{T}(\theta_1, \theta_4)$. The associativity of composition is inherited from the associativity of point transformations: $\varphi \circ (\psi \circ \rho) = (\varphi \circ \psi) \circ \rho$ whenever the composition is defined. We are thus led to the following definition:

Definition 3.3. The set of all admissible transformations with the groupiod structure described above is called the *equivalence groupoid* of the class $\mathcal{L}|_\mathcal{S}$ and is denoted by $\mathcal{G}^\sim$. Thus, $\mathcal{G}^\sim = \mathcal{G}^\sim(\mathcal{L}|_\mathcal{S})\colon = \{(\theta, \tilde{\theta}, \varphi) \mid \theta, \tilde{\theta} \in \mathcal{S}, \varphi \in \mathrm{T}(\theta, \tilde{\theta})\}$.

The equivalence groupoid $\mathcal{G}^\sim$ is computed by the use of the direct method. To do this, we fix two arbitrary systems $\mathcal{L}_\theta\colon = L(x, u_{(p)}, \theta_{(q)}(x, u_{(p)})) = 0$, and $\mathcal{L}_{\tilde{\theta}}\colon = L(\tilde{x}, \tilde{u}_{(p)}, \tilde{\theta}_{(q)}(\tilde{x}, \tilde{u}_{(p)})) = 0$, and search for invertible point transformations,

$$\varphi\colon \tilde{x}^i = X^i(x, u), \quad \tilde{u}^a = u^a(x, u), \quad i = 1, \ldots, n; \quad a = 1, \ldots, m,$$

that connect the two systems. Using the chain rule we extend these transformations to transformations of derivatives, expressing the derivatives of tilded variables in terms of untilded ones. In this way we obtain a transformed system which should be satisfied identically when substituted into the system $\mathcal{L}_\theta$. This substitution leads to the system of determining equations for transformational components of φ. Solving these determining equations gives the required $\mathcal{G}^\sim$. A particular case of such transformations is that of symmetries: they map a given system $\mathcal{L}_\theta$ to itself. The set of all such symmetries is denoted by G_θ.

Definition 3.4. The *maximal symmetry group* of the system of differential equations $\mathcal{L}_\theta$ for a fixed $\theta \in \mathcal{S}$ is the group G_θ of transformations in the space of variables (x, u) such that any solution of the system $\mathcal{L}_\theta$ is mapped to a solution of the same system. That is $G_\theta\colon = \mathrm{T}(\theta, \theta)$.

The intersection of the maximal point symmetry groups of all systems from this class is called the *kernel group*, $G^\cap = G^\cap(\mathcal{L}|_\mathcal{S})\colon = \bigcap_{\theta\in\mathcal{S}} G_\theta$.

The generators of one-parameter subgroups of the groups G_θ and $G^\cap$ form the Lie algebras $\mathfrak{g}_\theta$ and $\mathfrak{g}^\cap$, respectively, and are called the *maximal Lie invariance algebra* of the system $\mathcal{L}_\theta$ and the *kernel invariance algebra* of the class $\mathcal{L}|_\mathcal{S}$.

3.3 Equivalence groups

The notion of equivalence group, developed in more detail in [33], plays an important role in the group classification of differential equations. In this section we look at the equivalence group of the class of differential equations $\mathcal{L}|_\mathcal{S}$, some of its properties and generalization.

3.3.1 Usual equivalence group and equivalence algebra

Definition 3.5. The *usual equivalence group* $G^\sim$ of the class $\mathcal{L}|_\mathcal{S}$ is the set of transformations $\mathcal{T}$ satisfying the following properties:

1. $\mathcal{T}$ is a point transformation in the jet space $J^p(\mathbb{R}^n, \mathbb{R}^m) \times \mathcal{S}$ endowed with local coordinates $(x, u_{(p)}, \theta)$, which is *projectable* onto the space $J^q(\mathbb{R}^n, \mathbb{R}^m)$ with local coordinates $(x, u_{(q)})$ for each $0 \leqslant q \leqslant p$, so that $\mathcal{T}|_{(x,u_{(q)})}$ is the qth order prolongation of $\mathcal{T}|_{(x,u)}$ for all $\theta \in \mathcal{S}$ for which $\tilde\theta \in \mathcal{S}$ and $\mathcal{T}|_{(x,u)} \in \mathrm{T}(\theta, \tilde\theta)$.

2. $\mathcal{T}$ maps every system from the class $\mathcal{L}|_\mathcal{S}$ to a system from the same class.

Remark 3.1. From Definition 3.5 the notation $\mathcal{T}|_{(x,u_{(q)})}$ means the restriction of $\mathcal{T}$ to the space $J^q(\mathbb{R}^n, \mathbb{R}^m)$. The symbol π will stand for the projection operator

$$\begin{aligned}\pi\colon\ & J^p(\mathbb{R}^n, \mathbb{R}^m) \times \mathcal{S} \to J^p(\mathbb{R}^n, \mathbb{R}^m),\\ & \pi(x, u_{(p)}, \theta) \to (x, u_{(p)}).\end{aligned}$$

The admissible transformations $(\theta, \tilde\theta, \mathcal{T}|_{(x,u)})$, where θ, $\tilde\theta \in \mathcal{S}$ and $\mathcal{T} \in G^\sim$ obtained in this way are said to be *induced by the transformations* of $G^\sim$.

Example 3.4

For the class of equations $i\psi_t + \psi_{xx} + V(t,x)\psi = 0$, where ψ is unknown complex function of (t,x) and V is an arbitrary complex-valued smooth function depending on its arguments, the point transformations $\tilde{t} = t$, $\tilde{x} = -x$, $\tilde{\psi} = \psi$ and $\tilde{V} = V$ is an equivalence transformation.

Remark 3.2. It is important to note that the entire equivalence group $G^{\sim}$ of the class $\mathcal{L}|_{\mathcal{S}}$ is generated by two kinds of transformations: continuous transformations and discrete transformations. When studying the group classification of classes of differential equations, one usually looks at the continuous transformations and in particular the identity component, which contains one-parameter subgroups.

Both the equivalence groupoid $\mathcal{G}^{\sim}$ and the equivalence group $G^{\sim}$ of the class of differential equations $\mathcal{L}|_{\mathcal{S}}$ play a crucial role in the group classification of differential equations. The knowledge of the groupoid $\mathcal{G}^{\sim}$ of the class $\mathcal{L}|_{\mathcal{S}}$ simplifies the process of classifying Lie symmetry extensions of this class. Using $\mathcal{G}^{\sim}$, we can easily obtain the equivalence group $G^{\sim}$ of the class $\mathcal{L}|_{\mathcal{S}}$. The group classification of the class under study is easily investigated once these objects are related by the point transformations from this class. The properties of point transformations with respect to these objects are given in the section below.

We close this section with a description of the set of generators of the group $G^{\sim}$.

Definition 3.6. We call an *equivalence algebra* of the class $\mathcal{L}|_{\mathcal{S}}$ denoted by $\mathfrak{g}^{\sim}$, a Lie algebra formed by the set of generators of one-parameter subgroups of the equivalence group $G^{\sim}$.

It is convenient, for a given class of differential equations, to use the direct method to find the equivalence groupoid $\mathcal{G}^{\sim}$, which is used to construct the equivalence group $G^{\sim}$. Once the latter is known, we can find the infinitesimal generators of $G^{\sim}$, which constitute the algebra $\mathfrak{g}^{\sim}$.

3.3.2 Generalized and generalized extended equivalence groups

The notion of equivalence group can be generalized in several ways. For the "usual equivalence group", the transformations of independent and dependent variables (x,u) do not dependent on the arbitrary element θ. In this case the equivalence transformations are of the form $(\tilde{x},\tilde{u}) = (X(x,u),U(x,u))$, $\tilde{\theta} = \Theta(x,u,\theta)$. The class $\mathcal{L}|_{\mathcal{S}}$, with usual equivalence group $G^{\sim}$, may have other equivalence transformations which do not belong to $G^{\sim}$ and, together with this group, form different variations of equivalence groups [33], [44].

If the transformations of the variables (x,u) depend on the arbitrary element θ then the equivalence group is called the *generalized equivalence group* and is denoted by $G^{\sim}_{\rm gen}$. More specifically, this means that for any $\mathcal{T} \in G^{\sim}_{\rm gen}$ is a point transformation in the space (x,u,θ) such that

$$\forall\theta \in \mathcal{S}\colon \tilde{\theta} \in \mathcal{S} \text{ and } \mathcal{T}(x,u,\theta(x,u))|_{(x,u)} \in \mathrm{T}(\theta,\tilde{\theta}).$$

This is equivalent to saying that the generalized equivalence transformations are of the form $(\tilde{x}, \tilde{u}, \tilde{\theta}) = (X, U, \Theta)(x, u, \theta)$. When the transformations of the arbitrary elements are expressed in terms of the old arbitrary elements of the class in a non-local way (involving the integrals of the arbitrary elements) then the corresponding equivalence group is called the *extended equivalence group* and is denoted by $\hat{G}^{\sim}$. Furthermore, if, in addition, the transformational components of the equivalence transformations (both components for variables (x, u) and arbitrary elements) are expressed in terms of the arbitrary elements of the class in a non-local way then the equivalence group is said to be the *generalized extended equivalence group* and it is denoted by $\hat{G}^{\sim}_{\text{gen}}$. Different types of these equivalence groups are discussed in [22], [44].

Example 3.5

Consider the class of variable coefficient diffusion-convection equations of the form

$$f(x)u_t = (g(x)A(u)u_x)_x + h(x)B(u)u_x, \tag{3.1}$$

where $f = f(x)$, $g = g(x)$, $h = h(x)$, $A = A(u)$ and $B = B(u)$ are arbitrary smooth functions of their variables, $fghA \neq 0$ and $(A_u, B_u) \neq (0, 0)$ studied in [22]. The usual equivalence group $G^{\sim}$ of this class consists of the point transformations of the form:

$$\begin{aligned}
&\tilde{t} = \sigma_1 t + \sigma_2, \quad \tilde{x} = X(x), \quad \tilde{u} = \sigma_3 u + \sigma_4,\\
&\tilde{f} = \frac{\varepsilon_1 \sigma_1}{X_x} f, \quad \tilde{g} = \varepsilon_1 {\varepsilon_2}^{-1} X_x g, \quad \tilde{h} = \varepsilon_1 {\varepsilon_3}^{-1}, \quad \tilde{A} = \varepsilon_2 A, \quad \tilde{B} = \varepsilon_3 B,
\end{aligned}$$

where $\sigma_j (j = 1, \ldots, 4)$ and $\varepsilon_i (i = 1, \ldots, 3)$ are arbitrary constants, $\sigma_1\sigma_3\varepsilon_1\varepsilon_2\varepsilon_3 \neq 0$, X an arbitrary smooth function of x with $X_x \neq 0$ while the transformations

$$\begin{aligned}
&\tilde{t} = \sigma_1 t + \sigma_2, \quad \tilde{x} = X(x), \quad \tilde{u} = \sigma_3 u + \sigma_4,\\
&\tilde{f} = \frac{\varepsilon_1 \sigma_1 \varphi}{X_x} f, \quad \tilde{g} = \varepsilon_1 {\varepsilon_2}^{-1} X_x \varphi g, \quad \tilde{h} = \varepsilon_1 {\varepsilon_3}^{-1} \varphi h,\\
&\tilde{A} = \varepsilon_2 A, \quad \tilde{B} = \varepsilon_3 (B + \varepsilon_4 A),
\end{aligned}$$

where $\sigma_j (j = 1, \ldots, 4)$ and $\varepsilon_i (i = 1, \ldots, 4)$ are arbitrary constants, $\sigma_1\sigma_3\varepsilon_1\varepsilon_2\varepsilon_3 \neq 0$, X an arbitrary smooth function of x with $X_x \neq 0$, $\varphi = \exp\left(-\varepsilon_4 \int \frac{h(x)}{g(x)} \mathrm{d}x\right)$, constitute an extended equivalence group $\hat{G}^{\sim}$.

Example 3.6

The equivalence transformations corresponding to the class of variable-coefficient Korteweg de-Vries equations of the form $u_t + f(t, x)uu_x + g(t, x)u_{xxx} = 0$ with $fg \neq 0$ studied in [44] whose form is

$$\tilde{t} = T(t) \quad \tilde{x} = \frac{\sigma_1 x + \sigma_2}{\varphi(t)} + \sigma_5, \quad \tilde{u} = \varphi(t)u - \sigma_3 x - \frac{\sigma_2\sigma_3}{\sigma_1},$$

$$\tilde{f}(\tilde{t}) = \frac{\sigma_1}{T_t(\varphi(t))^2} f(t), \quad \tilde{g}(\tilde{t}, \tilde{x}) = \frac{\sigma_1^3}{T_t(\varphi(t))^3} g(t, x),$$

where T is an arbitrary function of t with $T_t \neq 0$, $\varphi = \sigma_3 \int f(t)\mathrm{d}t + \sigma_4$ and $\sigma_j (j = 1, \ldots, 5)$ are arbitrary constants such that $\sigma_1(\sigma_3^2 + \sigma_4^2) \neq 0$ constitute a generalized extended equivalence group $\hat{G}^{\sim}_{\text{gen}}$ for that class.

3.3.3 Gauge equivalence group

We continue here the discussion of different type of equivalence transformations for the given class $\mathcal{L}|_{\mathcal{S}}$. Here we are concerned with equivalence transformations which act only on the arbitrary elements and preserve the space of variables.

Definition 3.7. The equivalence transformations from the group $G^{\sim}$ which act only on the arbitrary elements (and do not change the systems of differential equations), generate *gauge admissible transformations.* Equivalence transformations of this form are considered as trivial (they are "gauges") and are called *gauge equivalence transformations.* They constitute the *gauge* (normal) subgroup $G^{g\sim}$, $G^{g\sim} = \{\mathcal{T} \in G^{\sim} | \mathcal{T}x = x, \mathcal{T}u = u, \mathcal{T}\theta \sim \theta\}$ of the equivalence group $G^{\sim}$. As in the previous consideration, we may have different equivalence groups and also their corresponding gauge subgroups. The values of the arbitrary elements θ and $\tilde{\theta}$ in $\mathcal{S}$ are *gauge equivalent* if the systems $\mathcal{L}_\theta$ and $\mathcal{L}_{\tilde{\theta}}$ are the same systems of differential equations. Here their sets of solutions coincide.

Example 3.7

Consider the class $\mathcal{V}$ of nonlinear Schrödinger equations with potentials and modular nonlinearity of the form

$$i\psi_t + \Delta\psi + f(|\psi|)\psi + V(t, x)\psi = 0, \quad f_{|\psi|} \neq 0$$

and consider an equation $\mathcal{L}_V$ from this class for any fixed V. Then the equivalence transformations

$$\tilde{t} = t, \quad \tilde{x} = x, \quad \tilde{\psi} = \psi,$$
$$\tilde{f} = f + \beta, \quad \tilde{V} = V - \beta$$

with the arbitrary complex number β, which map an equation $\mathcal{L}_V$ from the class $\mathcal{V}$ to an equation $\mathcal{L}_{\tilde{V}} \colon i\psi_t + \Delta\psi + \tilde{f}(|\psi|)\psi + \tilde{V}(t, x)\psi = 0$, they do not change the form of the class under study (they preserve the equations of differential equations). Thus, they constitute the gauge equivalence group of that class.

3.3.4 Conditional equivalence groups

We look at the structure of the equivalence transformations arising from the classes and their subclasses that are singled out from these larger classes by imposing some conditions on the set of arbitrary elements. For more details the reader is referred to [33].

Consider the class $\mathcal{L}|_{\mathcal{S}}$ whose equivalence group $G^{\sim}$ is known and let $\mathcal{L}|_{\mathcal{S}'}$ be a subclass singled out from the class $\mathcal{L}|_{\mathcal{S}}$ by imposing additional constraints on the sets $\mathcal{S}'$ and Σ' of the form $\mathcal{S}'(x, u_{(p)}, \theta_{(q')}) = 0, \quad \Sigma'(x, u_{(p)}, \theta_{(q')}) \neq 0$ with respect to the arbitrary elements $\theta = \theta(x, u_{(p)})$, where $\mathcal{S}' \subset \mathcal{S}$ is the set of solution of the united system $\mathcal{S} = 0$, $\Sigma \neq 0$, $\mathcal{S}' = 0$, $\Sigma' \neq 0$.

Definition 3.8. The equivalence group $G^{\sim}$ of the class $\mathcal{L}|_{\mathcal{S}'}$, $G^{\sim}(\mathcal{L}|_{\mathcal{S}'})$, is called the *conditional equivalence group* of the whole class $\mathcal{L}|_{\mathcal{S}}$ under the conditions $\mathcal{S}' = 0$ and $\Sigma' \neq 0$. The conditional equivalence group is called *nontrivial* if and only if it is not a subgroup of $G^{\sim}(\mathcal{L}|_{\mathcal{S}})$. It is said to be *maximal* under the above conditions on $\mathcal{S}'$ and Σ' if for any subclass $\mathcal{L}|_{\mathcal{S}''}$ of the class $\mathcal{L}|_{\mathcal{S}}$ containing the subclass $\mathcal{L}|_{\mathcal{S}'}$ we have $G^{\sim}(\mathcal{L}|_{\mathcal{S}'}) \nsubseteq G^{\sim}(\mathcal{L}|_{\mathcal{S}''})$. Throughout of this thesis we are concerned with the maximal conditional equivalence groups since they are nontrivial.

3.4 Normalization properties

The study of Lie symmetries in the class of differential equations $\mathcal{L}|_{\mathcal{S}}$ relies on the properties of point transformations that this class possesses. In this section our aim is to describe the property of normalization with respect to point transformations and we will see throughout the thesis how we can relate the equivalence groupoid, the equivalence group and normalization properties of the concerned class. More details on normalization property can be obtained in [24], [33].

Definition 3.9. A class of differential equations $\mathcal{L}|_{\mathcal{S}}$ is said to be *normalized* (in the usual sense) if the equivalence groupoid $\mathcal{G}^{\sim}$ for this class is generated by its equivalence group $G^{\sim}$. This means that for each triple $(\theta, \tilde{\theta}, \varphi)$ from the equivalence groupoid $\mathcal{G}^{\sim}$, one can find an equivalence transformation $\mathcal{T} \in G^{\sim}$ such that $\tilde{\theta} = \mathcal{T}\theta$ and $\varphi = \mathcal{T}|_{(x,u)}$. The class is called *normalized* in the generalized sense if for any triple $(\theta, \tilde{\theta}, \varphi) \in \mathcal{G}^{\sim}$, there exists an equivalence transformation $\mathcal{T} \in G^{\sim}_{\mathrm{gen}}$ with $\tilde{\theta} = \mathcal{T}\theta$ and $\varphi = \mathcal{T}(x, u, \theta(x, u))|_{(x,u)}$.

Example 3.8

Consider the class of second order ordinary differential equations,

$$y_{tt} + a_1(t)y_t + a_0(t)y = b(t), \tag{3.2}$$

where a_1, a_0 and b_0 are real-valued arbitrary functions of t and $y = y(t)$ is unknown function of t. Its equivalence groupoid $\mathcal{G}^{\sim}$ is constituted by triples $\big((a_1, a_0, b), (\tilde{a_1}, \tilde{a_0}, \tilde{b}), \varphi\big)$, where φ is a point transformations in the space of variables whose components are

$$T = T(t), \quad Y = Y^1(t)x + Y^0(t), \tag{3.3}$$

and the transformed arbitrary elements $(\tilde{a_1}, \tilde{a_0}, \tilde{b})$ are given by

$$\begin{aligned} \tilde{a_1} &= a_1 - 2\frac{Y_t^1}{Y^1} - \frac{T_{tt}}{T_t}, \quad \tilde{a_0} = a_0 + \frac{1}{Y^1}\left(T_{tt}Y_t^1 - Y_{tt}^1 - \tilde{a}_1 Y_t^1\right) \\ \tilde{b} &= \frac{1}{T_t^2}\left(bY^1 + Y_{tt}^0 + \tilde{a}_1 Y_t^0 - Y_t^0\frac{T_{tt}}{T_t} + \tilde{a}_0 Y^0\right). \end{aligned} \tag{3.4}$$

Here T, Y^1 and Y^0 are real-valued functions of t with $T_tY^1 \neq 0$. The class (3.2) is normalized and its equivalence group $G^\sim$ is constituted by the point transformations of the form (3.3) and (3.4). More generally, the class of homogeneous (resp. inhomogeneous) linear systems of differential equations is normalized.

Example 3.9

The class of Burgers equations,

$$u_t + uu_x + f(t,x)u_{xx} = 0, \quad f \neq 0$$

whose equivalence group $G^\sim$ consists of the point transformations of the form

$$\begin{aligned} \tilde{t} &= \frac{a_1 t + a_0}{a_3 t + a_2}, \quad \tilde{x} = \frac{\kappa x + \alpha_1 t + \alpha_0}{a_3 t + a_2}, \\ \tilde{u} &= \frac{\kappa(a_3 t + a_2)u - \kappa a_3 x + \alpha_1 a_2 - \alpha_0 a_3}{a_1 a_2 - a_0 a_3}, \quad \tilde{f} = \frac{\kappa^2}{a_1 a_2 - a_0 a_3} f, \end{aligned}$$

where a_1, a_0, a_3, a_2, α_1,α_0 and κ are constants with $a_1a_2 - a_0a_3 \neq 0$ and $\kappa \neq 0$, is normalized.

From the knowledge of the equivalence groupoid $\mathcal{G}^\sim$, one can check by inspection whether or not the class under study is normalized (in the usual sense). It is true when the transformational part φ of each admissible transformation does not depend on the fixed initial value θ of the arbitrary element tuple, and is hence appropriate for an arbitrary initial value of the arbitrary element tuple. Furthermore, the prolongation of φ to the space $(x, u_{(p)})$ and the further extension to the arbitrary elements according to the relation between the arbitrary elements θ and $\tilde{\theta}$ give a point transformation in the joint space with local coordinates $(x, u_{(p)}, \theta)$, see [5]. If the class $\mathcal{L}|_\mathcal{S}$ is normalized, then the projection of the equivalence algebra of this class contains the linear span of the the maximal Lie invariance algebras of equations from the class $\mathcal{L}|_\mathcal{S}$.

Definition 3.10. A class of differential equation $\mathcal{L}|_\mathcal{S}$ is called *semi-normalized* if its equivalence groupoid $\mathcal{G}^\sim$ is induced by transformations from its equivalence group $G^\sim$ and maximal point symmetry groups G_θ of equations from this class, i.e., for any triple $(\theta, \tilde{\theta}, \varphi) \in \mathcal{G}^\sim$, $\exists \mathcal{T} \in G^\sim$ and $\varphi^1 \in G_\theta$ such that $\tilde{\theta} = \mathcal{T}\theta$ and $\varphi = \mathcal{T}|_{(x,u)} \circ \varphi^1$.

Example 3.10

The class of second order ordinary differential equations of the form (3.2) with $b(t) = 0$ is semi-normalized. Its equivalence group $G^{\sim}$ consists of point transformations of the form (3.3) and (3.4) with $Y^0 = 0$.

Besides these two properties of the normalization, we define another useful property in the group classification of differential equations: *uniform semi-normalization.* The theory of uniform semi-normalization, which is new in group classification, is developed in the papers that make up this thesis. Here we give only the definition; the complete theory, together with the Theorem on splitting of symmetry groups in uniformly semi-normalized classes, is presented in the paper [24].

Definition 3.11. Let $\mathcal{G}^{\sim}$ and $G^{\sim}$ be the equivalence groupoid and the (usual) equivalence group for the class $\mathcal{L}|_{\mathcal{S}}$. Let for each $\theta \in \mathcal{S}$ the point symmetry group G_θ of the equation $\mathcal{L}_\theta \in \mathcal{L}|_{\mathcal{S}}$ contains a subgroup N_θ such that the family $\mathcal{N}_{\mathcal{S}} = \{N_\theta \mid \theta \in \mathcal{S}\}$ of all these subgroups satisfies the following properties:

1. $\mathcal{T}|_{(x,u)} \notin N_\theta$ for any $\theta \in \mathcal{S}$ and any nonidentity $\mathcal{T} \in G^{\sim}$,
2. $N_{\mathcal{T}\theta} = \mathcal{T}|_{(x,u)} N_\theta (\mathcal{T}|_{(x,u)})^{-1}$ for any $\theta \in \mathcal{S}$ and any $\mathcal{T} \in G^{\sim}$,
3. For any $(\theta^1, \theta^2, \varphi) \in \mathcal{G}^{\sim}$ there exist $\varphi^1 \in N_{\theta^1}$, $\varphi^2 \in N_{\theta^2}$ and $\mathcal{T} \in G^{\sim}$ such that $\theta^2 = \mathcal{T}\theta^1$ and $\varphi = \varphi^2(\mathcal{T}|_{(x,u)})\varphi^1$,

where $\mathcal{T}|_{(x,u)}$ denotes the restriction of $\mathcal{T}$ to the space with local coordinates (x, u). Then the class $\mathcal{L}|_{\mathcal{S}}$ is called *uniformly semi-normalized* with respect to the symmetry-subgroup family $\mathcal{N}_{\mathcal{S}}$.

We end this section by giving the relation between these three properties in terms of point transformations. If we denote by `N`, `USN` and `SN` the set of normalized, uniformly semi-normalized and semi-normalized classes, respectively, then the following relation holds: $\mathtt{N} \subset \mathtt{USN} \subset \mathtt{SN}$. From this relation we see that each normalized class of differential equations is uniformily semi-normalized and semi-normalized, and each uniformly semi-normalized class is semi-normalized. But there are semi-normalized classes that are not uniformly semi-normalized. For instance, the class of nonlinear diffusion equation of the form $u_t = (f(u)u_x)_x$ with $f \neq 0$ is semi-normalized but not uniformly semi-normalized.

4

Group classification problem

4.1 Formulation of the problem

The study of Lie symmetries of differential equations is in general quite complex. It is relatively simple when one investigates an ordinary differential equation since the symmetry analysis can be done by the use of Lie's infinitesimal method. However, this method is not effective when arbitrary elements are involved, and for this situation more advanced techniques are required. We will look at the *group classification problem* for Schrödinger equations. That is, we will look for the possible symmetries that these equations may possess and classify them according to the types of symmetries that are admitted. The solution of this problem is based on the properties of point transformations that the class under study possesses as well as their equivalence relations.

The group classification problem can be stated as follows: *given a class of differential equations $\mathcal{L}|_{\mathcal{S}}$ (in an appropriate jet space), find, for a given $\theta \in \mathcal{S}$, all possible inequivalent extensions of the maximal Lie invariance algebra $\mathfrak{g}_\theta$ together with all $G^{\sim}$-inequivalent families of the arbitrary element θ that satisfy the condition $\mathfrak{g}_\theta \neq \mathfrak{g}^{\cap}$.*

4.2 Methods to solve the problem of group classification

Here, we list different cases that may occur when one attempts to solve the group classification problem.

First of all, one finds, for the class $\mathcal{L}|_{\mathcal{S}}$, the equivalence groupoid $\mathcal{G}^{\sim}$ and the equivalence group $G^{\sim}$ whose associated equivalence algebra is $\mathfrak{g}^{\sim}$. It is in this step that we will know if the class under study possesses the normalization property in view of Section 3.4.

4.2.1 Infinitesimal Lie's method and direct integration of determining equations

For each system $\mathcal{L}_\theta$, θ fixed from the class $\mathcal{L}|_\mathcal{S}$, a vector field X,

$$X = \xi^i(x,u)\partial_{x_i} + \eta^a(x,u)\partial_{u^a},$$

belongs to the algebra $\mathfrak{g}_\theta$ if and only if it satisfies the infinitesimal Lie invariance criterion, which is the essence of the infinitesimal Lie's method, see Theorem 2.2. This invariance criterion, applied to X, leads to the determining equations for the coefficients of the symmetry vector fields in the algebra $\mathfrak{g}_\theta$ associated with that equation. This yields an overdetermined systems of linear homogeneous partial differential equations which can then be solved. When the class under study has a simple structure, possesses constant arbitrary elements or a single arbitrary element, the determining equations for Lie symmetries of equations from the class $\mathcal{L}|_\mathcal{S}$ can be solved by looking at the *compatibility analysis and direct integration* of these equations (up to equivalence relations). This framework leads to an exhaustive group classification for the class under study.

4.2.2 Algebraic method for group classification

For classes with a complex structure, the determining equations lead to overdetermined systems which can be split into two parts: The first which involves arbitrary elements and the second without arbitrary elements. Those which do not involve the arbitrary elements are integrated immediately and give the components of the symmetry vector fields from $\mathfrak{g}_\theta$. The remaining equations, those containing arbitrary elements, which we refer to as the *classifying equations*, constitute the essential part of the group classification of the class $\mathcal{L}|_\mathcal{S}$. They are investigated in order to obtain Lie symmetry extensions for the kernel invariance algebra $\mathfrak{g}^\cap$ (which is obtained by varying the arbitrary elements and further splitting the classifying equations with respect to various powers of the arbitrary element θ).

A deeper analysis of the classifying equations depends upon whether the class $\mathcal{L}|_\mathcal{S}$ is normalized or not. If the class is normalized then the maximal Lie symmetry extensions $\mathfrak{g}_\theta$ of the kernel invariance algebra $\mathfrak{g}^\cap$ are obtained via the classification of subalgebras of the equivalence algebra, whose projections to the space of independent and dependent variables (x,u) coincide with the maximal Lie invariance algebras of systems from the class $\mathcal{L}|_\mathcal{S}$ [24]. This process reduces the group classification problem to the algebraic problem of classifying inequivalent subalgebras of the equivalence algebra $\mathfrak{g}^\sim$. In this case, for each system $\mathcal{L}_\theta$ with a fixed arbitrary element θ, the solution space of the determining equations is associated with a Lie algebra of vector fields $\mathfrak{g}_\theta$, and $\mathfrak{g}_\theta$ is contained in the linear span $\mathfrak{g}_{\langle\,\rangle}\colon \sum_V \mathfrak{g}_\theta$ when θ varies. The subalgebras of this linear span are called *appropriate subalgebras* and they contain the kernel invariance algebra $\mathfrak{g}^\cap$. This technique for solving the problem of group classification is referred to as the *algebraic method.* If the class under consideration is uniformly semi-normalized, it follows from Definition 3.11 that the linear span $\mathfrak{g}_{\langle\,\rangle}$ has a representation of the form

$$\mathfrak{g}_{\langle\,\rangle} = \mathfrak{g}^{\rm ess}_{\langle\,\rangle} \Subset \mathfrak{g}^{\rm lin}_{\langle\,\rangle},$$

where $\mathfrak{g}^{\rm ess}_{\langle\,\rangle}$ and $g^{\rm lin}_{\langle\,\rangle}$ are a subalgebra and an ideal of $\mathfrak{g}_{\langle\,\rangle}$, respectively. It follows from this representation that the kernel algebra $\mathfrak{g}^{\cap}$ is an ideal in $\mathfrak{g}^{\rm ess}_{\langle\,\rangle}$ and in the whole algebra $\mathfrak{g}_{\langle\,\rangle}$. The form of the algebra $\mathfrak{g}_{\langle\,\rangle}$ given in such a way induces a similar form of the representation of the maximal Lie invariance algebra $\mathfrak{g}_\theta$,

$$\mathfrak{g}_\theta = \mathfrak{g}^{\rm ess}_\theta \Subset \mathfrak{g}^{\rm lin}_\theta$$

where $\mathfrak{g}^{\rm ess}_\theta\colon\ = \mathfrak{g}_\theta \cap \mathfrak{g}^{\rm ess}_{\langle\,\rangle}$ and $\mathfrak{g}^{\rm lin}_\theta\colon\ = \mathfrak{g}_\theta \cap \mathfrak{g}^{\rm lin}_{\langle\,\rangle}$ are a finite-dimensional subalgebra and an infinite-dimensional abelian ideal of $\mathfrak{g}_\theta$, respectively.

Definition 4.1. The algebra $\mathfrak{g}^{\rm ess}_\theta$ with $\mathfrak{g}^{\rm ess}_\theta := \mathfrak{g}_\theta \cap \mathfrak{g}^{\rm ess}_{\langle\,\rangle}$ is called *the essential invariance algebra* corresponding to the system $\mathcal{L}_\theta$ of the class $\mathcal{L}|_\mathcal{S}$.

In this case, the algebra $\mathfrak{g}^{\rm lin}_\theta$ (resp. $g^{\rm lin}_{\langle\,\rangle}$) is considered as the trivial part and the problem of group classification for the class $\mathcal{L}|_\mathcal{S}$ reduces to the classification of appropriate subalgebras of the algebra $\mathfrak{g}^{\rm ess}_\theta$ up to the equivalence relation generated by the action of $\pi G^\sim$.

We close the group classification for normalized classes with the following assertion:

Corollary 4.1. *If the class $\mathcal{L}|_\mathcal{S}$ is normalized in the usual sense then the kernel algebra $\mathfrak{g}^{\cap}$ is an ideal of the maximal Lie algebra $\mathfrak{g}_\theta$ for each $\theta \in \mathcal{S}$.*

For the proof see [4]. It was shown there that, in general, the kernel invariance algebra $\mathfrak{g}^{\cap}$ is not necessarily an ideal of the maximal Lie invariance algebra $\mathfrak{g}_\theta$. This was illustrated by considering the class of (1+1)-dimensional non-linear diffusion equation, $u_t = (F(u)u_x)_x$ with $F \neq 0$, where the algebras $\mathfrak{g}^{\cap}$ and $\mathfrak{g}_\theta$ associated with $F = u^{4/3}$ are $\mathfrak{g}^{\cap} = \langle \partial_t, \partial_x, 2t\partial_t + x\partial_x\rangle$ and $\mathfrak{g}_\theta = \langle \partial_t, \partial_x, 2t\partial_t + x\partial_x, 4t\partial_t + 3u\partial_u, \partial_x, x^2t\partial_x - 3xu\partial_u\rangle$, respectively. It is clear seen that $[\mathfrak{g}^{\cap}, \mathfrak{g}_\theta] \notin \mathfrak{g}^{\cap}$ since $[\partial_x, x^2t\partial_x - 3xu\partial_u] = 2x\partial_x - 3xu\partial_u \notin \mathfrak{g}^{\cap}$.

In contrast to normalized classes, the group classification of non-normalized classes is done by partitioning the class into disjoint normalized subclasses and then investigating each subclass separately. The solution set of inequivalent Lie symmetry extensions together with their corresponding $G^\sim$-inequivalent families of arbitrary elements is the union of all the lists obtained. Both normalized and non-normalized classes are investigated in our thesis and Lie's infinitesimal method, the algebraic method and direct integrability are used.

4.3 Algorithm for solving the problem of group classification

The solution of the problem of group classification of class of differential equation $\mathcal{L}|_\mathcal{S}$ requires several steps to reach completion. Here, we indicate the main steps for the complete solution of the group classification problem for Schrödinger equations presented in our thesis.

- Using direct method find the equivalence groupoid $\mathcal{G}^{\sim}$ of the class $\mathcal{L}|_{\mathcal{S}}$.
- Construct the equivalence group $G^{\sim}$ of the class $\mathcal{L}|_{\mathcal{S}}$ and the equivalence algebra $\mathfrak{g}^{\sim}$ corresponding to this group. This is the main step in the group classification.
- Check the normalization property.
- For an equation $\mathcal{L}_\theta$ with a fixed arbitrary element, find Lie symmetries from the determining equations for this equation. Here we derive the maximal Lie invariance algebra $\mathfrak{g}_\theta$, the kernel invariance Lie algebra $\mathfrak{g}^{\cap}$ and the classifying equations for Lie symmetry extensions.
- Find all possible $G^{\sim}$-inequivalent values of families of the arbitrary elements θ together with all inequivalent Lie symmetry extensions of the kernel invariance algebra, (i.e., the algebras $\mathfrak{g}_\theta$ such that $\mathfrak{g}_\theta \neq \mathfrak{g}^{\cap}$).

Summarizing, the problem of group classification is completely solved when the final group classification list contains all the possible inequivalent cases of extensions, all equations from this list are mutually inequivalent with respect to the transformations from the equivalence group $G^{\sim}$ (or the generalized equivalence group $G^{\sim}_{\mathrm{gen}}$) and the Lie symmetry algebras that are obtained are really the maximal invariance algebras of the corresponding equations.

5

Summary of the papers

Paper I: Algebraic method for group classification of (1+1)-dimensional linear Schrödinger equations

We carry out the group classification of the class of (1+1)-dimensional linear Schrödinger equations with complex potentials of the form

$$i\psi_t + \psi_{xx} + V(t,x)\psi = 0,$$

where ψ is the complex-valued unknown function of the real variables t and x, and V is the complex-valued smooth function of (t,x) interpreted as potential.

We find the equivalence groupoid $\mathcal{G}^\sim$, the equivalence group $G^\sim$ and the equivalence algebra $\mathfrak{g}^\sim$ of this class. We show that the class under study is uniformly semi-normalized with respect to linear superposition of solutions. This allows us to apply the algebraic method and reduce the group classification problem of linear Schrödinger equations to the classification of low-dimensional appropriate subalgebras of the equivalence algebra of this class up to the equivalence relations. The dimensional of any maximum Lie symmetry extension of an equation from the above class is proved not to be greater than seven.

Splitting the classification cases with respect to two invariant integers results in eight inequivalent families of potentials together with their corresponding inequivalent Lie symmetry extensions.

Paper II: Group classification of multidimensional linear Schrödinger equations with the algebraic method

We consider the group classification problem for Schrödinger equations with complex potentials in dimension $(1+n)$ with $n \geqslant 2$,

$$i\psi_t + \Delta\psi + V(t,x)\psi = 0,$$

where ψ is the complex-valued unknown function of the variables $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$, and the complex-valued smooth parameter-function V of (t,x) is interpreted as potential.

We find the equivalence groupoid and the equivalence group of the given class in closed form and then show that the class is uniformly semi-normalized. This class admits the similar kernel invariance algebra $\mathfrak{g}^{\cap}$ as the corresponding class investigated in Paper I. We proceed to the calculation of the maximal Lie invariance algebra $\mathfrak{g}_V$ of the above Schrödinger equation for each fixed V. The algebra $\mathfrak{g}_V$ may by essentially higher-dimensional than the algebras obtained in Paper I for the case of (1+1) variables. Moreover, the structure of $\mathfrak{g}_V$ may be much more complicated due to appearing rotations in higher-dimensional spaces.

We show that the span $\mathfrak{g}_{\langle\,\rangle} = \sum_V \mathfrak{g}_V$ is a Lie algebra and can be represented as the semi-direct sum $\mathfrak{g}^{\rm ess}_{\langle\,\rangle} \Subset \mathfrak{g}^{\rm lin}_{\langle\,\rangle}$, where $\mathfrak{g}^{\rm ess}_{\langle\,\rangle}$ and $\mathfrak{g}^{\rm lin}_{\langle\,\rangle}$ are a subalgebra and an abelian ideal of $\mathfrak{g}_{\langle\,\rangle}$, respectively.

In order to find Lie symmetry extensions of the kernel invariance algebra $\mathfrak{g}^{\cap}$, we apply the algebraic method and reduce the group classification problem to the classification of certain low-dimensional subalgebras of the associated equivalence algebra. An important role in the consideration is played by estimates of the dimension of the essential Lie invariance algebra $\mathfrak{g}^{\rm ess}_V$ and its various parts for any potential V. In particular,

$$\dim \mathfrak{g}^{\rm ess}_V \leqslant \frac{n(n+3)}{2} + 5.$$

The group classification of linear Schrödinger equations in the (1+2)-dimensional case is considered in more detail. For the exposition to be systematic, we then introduce the three $\pi G^{\sim}$-invariant integers $k_1 \in \{0,1,2\}$, $k_2 \in \{0,1\}$ and $k_3 \in \{0,1,2,3\}$, respectively. Using these integers, we single out the classification cases and obtain a complete classification of inequivalent Lie symmetry extensions for (1+2)-dimensional linear Schrödinger equations with complex-valued potentials.

Paper III: Group classification of multidimensional nonlinear Schrödinger equations

We study transformational properties for the class $\mathcal{S}$ of multidimensional nonlinear Schrödinger equations,

$$i\psi_t + \Delta\psi + S(t,x,\rho)\psi = 0, \quad S_\rho \neq 0,$$

where t and $x = (x_1,\dots,x_n)$ are real independent variables, ψ is the complex dependent variable and $\rho = |\psi|$, and completely carry out the group classification problem of the class $\mathcal{V}$ of (1+2)-dimensional Schrödinger equations with potentials and modular nonlinearities of the form $i\psi_t + \Delta\psi + f(\rho)\psi + V(t,x)\psi = 0$, where $f_\rho \neq 0$. We start by computing the equivalence groupoid and equivalence group of the class $\mathcal{S}$ and show that this class is normalized. For a fixed equation from this class we find the Lie symmetry properties of each equation and derive from this the classifying equation as well as the kernel invariance algebra of the class $\mathcal{S}$.

Specifying the form of S as $S = f(\rho) + V(t,x)$ with $f_\rho \neq 0$, we obtain the class $\mathcal{V}$, whose the set of admissible transformations does not belong to its equivalence group, i.e., the class $\mathcal{V}$ is not normalized, which makes its group classification more involved. To deal with this challenge, we partition the class $\mathcal{V}$ into the three disjoint normalized subclasses $\mathcal{V}'$, $\mathcal{P}_0$ and $\mathcal{P}_\lambda$, conditioned by $\rho f_{\rho\rho}/f_\rho \neq \lambda\ \forall\lambda \in \mathbb{R}$, $f = \sigma \ln \rho$, and $f = \sigma\rho^\lambda$, respectively, where $\sigma \in \mathbb{C} \setminus \{0\}$ and $\lambda \in \mathbb{R} \setminus \{0\}$.

From the knowledge of the equivalence groupoid and the equivalence group of the class $\mathcal{S}$ we derive, successively, the equivalence groupoids and the equivalence groups for the above restricted classes and show that each class is normalized. This allows us to apply the algebraic method for group classification as developed in [4] and reduce the group classification of each subclass to the classification of subalgebras of the associated equivalence algebra. We show that the dimension of the Lie symmetry group of any equation from each class is not greater than $n(n+3)/2+2$, $n(n+3)/2+3$ and $n(n+3)/2+4$ for $\mathcal{V}'$, $\mathcal{P}_0$ and $\mathcal{P}_\lambda$, respectively.

The complete group classification of the class $\mathcal{V}$ is carried out for $n = 2$. Splitting into different cases, we introduce three integers k_1, k_2 and k_3 with $(k_1, k_2) \in \{0, 1\}$ and $k_3 \in \{0, 1, 2\}$ for $\mathcal{V}'$, $\mathcal{P}_0$ and $k_1 \in \{0, 1, 2, 3\}$, $k_2 \in \{0, 1\}$ and $k_3 \in \{0, 1, 2\}$ for $\mathcal{P}_\lambda$, that characterize the dimension of specific subspaces of the corresponding maximal Lie invariance algebras. As a result, we obtain a complete list of inequivalent Lie symmetry extensions together with the corresponding families of potentials which is made by the union of the three obtained results lists corresponding to each consideration.

Paper IV: Admissible transformations of Schrödinger equations with variable mass and potentials

We study the admissible transformations in the class $\mathcal{A}$ of (1+1)-dimensional generalized nonlinear Schrodinger equations

$$i\psi_t + G(t,x,\psi,\psi^*,\psi_x,\psi_x^*)\psi_{xx} + F(t,x,\psi,\psi^*,\psi_x,\psi_x^*) = 0, \quad G \neq 0$$

where t and x are independent variables, ψ is the complex dependent variable of t and x, F and G are smooth complex-valued functions of their arguments. Then reduce this study to the class of linear Schrödinger equations with variable mass and complex potentials.

We start by computing the equivalence groupoid of the wide generalized class $\mathcal{H}$ of equations, $i\psi_t = H(t,x,\psi,\psi^*,\psi_x,\psi_x^*,\psi_{xx},\psi_{xx}^*)$ with $|H_{\psi_{xx}}| \neq |H_{\psi_{xx}^*}|$, where H is a complex-valued smooth function of its arguments, which covers the class $\mathcal{A}$, and then find the equivalence groupoid and the equivalence group of the class $\mathcal{A}$. From this, we show that the class $\mathcal{A}$ is not normalized. We partition it with respect to conditions $G = -G^*$ and $G \neq -G^*$ into two normalized classes, and then investigate the equivalence groupoids and the equivalence groups of the narrowed classes obtained by specifying the forms of the arbitrary functions G and F using the equivalence groupoid of $\mathcal{A}$. We continue this process until we reach the class of (1+1)-dimensional linear Schrödinger equations with variable mass and complex potentials singled out by $G = 1/m(t,x)$ and $F = V(t,x)\psi$. Using equivalence transformations of this class, we can gauge the arbitrary element G to the canonical value, i.e., we can set the arbitrary element G to be equal to one. Thus we obtain the class of (1+1)-dimensional linear Schrödinger equations with constant mass equal to one and complex potentials, which is well studied in

Paper 1. Therefore, the group classification of the class of Schrödinger equations with variable mass and complex potentials can be reduced to the group classification of (1+1)-dimensional linear Schrödinger equations with constant mass equal to one and complex potentials.

Paper V: Equivalence groupoid for (1+2)-dimensional linear Schrödinger equations with complex potentials

We compute the equivalence groupoid and the equivalence group of the class of linear Schrödinger equations with complex potentials in dimension (1+2). From the knowledge of these objects we show that this class is semi-normalized. More specifically, any admissible transformations in this class is the composition of a symmetry transformation of the initial equation and an equivalence transformation of the class under study.

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Research papers

Paper I

Algebraic method for group classification of (1+1)-dimensional linear Schrödinger equations

Algebraic method for group classification of (1+1)-dimensional linear Schrödinger equations

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We carry out the complete group classification of the class of (1+1)-dimensional linear Schrödinger equations with complex-valued potentials. After introducing the notion of uniformly semi-normalized classes of differential equations, we compute the equivalence groupoid of the class under study and show that it is uniformly semi-normalized. More specifically, each admissible transformation in the class is the composition of a linear superposition transformation of the corresponding initial equation and an equivalence transformation of this class. This allows us to apply the new version of the algebraic method based on uniform semi-normalization and reduce the group classification of the class under study to the classification of low-dimensional appropriate subalgebras of the associated equivalence algebra. The partition into classification cases involves two integers that characterize Lie symmetry extensions and are invariant with respect to equivalence transformations.

1 Introduction

A standard assumption of quantum mechanics requires that the Hamiltonian of a quantum system be Hermitian since this guarantees that the energy spectrum is real and that the time evolution of the system is unitary and hence probability-preserving [10]. For linear Shrödinger equations, this means that only equations with real-valued potentials are considered to be physically relevant. Since the above assumption is, unlike the other axioms of quantum mechanics, more mathematical than physical, attempts at weakening or modifying the Hermitian property of Hamiltonians have recently been made by looking at so-called $\mathcal{PT}$-symmetric Hamiltonians [9], [10], [25]. Here $\mathcal{P}$ is the space reflection (or parity) operator and $\mathcal{T}$ is the time reversal operator. Some complex potentials are associated with non-Hermitian $\mathcal{PT}$-symmetric Hamiltonians. Non-Hermitian $\mathcal{PT}$-symmetric Hamiltonians have been used to describe (observable) phenomena in quantum mechanics, such as systems interacting with electromagnetic fields, dissipative processes such as radioactive decay, the ground state of Bose systems of hard spheres and both bosonic and fermionic degrees of freedom. Other important applications of non-Hermitian $\mathcal{PT}$-symmetric Hamiltonians are to be found in scattering theory which include numerical investigations of various physical phenomena in optics, condensed matter physics, scalar wave equations (acoustical scattering) and Maxwell's equations (electromagnetic scattering), in quasi-exactly solvable Hamiltonians, complex crystals and quantum field theory [9], [10], [25]. In general, however, the physical interpretation of linear Schrödinger equations with complex potentials is not completely clear.

The study of Lie symmetries of Schrödinger equations was begun in the early 1970's after the revival of Lie's classical methods (see for instance [33]). Lie symmetries of the free (1+3)-dimensional Schrödinger equations were first considered in [27]. Therein it was suggested to call the essential part of the maximal Lie symmetry group of the free Schrödinger equation the Schrödinger group. In [28] it was noted that the results of [27] could be extended directly to any number of space variables, and the isomorphism of the Lie symmetry groups of the Schrödinger equations of the n-dimensional harmonic oscillator and of the n-dimensional free fall to the symmetry group for the $(1+n)$-dimensional free Schrödinger equation was proved in [28], [29], [30]. This gave a hint for the construction of point transformations connecting these equations. The problem of finding Lie symmetries of $(1+n)$-dimensional linear Schrödinger equations with real-valued potentials was considered in [11], [30]. In particular, in [30] the general "potential-independent" form of point symmetry transformations of these equations was found under the a priori assumption of fibre preservation. The classifying equation involving both transformation components and the potential was derived and used to obtain an upper bound of dimensions of Lie symmetry groups admitted by linear Schrödinger equations. Then some static potentials of physical relevance were considered, including the harmonic oscillator, the free fall, the inverse square potential, the anisotropic harmonic oscillator and the time-dependent Kepler problem. The case of arbitrary time-independent real-valued potential was studied in [11]. Although it was claimed there that "the general solution and a complete list of such potentials and their symmetry groups are then given for the cases $n = 1, 2, 3$", it is now considered that this list is not complete. Note that in the papers cited above, phase translations and amplitude scalings were ignored, which makes certain points inconsistent.

Similar studies were carried out for the time-dependent Schrödinger equation for the two-dimensional harmonic oscillator and for the two- and three-dimensional hydrogen-like atom in [1], [2]. Closely related research on both first- and higher-order symmetry operators of linear Schrödinger equations and separation variables for such equations was initiated in the same time (see [24] and references therein).

After this "initial" stage of research into linear Schrödinger equations, the study of Lie symmetries was extended to various nonlinear Schrödinger equations [17], [19], [21], [36], [37], [42]. However, the group classification of linear Schrödinger equations with arbitrary complex-valued potentials still remains an open problem.

Our philosophy is that symmetries underlie physical theories and that it is therefore reasonable to look for physically relevant models from a set of models (with undetermined parameters) using symmetry criteria. The selection of possible models is made first by solving the group classification problem for the (class of) models at hand and then choosing a suitable model (or set of models) from the list of models obtained in the classification procedure. This procedure consists essentially of two parts: given a parameterized class of models, first determine the symmetry group that is common for all models from the class and then describe models admitting symmetry groups that are extensions of this common symmetry group [33].

In this paper we carry out the group classification of (1+1)-dimensional linear Schrödinger equations with complex-valued potentials, having the general form

$$i\psi_t + \psi_{xx} + V(t,x)\psi = 0, \tag{1}$$

where ψ is an unknown complex-valued function of two real independent variables t and x and V is an arbitrary smooth complex-valued potential also depending on t and x. To achieve this, we apply the algebraic method of group classification (which we describe further on in this paper) and reduce the problem of the group classification of the class (1) to the classification of appropriate subalgebras of the associated equivalence algebra [6], [38]. In order to reduce the standard form of Schrödinger equations to the form (1), we scale t and x and change the sign of V. Note that the larger class of (1+1)-dimensional linear Schrödinger equations with "real" variable mass $m = m(t,x) \neq 0$ can be mapped to the class (1) by a family of point equivalence transformations in a way similar to that of gauging coefficients in linear evolution equations, cf. [33], [39]. Hence the group classification of the class (1) also provides the group classification of this larger class.

A particular feature of the above equations is that the independent variables t, x, on the one hand, and the dependent variable ψ and arbitrary element V, on the other hand, belong to different fields. This feature needs a delicate treatment of objects involving ψ or V. It is possible to consider Schrödinger equations from a "real perspective" by representing them as systems of two equations for the real and the imaginary parts of ψ, but such a representation will only complicate the whole discussion. The use of the absolute value and the argument of ψ instead of the real and the imaginary parts is even less convenient since it leads to nonlinear systems instead of linear ones. This is why we work with complex-valued functions. We then need to formally extend the space of variables (t,x,ψ) with ψ^* and the space of the arbitrary element V with V^*. Here and in what follows star denotes the complex conjugate. In particular, we consider ψ^* (resp. V^*) as an argument for all functions depending on ψ (resp. V), including components of point transformations and of vector fields. When we restrict a differential function of ψ to the solution set of an equation from the class (1), we also take into account the complex conjugate of the equation, that is $-i\psi_t^* + \psi_{xx}^* + V^*(t,x)\psi^* = 0$. However, it is sufficient to test invariance and equivalence conditions only for the original equations since the results of this testing will be the same for their complex conjugate counterparts. Presenting point transformations, we omit the transformation components for ψ^* and V^* since they are obtained by conjugating those for ψ and V.

The structure of this paper is the following: In Section 2 we describe the general framework of the group classification of classes of differential equations. We define various objects related to point transformations and discuss their properties. In Section 3 we extend the algebraic method of group classification to uniformly semi-normalized classes of differential equations. We compute the equivalence groupoid, the equivalence group and the equivalence algebra of the class (1) in Section 4. It turns out that the class (1) has rather good transformational properties: it is uniformly semi-normalized with respect to linear superposition of solutions. In Section 5 we then analyze the determining equations for the Lie symmetries of equations from the class (1), find the kernel Lie invariance algebra of this class and single out the classifying condition for admissible Lie symmetry extensions. In Section 6 we study properties of appropriate subalgebras of the equivalence algebra, classify them and complete the group classification of the class (1). In Section 7 we illustrate the advantages of the algebraic method of group classification by performing the group classification of the class (1) in a different way. The group classification of (1+1)-dimensional linear Schrödinger equations with real potentials is presented in Section 8. In the final section we summarize results of the paper.

2 Group classification in classes of differential equations

In this section we give the definitions and notation needed for the group classification of differential equations. For more details see [6], [33] [38].

We begin with a definition of the notion of *class* of differential equations. Let $\mathcal{L}_\theta$ be a system

$$L(x, u_{(p)}, \theta_{(q)}(x, u_{(p)})) = 0$$

of l differential equations $L^1 = 0, \ldots, L^l = 0$ parameterized by a tuple of arbitrary elements $\theta(x, u_{(p)}) = (\theta^1(x, u_{(p)}), \ldots, \theta^k(x, u_{(p)}))$, where $x = (x_1, \ldots, x_n)$ is the tuple of independent variables and $u_{(p)}$ is the set of the dependent variables $u = (u^1, \ldots, u^m)$ together with all derivatives of u with respect to x of order less than or equal to p. The symbol $\theta_{(q)}$ stands for the set of partial derivatives of θ of order less than or equal to q with respect to the variables x and $u_{(p)}$. The tuple of arbitrary elements θ runs through the set $\mathcal{S}$ of solutions of an auxiliary system of differential equations $S(x, u_{(p)}, \theta_{(q')}(x, u_{(p)})) = 0$ and differential inequalities $\Sigma(x, u_{(p)}, \theta_{(q')}(x, u_{(p)})) \neq 0$ (other kinds of inequalities may also appear here), in which both x and $u_{(p)}$ play the role of independent variables and S and Σ are tuples of smooth functions depending on $x, u_{(p)}$ and $\theta_{(q')}$. The set $\{\mathcal{L}_\theta \mid \theta \in \mathcal{S}\} =: \mathcal{L}|_\mathcal{S}$ is called a *class (of systems) of differential equations* that is defined by the parameterized form of systems $\mathcal{L}_\theta$ and the set $\mathcal{S}$ run by the arbitrary elements θ.

Thus, for the class (1) we have two partial differential equations (including the complex conjugate equation) for two (formally unrelated) dependent variables ψ and ψ^* of two independent variables t and x, and two (formally unrelated) arbitrary elements $\theta = (V, V^*)$, which depend only on t and x. Therefore, the auxiliary system for the arbitrary elements of the class (1) is

$$
\begin{aligned}
&V_\psi = V_{\psi^*} = V_{\psi_t} = V_{\psi^*_t} = V_{\psi_x} = V_{\psi^*_x} = 0,\\
&V_{\psi_{tx}} = V_{\psi^*_{tx}} = V_{\psi_{tt}} = V_{\psi^*_{tt}} = V_{\psi_{xx}} = V_{\psi^*_{xx}} = 0,\\
&V^*_\psi = V^*_{\psi^*} = V^*_{\psi_t} = V^*_{\psi^*_t} = V^*_{\psi_x} = V^*_{\psi^*_x} = 0,\\
&V^*_{\psi_{tx}} = V^*_{\psi^*_{tx}} = V^*_{\psi_{tt}} = V^*_{\psi^*_{tt}} = V^*_{\psi_{xx}} = V^*_{\psi^*_{xx}} = 0.
\end{aligned}
$$

For a class of differential equations $\mathcal{L}|_\mathcal{S}$, there are objects of various structures that consist of point transformations related to this class. Let $\mathcal{L}_\theta$ and $\mathcal{L}_{\tilde\theta}$ be systems belonging to $\mathcal{L}|_\mathcal{S}$. We denote by $\mathrm{T}(\theta,\tilde\theta)$ the set of point transformations in the space of the variables (x,u) that map $\mathcal{L}_\theta$ to $\mathcal{L}_{\tilde\theta}$.

An *admissible transformation* of the class $\mathcal{L}|_\mathcal{S}$ is a triple $(\theta,\tilde\theta,\varphi)$ consisting of two arbitrary elements $\theta,\tilde\theta\in\mathcal{S}$ such that $\mathrm{T}(\theta,\tilde\theta)\neq\varnothing$ and a point transformation $\varphi\in\mathrm{T}(\theta,\tilde\theta)$. The set of all admissible transformations of the class $\mathcal{L}|_\mathcal{S}$,

$$
\mathcal{G}^\sim = \mathcal{G}^\sim(\mathcal{L}|_\mathcal{S}) := \{(\theta,\tilde\theta,\varphi) \mid \theta,\tilde\theta\in\mathcal{S}, \varphi\in\mathrm{T}(\theta,\tilde\theta)\},
$$

has the structure of a groupoid: for any $\theta\in\mathcal{S}$ the triple $(\theta,\theta,\mathrm{id})$, where id is the identity point transformation, is an element of $\mathcal{G}^\sim$, every $(\theta,\tilde\theta,\varphi)\in\mathcal{G}^\sim$ is invertible and $\mathcal{G}^\sim$ is closed under composition. This is why the set $\mathcal{G}^\sim$ is called the *equivalence groupoid* of the class $\mathcal{L}|_\mathcal{S}$.

The *(usual) equivalence (pseudo)group* $G^\sim = G^\sim(\mathcal{L}|_\mathcal{S})$ of the class $\mathcal{L}|_\mathcal{S}$ is defined as being the set of point transformations in the joint space of independent and dependent variables, their derivatives and arbitrary elements with local coordinates $(x,u_{(p)},\theta)$ that are projectable to the space of $(x,u_{(p')})$ for any $0\leqslant p'\leqslant p$, preserve the contact structure on the space with local coordinates $(x,u_{(p)})$, and map every system from the class $\mathcal{L}|_\mathcal{S}$ to a system from the same class. Elements of the group $G^\sim$ are called *equivalence transformations*. This definition includes two fundamental conditions for general equivalence transformations: the preservation of the class $\mathcal{L}|_\mathcal{S}$ and the preservation of the contact structure on the space with local coordinates $(x,u_{(p)})$. The conditions of projectability and locality with respect to arbitrary elements can be weakened, and this leads to various generalizations of the notion of equivalence group (see [38]). Note that each equivalence transformation $\mathcal{T}\in G^\sim$ generates a family of admissible transformations from $\mathcal{G}^\sim$, $G^\sim\ni\mathcal{T}\to\{(\theta,\mathcal{T}\theta,\mathcal{T}|_{(x,u)})\mid\theta\in\mathcal{S}\}\subset\mathcal{G}^\sim$, where $\mathcal{T}|_{(x,u)}$ is the restriction of $\mathcal{T}$ to the space of (x,u). For a generalized equivalence group, the restriction of $\mathcal{T}$ is made after fixing a value of θ, which can be denoted as $\mathcal{T}^\theta|_{(x,u)}$.

The equivalence group of a subclass $\mathcal{L}|_{\mathcal{S}'}$, $\mathcal{S}'\subset\mathcal{S}$, of the class $\mathcal{L}|_\mathcal{S}$ is called a *conditional equivalence group* of the class $\mathcal{L}|_\mathcal{S}$ that is associated with the subclass $\mathcal{L}|_{\mathcal{S}'}$. A useful way of describing the equivalence groupoid $\mathcal{G}^\sim$ is to classify maximal conditional equivalence groups of the class $\mathcal{L}|_\mathcal{S}$ up to $G^\sim$-equivalence and then to classify (up to an appropriate conditional equivalence) the admissible transformations that are not generated by conditional equivalence transformations (see [38] for more details).

The *equivalence algebra* $\mathfrak{g}^\sim = \mathfrak{g}^\sim(\mathcal{L}|_\mathcal{S})$ of the class $\mathcal{L}|_\mathcal{S}$ is defined as the set of generators of one-parameter groups of equivalence transformations of the class $\mathcal{L}|_\mathcal{S}$.

These generators are vector fields in the space of $(x, u_{(p)}, \theta)$, that are projectable to the space of $(x, u_{(p')})$ for any $0 \leqslant p' \leqslant p$ and whose projections to the space of $(x, u_{(p)})$ are the pth order prolongations of the corresponding projections to the space of (x, u).

The *maximal point symmetry (pseudo)group* G_θ of the system $\mathcal{L}_\theta$ (for a fixed $\theta \in \mathcal{S}$) is a (pseudo)group of transformations that act in the space of independent and dependent variables that preserve the solution set of the system $\mathcal{L}_\theta$. Each G_θ can be interpreted as a vertex group of the equivalence groupoid $\mathcal{G}^\sim$. The intersection $G^\cap = G^\cap(\mathcal{L}|_\mathcal{S}) := \bigcap_{\theta\in\mathcal{S}} G_\theta$ of all G_θ, $\theta \in \mathcal{S}$, is called the *kernel of the maximal point symmetry groups* of systems from $\mathcal{L}|_\mathcal{S}$.

The vector fields in the space of (x, u) generating one-parameter subgroups of the maximal point symmetry group G_θ of the system $\mathcal{L}_\theta$ form a Lie algebra $\mathfrak{g}_\theta$ with the Lie bracket defined by commutators of vector fields. It is called the *maximal Lie invariance algebra* of $\mathcal{L}_\theta$. The *kernel invariance algebra* of the class $\mathcal{L}|_\mathcal{S}$ is the intersection $\mathfrak{g}^\cap = \mathfrak{g}^\cap(\mathcal{L}|_\mathcal{S}) := \bigcap_{\theta\in\mathcal{S}} \mathfrak{g}_\theta$ of the algebras $\mathfrak{g}_\theta$, $\theta \in \mathcal{S}$.

The classical *group classification problem* for the class $\mathcal{L}|_\mathcal{S}$ is to list all $G^\sim$-inequivalent values of $\theta \in \mathcal{S}$ for which the corresponding maximal Lie invariance algebras, $\mathfrak{g}_\theta$, are larger than the kernel invariance algebra $\mathfrak{g}^\cap$. There may be additional point equivalences between the cases obtained in this way and these additional equivalences have then to be incorporated into the results. This solves the group classification problem for the class $\mathcal{L}|_\mathcal{S}$ up to $\mathcal{G}^\sim$-equivalence.

Summing up, objects to be found in the course of group classification of the class $\mathcal{L}|_\mathcal{S}$ include the equivalence groupoid $\mathcal{G}^\sim$, the equivalence group $G^\sim$, the equivalence algebra $\mathfrak{g}^\sim$, the kernel invariance algebra $\mathfrak{g}^\cap$ and a complete list of $G^\sim$-inequivalent (resp. $\mathcal{G}^\sim$-inequivalent) values of θ with the corresponding Lie symmetry extensions of $\mathfrak{g}^\cap$. Additional point equivalences between classification cases can be computed directly via looking for pairs of cases with similar Lie invariance algebras (if two systems are equivalent under an invertible point transformation, then their Lie symmetry algebras are isomorphic). Therefore, the construction of the equivalence groupoid $\mathcal{G}^\sim$ can be excluded from the procedure of group classification if this groupoid is of complicated structure, e.g., due to the involved hierarchy of maximal conditional equivalence groups of the class $\mathcal{L}|_\mathcal{S}$.

The classical way of performing a group classification of a class $\mathcal{L}|_\mathcal{S}$ is to use the *infinitesimal invariance criterion* [31], [33]: under an appropriate nondegeneracy condition for the system $\mathcal{L}_\theta \in \mathcal{L}|_\mathcal{S}$, a vector field $Q = \xi^j(x, u)\partial_{x_j} + \eta^a(x, u)\partial_{u^a}$ belongs to the maximal Lie invariance algebra $\mathfrak{g}_\theta$ of $\mathcal{L}_\theta$ if and only if the condition

$$Q_{(p)} L(x, u_{(p)}, \theta_{(q)}(x, u_{(p)})) = 0,$$

holds on the manifold $\mathcal{L}^p_\theta$ defined by the system $\mathcal{L}_\theta$ together with its differential consequences in the jet space $J^{(p)}$. Here the indices j and a run from 1 to n and from 1 to m, respectively, and we use the summation convention for repeated indices. $Q_{(p)}$ denotes the standard pth prolongation of the vector field Q,

$$Q_{(p)} = Q + \sum_{0<|\alpha|\leqslant p} \Big(D_1^{\alpha_1} \cdots D_n^{\alpha_n} \left(\eta^a - \xi^j u^a_j\right) + \xi^j u^a_{\alpha+\delta_j} \Big) \partial_{u^a_\alpha}.$$

The tuple $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multiindex, $\alpha_j \in \mathbb{N} \cup 0$, $|\alpha| := \alpha_1 + \dots + \alpha_n$, and δ_j is the multiindex whose ith entry equals 1 and whose other entries are zero. The variable u^a_α of the jet space $J^{(p)}$ is identified with the derivative $\partial^{|\alpha|} u^a / \partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}$. $D_j = \partial_j + u^a_{\alpha+\delta_j} \partial_{u^a_\alpha}$ is the total derivative operator with respect to the variable x_j.

The infinitesimal invariance criterion yields the *system of determining equations* for the components of the generators of the one-parameter Lie symmetry groups of systems from the class $\mathcal{L}|_\mathcal{S}$, where the arbitrary elements θ play the role of parameters. Integrating those determining equations that do not involve arbitrary elements gives a preliminary form of the generator components, and one must then solve the remaining equations. The solution of these remaining equations depends on the values of the arbitrary elements. We call these equations the *classifying equations* for the class $\mathcal{L}|_\mathcal{S}$.

In order to find the kernel invariance algebra $\mathfrak{g}^\cap$, one must first split the determining equations with respect to parametric derivatives of arbitrary elements and of dependent variables and then solve the system obtained.

Finding Lie symmetry extensions of the kernel Lie algebra depends on an analysis of the classifying equations. This part of solving the group classification problem is intricate and various techniques are used to obtain the solution. There are two main approaches. If the class considered has a simple structure (for example, when arbitrary elements are constants or are functions of just one argument), then the techniques used rely on the study of the compatibility of the classifying equations and their direct solution with respect to both the components of Lie symmetry generators and the arbitrary elements (up to the equivalence defined by the equivalence group). See, for instance, [8], [33], [40], [41] and the references given there. For more complicated classes, the direct approach seems to be irrelevant, and more advanced algebraic techniques need to be used.

3 Uniformly semi-normalized classes

In the most general setting, the main point of the algebraic approach to group classification is to classify (up to certain equivalence relation induced by point transformations between systems belonging to the class $\mathcal{L}|_\mathcal{S}$ under study) certain Lie algebras of vector fields related to these systems. The key problem is to select sets of vector fields to be classified and the equivalence relation to be used in this classification [6]. For the application of the algebraic method to be effective, the selected objects have to satisfy certain consistency conditions which then require particular properties of the equivalence groupoid $\mathcal{G}^\sim$ of $\mathcal{L}|_\mathcal{S}$. To this end, we begin with some definitions which enable us to formulate our approach.

We say that the class $\mathcal{L}|_\mathcal{S}$ is *normalized* if its equivalence groupoid $\mathcal{G}^\sim$ is generated by its equivalence group $G^\sim$. We say that it is *semi-normalized* if the equivalence groupoid $\mathcal{G}^\sim$ is generated by transformations from $G^\sim$ and point symmetry transformations of the corresponding source or target systems. It is clear that any normalized class of differential equations is semi-normalized. Normalized classes are especially convenient when one applies the algebraic method of group classification. If the class $\mathcal{L}|_{\mathcal{S}}$ is normalized, then the Lie symmetry extensions of its kernel invariance algebra are obtained via the classification of appropriate subalgebras of the equivalence algebra whose projections onto the space with local coordinates (x, u) coincide with the maximal Lie invariance algebras of systems from $\mathcal{L}|_{\mathcal{S}}$. The property of semi-normalization is useful for determining equivalences between Lie symmetry extensions but not for finding such extensions. For rigorous definitions and more details, we refer the reader to [6], [38].

Classes of differential equations that are not normalized but have stronger normalization properties than semi-normalization often appear in physical applications. This is why it is important to weaken the normalization property in such a way that still allows us to apply group classification techniques analogous to those developed for normalized classes.

Definition I.1. Given a class of differential equations $\mathcal{L}|_{\mathcal{S}}$ with the equivalence groupoid $\mathcal{G}^{\sim}$ and the (usual) equivalence group $G^{\sim}$,[1] suppose that for each $\theta \in \mathcal{S}$ the point symmetry group G_θ of the system $\mathcal{L}_\theta \in \mathcal{L}|_{\mathcal{S}}$ contains a subgroup N_θ such that the family $\mathcal{N}_{\mathcal{S}} = \{N_\theta \mid \theta \in \mathcal{S}\}$ of all these subgroups satisfies the following properties:

1. $\mathcal{T}|_{(x,u)} \notin N_\theta$ for any $\theta \in \mathcal{S}$ and any $\mathcal{T} \in G^{\sim}$ with $\mathcal{T} \neq \mathrm{id}$.
2. $N_{\mathcal{T}\theta} = \mathcal{T}|_{(x,u)} N_\theta (\mathcal{T}|_{(x,u)})^{-1}$ for any $\theta \in \mathcal{S}$ and any $\mathcal{T} \in G^{\sim}$.
3. For any $(\theta^1, \theta^2, \varphi) \in \mathcal{G}^{\sim}$ there exist $\varphi^1 \in N_{\theta^1}$, $\varphi^2 \in N_{\theta^2}$ and $\mathcal{T} \in G^{\sim}$ such that $\theta^2 = \mathcal{T}\theta^1$ and $\varphi = \varphi^2 (\mathcal{T}|_{(x,u)}) \varphi^1$.

Here $\mathcal{T}|_{(x,u)}$ denotes the restriction of $\mathcal{T}$ to the space with local coordinates (x, u). We then say that the class of differential equations $\mathcal{L}|_{\mathcal{S}}$ is *uniformly semi-normalized* with respect to the symmetry-subgroup family $\mathcal{N}_{\mathcal{S}}$.

The qualification "uniformly" is justified by the fact that in practically relevant examples of such classes all the subgroups N_θ's are isomorphic or at least of a similar structure (in particular, of the same dimension). The first property in Definition I.1 means that the intersection of each subgroup N_θ with the restriction of $G^{\sim}$ to the space of (x, u) is just the identity transformation. The second property can be interpreted as equivariance of equivalence transformations with respect to $\mathcal{N}_{\mathcal{S}}$. The third property means, essentially, that the entire equivalence groupoid $\mathcal{G}^{\sim}$ is generated by equivalence transformations and transformations from uniform point symmetry groups. One of the symmetry transformations φ^1 or φ^2 in the last property may be taken to be the identity.

Each normalized class of differential equations is uniformly semi-normalized with respect to the trivial family $\mathcal{N}_{\mathcal{S}}$, where for each θ the group N_θ consists of just the identity transformation. It is also obvious that each uniformly semi-normalized class is semi-normalized. At the same time, there exist semi-normalized classes that are not uniformly semi-normalized. A simple example of such a class is given by the class ND of nonlinear diffusion equations of the form $u_t = (f(u)u_x)_x$

[1] A subgroup of the equivalence group can be considered here instead of the entire group.

with $f_u \neq 0$, which is a classic example in the group analysis of differential equations [32], [33]. Such equations with special power nonlinearities of the form $f = c_1(u+c_0)^{-4/3}$ have singular symmetry properties within the class ND. This fact does not allow the class ND to be normalized, although it is semi-normalized. The elements from $\mathcal{G}^{\sim}(\mathrm{ND})$ that are not generated by elements of $G^{\sim}(\mathrm{ND})$ are given by the equivalence transformations of equations with the above power nonlinearities that are composed with conformal symmetry transformations of these equations. The nonlinear diffusion equation with $f = c_1(u+c_0)^{-4/3}$ admits the conformal symmetry group with infinitesimal generator $x^2\partial_x - 3x(u+c_0)\partial_u$, but this is not a normal subgroup of the point symmetry group of the equation.

The following result, which we call *the theorem on splitting symmetry groups in uniformly semi-normalized classes*, provides a theoretical basis for the algebraic method of group classification of such classes.

Theorem I.1. *Let a class of differential equations $\mathcal{L}|_{\mathcal{S}}$ be uniformly semi-normalized with respect to a symmetry-subgroup family $\mathcal{N}_{\mathcal{S}} = \{N_\theta \mid \theta \in \mathcal{S}\}$. Then for each $\theta \in \mathcal{S}$ the point symmetry group G_θ of the system $\mathcal{L}_\theta \in \mathcal{L}|_{\mathcal{S}}$ splits over N_θ. More specifically, N_θ is a normal subgroup of G_θ, $G^{\mathrm{ess}}_\theta = G^{\sim}|_{(x,u)} \cap G_\theta$ is a subgroup of G_θ, and the group G_θ is the semidirect product of G^{ess}_θ acting on N_θ, $G_\theta = G^{\mathrm{ess}}_\theta \ltimes N_\theta$. Here $G^{\sim}|_{(x,u)}$ denotes the restriction of $G^{\sim}$ to the space with local coordinates (x,u), $G^{\sim}|_{(x,u)} = \{\mathcal{T}|_{(x,u)} \mid \mathcal{T} \in G^{\sim}\}$.*

Proof. We fix an arbitrary $\theta \in \mathcal{S}$ and take an arbitrary $\varphi \in G_\theta$. Then $(\theta,\theta,\varphi) \in \mathcal{G}^{\sim}$ and, by the third property in Definition I.1, the transformation φ admits the factorization $\varphi = \mathcal{T}|_{(x,u)}\varphi^1$ for some $\mathcal{T} \in G^{\sim}$ and some $\varphi^1 \in N_\theta$. The element N_θ of the family $\mathcal{N}_{\mathcal{S}}$ is a subgroup of G_θ, $N_\theta < G_\theta$ and hence the transformation $\varphi^0 := \mathcal{T}|_{(x,u)} = \varphi(\varphi^1)^{-1}$ also belongs to G_θ, and consequently to $G^{\sim}|_{(x,u)} \cap G_\theta =: G^{\mathrm{ess}}_\theta$, which is a subgroup of G_θ as it is the intersection of two groups. From this, it follows that for any $\varphi \in G_\theta$ we have the representation $\varphi = \varphi^0\varphi^1$, where $\varphi^0 \in G^{\mathrm{ess}}_\theta$ and $\varphi^1 \in N_\theta$.

The first property of Definition I.1 means that the intersection of $G^{\sim}|_{(x,u)}$ and N_θ consists of just the identity transformation so that the intersection $G^{\mathrm{ess}}_\theta \cap N_\theta$ contains only the identity transformation.

For an arbitrary $\varphi \in G_\theta$ and an arbitrary $\tilde\varphi \in N_\theta$, we consider the composition $\varphi\tilde\varphi\varphi^{-1}$. As an element of G_θ, the transformation φ admits the factorization $\varphi = \varphi^0\varphi^1$ with some $\varphi^0 \in G^{\mathrm{ess}}_\theta$ and some $\varphi^1 \in N_\theta$. Since $G^{\mathrm{ess}}_\theta < G^{\sim}|_{(x,u)}$, there exists a $\mathcal{T} \in G^{\sim}$ such that $\mathcal{T}\theta = \theta$ and $\varphi^0 = \mathcal{T}|_{(x,u)}$. By property 2 of Definition I.1, we obtain $N_\theta = \varphi^0 N_\theta (\varphi^0)^{-1}$. Hence the composition $\varphi\tilde\varphi\varphi^{-1} = \varphi^0\varphi^1\tilde\varphi(\varphi^1)^{-1}(\varphi^0)^{-1}$ belongs to N_θ. Thus we have that N_θ is a normal subgroup of G_θ, $N_\theta \lhd G_\theta$, and so $G_\theta = G^{\mathrm{ess}}_\theta \ltimes N_\theta$. □

The members of the family $\mathcal{N}_{\mathcal{S}} = \{N_\theta \mid \theta \in \mathcal{S}\}$ are called *uniform point symmetry groups* of the equations from the class $\mathcal{L}|_{\mathcal{S}}$, and the subgroup G^{ess}_θ is called the *essential point symmetry group* of the system $\mathcal{L}_\theta$ associated with the uniform point symmetry group N_θ. The knowledge of a family of uniform point symmetry groups trivializes them in the following sense: since G_θ splits over N_θ for each θ, then we only need to find the subgroup G^{ess}_θ in order to construct G_θ.

The infinitesimal version of Theorem I.1 may be called *the theorem on splitting invariance algebras in uniformly semi-normalized classes*. This version follows immediately from Theorem I.1 if we replace the groups with the corresponding algebras of generators of the one-parameter subgroups of these groups.

Theorem I.2. *Suppose that a class of differential equations $\mathcal{L}|_\mathcal{S}$ is uniformly semi-normalized with respect to a family of symmetry subgroups $\mathcal{N}_\mathcal{S} = \{N_\theta \mid \theta \in \mathcal{S}\}$. Then for each $\theta \in \mathcal{S}$ the Lie algebras $\mathfrak{g}^{\rm ess}_\theta$ and $\mathfrak{n}_\theta$ that are associated with the groups $G^{\rm ess}_\theta$ and N_θ are, respectively, a subalgebra and an ideal of the maximal Lie invariance algebra $\mathfrak{g}_\theta$ of the system $\mathcal{L}_\theta \in \mathcal{L}|_\mathcal{S}$. Moreover, the algebra $\mathfrak{g}_\theta$ is the semi-direct sum $\mathfrak{g}_\theta = \mathfrak{g}^{\rm ess}_\theta \mathbin{\Subset} \mathfrak{n}_\theta$, and $\mathfrak{g}^{\rm ess}_\theta = \mathfrak{g}^\sim|_{(x,u)} \cap \mathfrak{g}_\theta$, where $\mathfrak{g}^\sim|_{(x,u)}$ denotes the restriction of $\mathfrak{g}^\sim$ to the space with local coordinates (x,u).*

The group classification problem for a uniformly semi-normalized class $\mathcal{L}|_\mathcal{S}$ is solved in the following way: when computing the equivalence groupoid $\mathcal{G}^\sim$ and analyzing its structure, we construct a family of uniform point symmetry groups $\mathcal{N}_\mathcal{S} = \{N_\theta \mid \theta \in \mathcal{S}\}$, which then establishes the uniformly semi-normalization of the class $\mathcal{L}|_\mathcal{S}$ and yields the corresponding uniform Lie invariance algebras $\mathfrak{n}_\theta$'s. The subgroup $G^{\rm ess}_\theta = G^\sim|_{(x,u)} \cap G_\theta$ and the subalgebra $\mathfrak{g}^{\rm ess}_\theta = \mathfrak{g}^\sim|_{(x,u)} \cap \mathfrak{g}_\theta$ which are the complements of N_θ and $\mathfrak{n}_\theta$ respectively, are in general not known on this step. By Theorem I.2, we have for each $\theta \in \mathcal{S}$ that the maximal Lie invariance algebra $\mathfrak{g}_\theta$ of the system $\mathcal{L}_\theta$ is given by the semi-direct sum $\mathfrak{g}_\theta = \mathfrak{g}^{\rm ess}_\theta \mathbin{\Subset} \mathfrak{n}_\theta$. Essential Lie invariance algebras are subalgebras of $\mathfrak{g}^\sim|_{(x,u)}$ and are mapped onto each other by the pushforwards of restrictions of equivalence transformations: $\mathfrak{g}^{\rm ess}_{\mathcal{T}\theta} = (\mathcal{T}|_{(x,u)})_*\mathfrak{g}^{\rm ess}_\theta$. Consequently, the group classification of the class $\mathcal{L}|_\mathcal{S}$ reduces to the classification of appropriate subalgebras of $\mathfrak{g}^\sim|_{(x,u)}$ or, equivalently, of the equivalence algebra $\mathfrak{g}^\sim$ itself.

An important case of uniformly semi-normalized classes, which is relevant to the present paper, is given by classes of homogeneous linear systems of differential equations.

Consider a normalized class $\mathcal{L}^{\rm inh}|_{\mathcal{S}^{\rm inh}}$ of (generally) inhomogeneous linear systems of differential equations $\mathcal{L}^{\rm inh}_{\theta,\zeta}$'s of the form $L(x, u_{(p)}, \theta_{(q)}(x)) = \zeta(x)$, where θ is a tuple of arbitrary elements parameterizing the homogeneous linear left hand side and depending only on x and the right hand side ζ is a tuple of arbitrary functions of x. Suppose that the class $\mathcal{L}^{\rm inh}|_{\mathcal{S}^{\rm inh}}$ also satisfies the following conditions:

- Each system from $\mathcal{L}^{\rm inh}|_{\mathcal{S}^{\rm inh}}$ is locally solvable.
- The zero function is the only common solution of the homogeneous systems from $\mathcal{L}^{\rm inh}|_{\mathcal{S}^{\rm inh}}$.
- Restrictions of elements of the equivalence group $G^\sim_{\rm inh} = G^\sim(\mathcal{L}^{\rm inh}|_{\mathcal{S}^{\rm inh}})$ to the space of (x,u) are fibre-preserving transformations whose components for u are affine in u, that is they are of the form

  $$\tilde x_j = X^j(x),\ \tilde u^a = M^{ab}(x)(u^b + h^b(x)),$$

  where $\det(X^j_{x_{j'}}) \neq 0$ and $\det(M^{ab}) \neq 0$.

Here the indices j and j' run from 1 to n and the indices a and b run from 1 to m. The functions X^j's and M^{ab}'s may satisfy additional constraints but the h^a's are arbitrary smooth functions of x.

Any system $\mathcal{L}_{\theta\zeta}$ from the class $\mathcal{L}^{\rm inh}|_{\mathcal{S}^{\rm inh}}$ is mapped to the associated homogeneous system $\mathcal{L}_{\theta 0}$ by the equivalence transformation

$$\mathcal{T}(\zeta)\colon\quad \tilde x_j=x_j,\quad \tilde u^a=u^a+h^a(x),\quad \tilde\theta=\theta,\quad \tilde\zeta=\zeta-L(x,h_{(p)}(x),\theta_{(q)}(x)),$$

where $h=(h^1,\dots,h^m)$ is a solution of $\mathcal{L}_{\theta\zeta}$. In other words, the class $\mathcal{L}^{\rm inh}|_{\mathcal{S}^{\rm inh}}$ is mapped, by the family $\{\mathcal{T}_{\theta\zeta}\}$ of equivalence transformations that are nonlocally parameterized by the arbitrary elements θ's and ζ's to the corresponding class $\mathcal{L}^{\rm hmg}|_{\mathcal{S}^{\rm hmg}}$ of homogeneous systems. The transformations from the equivalence group $G^\sim_{\rm inh}$ with (x,u)-components $\tilde x_j=x_j$, $\tilde u^a=u^a+h^a(x)$, where the h^a's run through the set of smooth functions of x, constitute a normal subgroup $N^\sim_{\rm inh}$ of this group. Furthermore, $G^\sim_{\rm inh}$ splits over $N^\sim_{\rm inh}$ since $G^\sim_{\rm inh}=H^\sim_{\rm inh}\ltimes N^\sim_{\rm inh}$, where $H^\sim_{\rm inh}$ is the subgroup of $G^\sim_{\rm inh}$ consisting of those elements with $h^a=0$. The restriction of $H^\sim_{\rm inh}$ to the space with local coordinates (x,u,θ) coincides with the equivalence group $G^\sim_{\rm hmg}$ of the class $\mathcal{L}^{\rm hmg}|_{\mathcal{S}^{\rm hmg}}$. The equivalence groupoid $\mathcal{G}^\sim_{\rm hmg}$ of this class can be considered as the subgroupoid of the equivalence groupoid $\mathcal{G}^\sim_{\rm inh}$ of the class $\mathcal{L}^{\rm inh}|_{\mathcal{S}^{\rm inh}}$ that is singled out by the constraints $\zeta=0$, $\tilde\zeta=0$ for the source and target systems of admissible transformations, respectively. For each relevant transformational part, the tuple of parameter-functions h is an arbitrary solution of the corresponding source system. The systems $\mathcal{L}_{\theta\zeta}$ and $\mathcal{L}_{\tilde\theta\tilde\zeta}$ are $G^\sim_{\rm inh}$-equivalent if and only if their homogeneous counterparts $\mathcal{L}_{\theta 0}$ and $\mathcal{L}_{\tilde\theta 0}$ are $G^\sim_{\rm hmg}$-equivalent. Thus the group classification of systems from the class $\mathcal{L}^{\rm inh}|_{\mathcal{S}^{\rm inh}}$ reduces to the group classification of systems from the class $\mathcal{L}^{\rm hmg}|_{\mathcal{S}^{\rm hmg}}$.

For each $\theta\in\mathcal{S}^{\rm hmg}$ we denote by $G^{\rm lin}_{\theta 0}$ the subgroup of the point symmetry group $G_{\theta 0}$ of $\mathcal{L}_{\theta 0}$ that consists of the linear superposition transformations:

$$\tilde x_j=x_j,\quad \tilde u^a=u^a+h^a(x),$$

where the tuple h is a solution of $\mathcal{L}_{\theta 0}$. The family $\mathcal{N}_{\rm lin}=\{G^{\rm lin}_{\theta 0}\mid\theta\in\mathcal{S}_{\rm hmg}\}$ of all these subgroups satisfies the properties of Definition I.1. Therefore, the class $\mathcal{L}^{\rm hmg}|_{\mathcal{S}^{\rm hmg}}$ is uniformly semi-normalized with respect to the family $\mathcal{N}_{\rm lin}$. We call this kind of semi-normalization, which is characteristic for classes of homogeneous linear systems of differential equations, *uniform semi-normalization with respect to linear superposition of solutions*. By Theorem I.1, for each $\theta\in\mathcal{S}^{\rm hmg}$ the group $G_{\theta 0}$ splits over $G^{\rm lin}_{\theta 0}$, and $G_{\theta 0}=G^{\rm ess}_{\theta 0}\ltimes G^{\rm lin}_{\theta 0}$, where $G^{\rm ess}_{\theta 0}=G^\sim_{\rm hmg}|_{(x,u)}\cap G_{\theta 0}$. By Theorem I.2, the splitting of the point symmetry group induces a splitting of the corresponding maximal Lie invariance algebra: $\mathfrak g_{\theta 0}=\mathfrak g^{\rm ess}_{\theta 0}\mathbin{\supset\!\!\!\!+}\mathfrak g^{\rm lin}_{\theta 0}$, where $\mathfrak g^{\rm ess}_{\theta 0}$ is the essential Lie invariance algebra of $\mathcal{L}_{\theta 0}$, $\mathfrak g^{\rm ess}_{\theta 0}=\mathfrak g^\sim_{\rm hmg}|_{(x,u)}\cap\mathfrak g_{\theta 0}$, and the ideal $\mathfrak g^{\rm lin}_{\theta 0}$, being the trivial part of $\mathfrak g_{\theta 0}$, consists of vector fields generating one-parameter symmetry groups of linear superposition of solutions. Thus, the group classification problem for the class $\mathcal{L}^{\rm hmg}|_{\mathcal{S}^{\rm hmg}}$ reduces to the classification of appropriate subalgebras of the equivalence algebra $\mathfrak g^\sim_{\rm hmg}$ of this class. The qualification “appropriate” means that the restrictions of these subalgebras to the space of (x,u) are essential Lie invariance algebras of systems from $\mathcal{L}^{\rm hmg}|_{\mathcal{S}^{\rm hmg}}$.

For a class $\mathcal{L}^{\rm hmg}|_{\mathcal{S}^{\rm hmg}}$ of linear homogeneous systems of differential equations that is uniformly semi-normalized with respect to the linear superposition of solutions, it is not necessary to start by considering the associated normalized superclass $\mathcal{L}^{\rm inh}|_{\mathcal{S}^{\rm inh}}$ of generally inhomogeneous linear systems. The class $\mathcal{L}^{\rm hmg}|_{\mathcal{S}^{\rm hmg}}$ itself can be the starting point of the analysis. In order to get directly its uniform semi-normalization, we need to suppose the following properties of $\mathcal{L}^{\rm hmg}|_{\mathcal{S}^{\rm hmg}}$, which are counterparts of the above properties of $\mathcal{L}^{\rm hmg}|_{\mathcal{S}^{\rm hmg}}$:

- Each system from $\mathcal{L}^{\rm hmg}|_{\mathcal{S}^{\rm hmg}}$ is locally solvable.
- The zero function is the only common solution of systems from $\mathcal{L}^{\rm hmg}|_{\mathcal{S}^{\rm hmg}}$.
- For any admissible transformation $(\theta^1,\theta^2,\varphi)\in\mathcal{G}^\sim_{\rm hmg}$, its transformational part φ is of the form $\tilde x_j=X^j(x)$, $\tilde u^a=M^{ab}(x)(u^b+h^b(x))$, where $h=(h^1,\dots,h^m)$ is a solution of $\mathcal{L}_{\theta^1 0}$, $\det(X^j_{x_{j'}})\neq0$ and $\det(M^{ab})\neq0$. The functions X^j's and M^{ab}'s may satisfy only additional constraints that do not depend on both θ^1 and θ^2.

The class (1) of linear Schrödinger equations fits very well into the framework which we use in the present paper to solve the group classification problem for this class.

Remark I.1. There exist classes of homogeneous linear systems of differential equations that are uniformly semi-normalized with respect to symmetry-subgroup families different from the corresponding families of subgroups of linear superposition transformations. See Corollary I.9 below.

Remark I.2. A technique similar to factoring out uniform Lie invariance algebras can be applied to kernel invariance algebras in the course of group classification of some normalized classes using the algebraic method. It is well known that for any class of differential equations $\mathcal{L}|_{\mathcal{S}}$ the kernel $G^\cap$ of the maximal point symmetry groups G_θ's of systems $\mathcal{L}_\theta$'s from $\mathcal{L}|_{\mathcal{S}}$ is a normal subgroup of the restriction $G^\sim|_{(x,u)}$ of the (usual) equivalence group $G^\sim$ of $\mathcal{L}|_{\mathcal{S}}$ to the space with local coordinates (x,u), $G^\cap\lhd G^\sim|_{(x,u)}$. Analogues of this result can be found in the references [5, Proposition 3], [23, p. 52, Proposition 3.3.9], [34] and [33, Section II.6.5]. Furthermore, if the class $\mathcal{L}|_{\mathcal{S}}$ is normalized, then the kernel $G^\cap$ is a normal subgroup of each G_θ, $G^\cap\lhd G_\theta$, $\theta\in\mathcal{S}$ [5, Corollary 2]. However, the groups $G^\sim|_{(x,u)}$ and G_θ's do not in general split over $G^\cap$ even if the class $\mathcal{L}|_{\mathcal{S}}$ is normalized. Similar assertions for the associated algebras are also true. See, in particular, Remark 9 and the subsequent subalgebra classification in [6] for a physically relevant example. Thus, the splitting of $G^\sim$ over $G^\cap$ does not follow from the normalization of $\mathcal{L}|_{\mathcal{S}}$ and is an additional requirement for the kernel invariance algebra $\mathfrak{g}^\cap$ to be factored out when the group classification of $\mathcal{L}|_{\mathcal{S}}$ is carried out.

4 Equivalence groupoid

In this section we find the equivalence groupoid $\mathcal{G}^\sim$ and the complete equivalence group $G^\sim$ of the class (1) in finite form (that is, not using the infinitesimal method).

In the following $\mathcal{L}_V$ denotes the Schrödinger equation from the class (1) with potential $V = V(t,x)$. We look for all (locally) invertible point transformations of the form

$$\tilde{t} = T(t,x,\psi,\psi^*), \quad \tilde{x} = X(t,x,\psi,\psi^*),$$
$$\tilde{\psi} = \Psi(t,x,\psi,\psi^*), \quad \tilde{\psi}^* = \Psi^*(t,x,\psi,\psi^*) \tag{2}$$

(that is, $dT \wedge dX \wedge d\Psi \wedge d\Psi^* \neq 0$) that map a fixed equation $\mathcal{L}_V$ from the class (1) to an equation $\mathcal{L}_{\tilde{V}}$: $i\tilde{\psi}_{\tilde{t}} + \tilde{\psi}_{\tilde{x}\tilde{x}} + \tilde{V}\tilde{\psi} = 0$ of the same class.

In the rest of this paper, we use the following notation: for any given complex number β

$$\hat{\beta} = \beta \quad \text{if} \quad T_t > 0 \quad \text{and} \quad \hat{\beta} = \beta^* \quad \text{if} \quad T_t < 0.$$

Theorem I.3. *The equivalence groupoid $\mathcal{G}^\sim$ of the class* (1) *consists of triples of the form $(V, \tilde{V}, \varphi)$, where φ is a point transformation acting on the space with local coordinates (t,x,ψ) and given by*

$$\tilde{t} = T, \quad \tilde{x} = \varepsilon|T_t|^{1/2}x + X^0, \tag{3a}$$

$$\tilde{\psi} = \exp\left(\frac{i}{8}\frac{T_{tt}}{|T_t|}\,x^2 + \frac{i}{2}\frac{\varepsilon\varepsilon' X^0_t}{|T_t|^{1/2}}\,x + i\Sigma + \Upsilon\right)(\hat{\psi} + \hat{\Phi}), \tag{3b}$$

V is an arbitrary potential and the transformed potential $\tilde{V}$ is related to V by the equation

$$\begin{aligned}\tilde{V} = {} & \frac{\hat{V}}{|T_t|} + \frac{2T_{ttt}T_t - 3T_{tt}^2}{16\varepsilon' T_t^3}x^2 + \frac{\varepsilon\varepsilon'}{2|T_t|^{1/2}}\left(\frac{X^0_t}{T_t}\right)_t x \\ & - \frac{iT_{tt} + (X^0_t)^2}{4T_t^2} + \frac{\Sigma_t - i\Upsilon_t}{T_t},\end{aligned} \tag{3c}$$

$T = T(t)$, $X^0 = X^0(t)$, $\Sigma = \Sigma(t)$ and $\Upsilon = \Upsilon(t)$ are arbitrary smooth real-valued functions of t with $T_t \neq 0$ and $\Phi = \Phi(t,x)$ denotes an arbitrary solution of the initial equation. $\varepsilon = \pm 1$ and $\varepsilon' = \operatorname{sgn} T_t$.

Proof. The class (1) is a subclass of the class of generalized Schrödinger equations of the form $i\psi_t + \psi_{xx} + F = 0$, where ψ is a complex dependent variable of two real independent variables t and x and $F = F(t,x,\psi,\psi^*,\psi_x,\psi^*_x)$ is an arbitrary smooth complex-valued function of its arguments. This superclass is normalized, see [38], where it was also shown that any admissible transformation of the superclass satisfies the conditions

$$T_x = T_\psi = T_{\psi^*} = 0, \quad X_\psi = X_{\psi^*} = 0, \quad X_x^2 = |T_t|, \quad \Psi_{\hat{\psi}^*} = 0. \tag{4}$$

Hence the same is true for the class (1). The equations (4) give us

$$T = T(t), \quad X = \varepsilon|T_t|^{1/2}x + X^0(t), \quad \Psi = \Psi(t,x,\hat{\psi}),$$

where T and X^0 are arbitrary smooth real-valued functions of t and Ψ is an arbitrary smooth complex-valued function of its arguments. Then the invertibility of the transformation gives $T_t \neq 0$ and $\Psi_{\hat\psi} \neq 0$. Using the chain rule, we take total derivatives of the equation $\tilde\psi(\tilde t, \tilde x) = \Psi(t, x, \hat\psi)$ with respect to t and x, with $\tilde t = T$ and $\tilde x = X$ and we find the following expressions for the transformed derivatives:

$$\tilde\psi_{\tilde x} = \frac{1}{X_x}(\Psi_x + \Psi_{\hat\psi}\hat\psi_x), \quad \tilde\psi_{\tilde t} = \frac{1}{T_t}(\Psi_t + \Psi_{\hat\psi}\hat\psi_t) - \frac{X_t}{T_t X_x}(\Psi_x + \Psi_{\hat\psi}\hat\psi_x),$$
$$\tilde\psi_{\tilde x\tilde x} = \frac{1}{X_x^2}(\Psi_{xx} + 2\Psi_{x\hat\psi}\hat\psi_x + \Psi_{\hat\psi\hat\psi}\hat\psi_x^2 + \Psi_{\hat\psi}\hat\psi_{xx}).$$

We substitute these expressions into the equation $\mathcal{L}_{\tilde V}$, and then take into account that ψ satisfies the equation (1) so that $\hat\psi_{xx} = -\varepsilon' i\hat\psi_t - \hat V\hat\psi$. We then split with respect to $\hat\psi_t$ and $\hat\psi_x$, which yields

$$\Psi_{\hat\psi\hat\psi} = 0, \quad \Psi_{x\hat\psi} = \frac{i}{2}\frac{X_x}{T_t}X_t\Psi_{\hat\psi}, \tag{5}$$
$$\varepsilon' i\Psi_t + \Psi_{xx} - \varepsilon' i\frac{X_t}{X_x}\Psi_x + |T_t|\tilde V\Psi - \hat V\Psi_{\hat\psi}\hat\psi = 0. \tag{6}$$

The general solution of the first equation in (5) is $\Psi = \Psi^1(t,x)\hat\psi + \Psi^0(t,x)$, where Ψ^0 and Ψ^1 are smooth complex-valued functions of t and x. The second equation in (5) then reduces to a linear ordinary differential equation with respect to Ψ^1 with the independent variable x, and the variable t plays the role of a parameter. Integrating this equation gives the following expression for Ψ^1:

$$\Psi^1 = \exp\left(\frac{i}{8}\frac{T_{tt}}{|T_t|}x^2 + \frac{i}{2}\frac{\varepsilon\varepsilon' X^0_t}{|T_t|^{1/2}}x + i\Sigma(t) + \Upsilon(t)\right),$$

where Σ and Υ are arbitrary smooth real-valued functions of t. We substitute the expression for Ψ into the equation (6) and then split this equation with respect to $\hat\psi$ and this then gives us the equation

$$\tilde V = \frac{\hat V}{|T_t|} - \frac{1}{|T_t|}\frac{\Psi^1_{xx}}{\Psi^1} - \frac{i}{|T_t|\Psi^1}\left(\Psi^1_t - \frac{X_t}{X_x}\Psi^1_x\right),$$

which represents the component of the transformation (3) for V. We introduce the function $\Phi = \hat\Psi^0/\hat\Psi^1$, i.e., $\Psi^0 = \Psi^1\hat\Phi$. The terms in (6) not containing $\hat\psi$ give an equation in Ψ^0, which is equivalent to the initial linear Schrödinger equation in terms of Φ. □

Corollary I.1. *A (1+1)-dimensional linear Schrödinger equation of the form* (1) *is equivalent to the free linear Schrödinger equation with respect to a point transformation if and only if* $V = \gamma^2(t)x^2 + \gamma^1(t)x + \gamma^0(t) + i\tilde\gamma^0(t)$ *for some real-valued functions* γ^2, γ^1, γ^0 *and* $\tilde\gamma^0$ *of* t.

Corollary I.2. *The usual equivalence group* $G^\sim$ *of the class* (1) *consists of point transformations of the form* (3) *with* $\Phi = 0$.

Proof. Each transformation from $G^\sim$ generates a family of admissible transformations for the class (1) and hence is of the form (3). As a usual equivalence group, the group $G^\sim$ merely consists of point transformations in the variables (t, x, ψ) and the arbitrary element V that can be applied to each equation from the class (1) and whose components for the variables are independent of V. The only transformations of the form (3) that satisfy these requirements are those for which Φ runs through the set of common solutions of all equations from the class (1). $\Phi = 0$ is the only common solution. □

Remark I.3. Consider the natural projection π of the joint space of the variables and the arbitrary element V on the space of the variables only. For each transformation $\mathcal{T}$ from the equivalence group $G^\sim$ its components for the variables do not depend on V and are uniquely extended to V. These components define a transformation φ, which can be taken to be the pushforward of $\mathcal{T}$ by the projection π, $\varphi = \pi_*\mathcal{T}$. Therefore, there is a one-to-one correspondence between the equivalence group $G^\sim$ and the group $\pi_*G^\sim$ consisting of the projected equivalence transformations, $G^\sim \ni \mathcal{T} \mapsto \pi_*\mathcal{T} \in \pi_*G^\sim$.

Remark I.4. The identity component of the equivalence group $G^\sim$ consists of transformations of the form (3) with $T_t > 0$ and $\varepsilon = 1$. This group also contains two discrete transformations that are involutions: the space reflection $\tilde t = t$, $\tilde x = -x$, $\tilde\psi = \psi$, $\tilde V = V$ and the Wigner time reflection $\tilde t = -t$, $\tilde x = x$, $\tilde\psi = \psi^*$, $\tilde V = V^*$, which are independent up to composition with each other and with continuous transformations. These continuous and discrete transformations generate the entire equivalence group $G^\sim$.

Corollary I.3. *The equivalence algebra of the class* (1) *is the algebra*

$$\mathfrak{g}^\sim = \langle \hat D(\tau), \hat G(\chi), \hat M(\sigma), \hat I(\rho)\rangle$$

where τ, χ, σ and ρ run through the set of smooth real-valued functions of t. The vector fields that span $\mathfrak{g}^\sim$ are given by

$$\begin{aligned}
\hat D(\tau) &= \tau\partial_t + \frac12\tau_t x\partial_x + \frac{i}{8}\tau_{tt}x^2(\psi\partial_\psi - \psi^*\partial_{\psi^*})\\
&\qquad - \left(\tau_t V - \frac18\tau_{ttt}x^2 - i\frac{\tau_{tt}}{4}\right)\partial_V - \left(\tau_t V^* - \frac18\tau_{ttt}x^2 + i\frac{\tau_{tt}}{4}\right)\partial_{V^*},\\
\hat G(\chi) &= \chi\partial_x + \frac{i}{2}\chi_t x(\psi\partial_\psi - \psi^*\partial_{\psi^*}) + \frac{\chi_{tt}}{2}x(\partial_V + \partial_{V^*}),\\
\hat M(\sigma) &= i\sigma(\psi\partial_\psi - \psi^*\partial_{\psi^*}) + \sigma_t(\partial_V + \partial_{V^*}),\\
\hat I(\rho) &= \rho(\psi\partial_\psi + \psi^*\partial_{\psi^*}) - i\rho_t(\partial_V + \partial_{V^*}).
\end{aligned}$$

Proof. The equivalence algebra $\mathfrak{g}^\sim$ can be computed using the infinitesimal Lie method in a way similar to that for finding the Lie invariance algebra of a single system of differential equations [33]. However, we can avoid these calculations by noting that we have already constructed the complete point equivalence group $G^\sim$. The algebra $\mathfrak{g}^\sim$ is just the set of infinitesimal generators of one-parameter subgroups of the group $G^\sim$. In order to find all such generators, in the transformation form (3) we set $\Phi = 0$ (to single out equivalence transformations), $\varepsilon = 1$ and $T_t > 0$, i.e., $\varepsilon' = 1$ (to restrict to the continuous component of the identical transformation in $G^\sim$), and represent the parameter function Σ in the form $\Sigma = \frac{1}{4}X^0X^0_t + \bar\Sigma$ with some function $\bar\Sigma$ of t (to make the group parameterization more consistent with the one-parameter subgroup structure of $G^\sim$). Then we successively take one of the parameter-functions T, X^0, $\bar\Sigma$ and Υ to depend on a continuous subgroup parameter δ, set the other parameter-functions to their trivial values, which are t for T and zeroes for X^0, $\bar\Sigma$ and Υ, differentiate the transformation components with respect to δ and evaluate the result at $\delta = 0$. The corresponding infinitesimal generator is the vector field $\tau\partial_t + \xi\partial_x + \eta\partial_\psi + \eta^*\partial_{\psi^*} + \theta\partial_V + \theta^*\partial_{V^*}$, where

$$\tau = \frac{\mathrm{d}\tilde t}{\mathrm{d}\delta}\Big|_{\delta=0}, \quad \xi = \frac{\mathrm{d}\tilde x}{\mathrm{d}\delta}\Big|_{\delta=0}, \quad \eta = \frac{\mathrm{d}\tilde \psi}{\mathrm{d}\delta}\Big|_{\delta=0}, \quad \theta = \frac{\mathrm{d}\tilde V}{\mathrm{d}\delta}\Big|_{\delta=0}.$$

The above procedure gives the vector fields $\hat D(\tau)$, $\hat G(\chi)$, $\hat M(\sigma)$ and $\hat I(\rho)$ for the parameter-functions T, X^0, $\bar\Sigma$ and Υ, respectively. $\square$

Consider the point symmetry group G_V of an equation $\mathcal L_V$ from the class (1). Each element φ of G_V generates an admissible point transformation in the class (1) with the same initial and target arbitrary element V. Therefore, the components of φ necessarily have the form given in (3a)–(3b), and the parameter-functions satisfy the equation (3c) with $\tilde V(\tilde t, \tilde x) = V(\tilde t, \tilde x)$. The symmetry transformations defined by linear superposition of solutions to the equation $\mathcal L_V$ are of the form given in (3a)–(3b) with $T = t$ and $X^0 = \Sigma = \Upsilon = 0$. They constitute a normal subgroup $G^{\rm lin}_V$ of the group G_V, which can be assumed to be the trivial part of G_V. The factor group $G_V/G^{\rm lin}_V$ is isomorphic to the subgroup $G^{\rm ess}_V$ of G_V that is singled out from G_V by the constraint $\Phi = 0$ and will be considered as the only essential part of G_V.

Corollary I.4. *The essential point symmetry group $G^{\rm ess}_V$ of any equation $\mathcal L_V$ from the class* (1) *is contained in the projection $\pi_*G^\sim$ of the equivalence group $G^\sim$ to the space with local coordinates (t, x, ψ, ψ^*).*

Corollary I.5. *The class* (1) *is uniformly semi-normalized with respect to linear superposition of solutions.*

Proof. We need to show that any admissible transformation in the class (1) is the composition of a specific symmetry transformation of the initial equation and a transformation from $G^\sim$. We consider two fixed similar equations $\mathcal L_V$ and $\mathcal L_{\tilde V}$ in the class (1) and let φ be a point transformation connecting these equations. Then φ is of the form (3a)–(3b), and the potentials V and $\tilde V$ are related by (3c). The point transformation φ^1 given by $\tilde t = t$, $\tilde x = x$, $\tilde\psi = \psi + \Phi$ with the same function Φ as in φ is a symmetry transformation of the initial equation, which is related to the linear superposition principle. We choose the transformation φ^2 to be of the same form (3a)–(3b) but with $\Phi = 0$. By (3c), its extension to the arbitrary element belongs to the group $G^\sim$. The transformation φ is the composition of φ^1 and φ^2.

It is obvious that the transformation φ maps the subgroup of linear superposition transformations of the equation $\mathcal L_V$ onto that of the equation $\mathcal L_{\tilde V}$. $\square$

5 Analysis of determining equations for Lie symmetries

We derive the determining equations for elements from the maximal Lie invariance algebra $\mathfrak{g}_V$ of an equation $\mathcal{L}_V$ from the class (1) with potential $V = V(t,x)$. The general form of a vector field Q in the space with local coordinates (t,x,ψ,ψ^*) is $Q = \tau\partial_t + \xi\partial_x + \eta\partial_\psi + \eta^*\partial_{\psi^*}$, where the components of Q are smooth functions of (t,x,ψ,ψ^*). The vector field Q belongs to the algebra $\mathfrak{g}_V$ if and only if it satisfies the infinitesimal invariance criterion for the equation $\mathcal{L}_V$, which gives

$$i\eta^t + \eta^{xx} + \tau V_t\psi + \xi V_x\psi + V\eta = 0 \tag{7}$$

for all solutions of $\mathcal{L}_V$. Here

$$\begin{aligned}
\eta^t &= D_t\left(\eta - \tau\psi_t - \xi\psi_x\right) + \tau\psi_{tt} + \xi\psi_{tx},\\
\eta^{xx} &= D_x^2\left(\eta - \tau\psi_t - \xi\psi_x\right) + \tau\psi_{txx} + \xi\psi_{xxx},
\end{aligned}$$

D_t and D_x denote the total derivative operators with respect to t and x, respectively. After substituting $\psi_{xx} = -i\psi_t - V\psi$ and $\psi^*_{xx} = i\psi^*_t - V^*\psi^*$ into (7) and splitting with respect to the other derivatives of ψ and ψ^* that occur, we obtain a linear overdetermined system of determining equations for the components of Q,

$$\begin{aligned}
&\tau_\psi = \tau_{\psi^*} = 0, \quad \tau_x = 0, \quad \xi_\psi = \xi_{\psi^*} = 0, \quad \tau_t = 2\xi_x,\\
&\eta_{\psi^*} = \eta_{\psi\psi} = 0, \quad 2\eta_{\psi x} = i\xi_t,\\
&i\eta_t + \eta_{xx} + \tau V_t\psi + \xi V_x\psi + V\eta - (\eta_\psi - \tau_t)V\psi = 0.
\end{aligned}$$

We solve the determining equations in the two first lines to obtain

$$\begin{aligned}
&\tau = \tau(t), \quad \xi = \frac{1}{2}\tau_t x + \chi(t),\\
&\eta = \left(\frac{i}{8}\tau_{tt}x^2 + \frac{i}{2}\chi_t x + \rho(t) + i\sigma(t)\right)\psi + \eta^0(t,x),
\end{aligned}$$

where τ, χ, ρ and σ are smooth real-valued functions of t, and η^0 is a complex-valued function of t and x. Then splitting the last determining equation with respect to ψ, we derive two equations:

$$i\eta^0_t + \eta^0_{xx} + \eta^0 V = 0, \tag{8}$$

$$\tau V_t + \left(\frac{1}{2}\tau_t x + \chi\right)V_x + \tau_t V = \frac{1}{8}\tau_{ttt}x^2 + \frac{1}{2}\chi_{tt}x + \sigma_t - i\rho_t - \frac{i}{4}\tau_{tt}. \tag{9}$$

Although both these equations contain the potential V, the first equation just means that the parameter-function η^0 satisfies the equation $\mathcal{L}_V$, which does not affect the structure of the algebra $\mathfrak{g}_V$ when the potential V varies. This is why we only consider the second equation as the real *classifying condition* for Lie symmetry operators of equations from the class (1).

We have thus proved the following result:

Theorem I.4. *The maximal Lie invariance algebra $\mathfrak{g}_V$ of the equation $\mathcal{L}_V$ from the class* (1) *consists of the vector fields of the form $Q = D(\tau) + G(\chi) + \sigma M + \rho I + Z(\eta^0)$, where*

$$D(\tau) = \tau\partial_t + \frac{1}{2}\tau_t x\partial_x + \frac{1}{8}\tau_{tt}x^2 M, \quad G(\chi) = \chi\partial_x + \frac{1}{2}\chi_t x M,$$
$$M = i\psi\partial_\psi - i\psi^*\partial_{\psi^*}, \quad I = \psi\partial_\psi + \psi^*\partial_{\psi^*}, \quad Z(\eta^0) = \eta^0\partial_\psi + \eta^{0*}\partial_{\psi^*},$$

the parameters τ, χ, ρ, σ run through the set of real-valued smooth functions of t satisfying the classifying condition (9)*, and η^0 runs through the solution set of the equation $\mathcal{L}_V$.*

Note that Theorem I.4 can be derived from Theorem I.3 using the same technique as in Corollary I.3. The algebra $\mathfrak{g}_V$ consists of infinitesimal generators of one-parameter subgroups of the point symmetry group G_V of the equation $\mathcal{L}_V$. In considering one-parameter subgroups of G_V, we set $\varepsilon = 1$ and $T_t > 0$, i.e., $\varepsilon' = 1$ since one-parameter subgroups are contained in the identity component of G_V. We also represent Σ in the form $\Sigma = \frac{1}{4}X^0X^0_t + \bar{\Sigma}$. We let the parameter functions T, X^0, $\bar{\Sigma}$, Υ and Φ properly depend on a continuous subgroup parameter δ. Then we differentiate the equations (3) with respect to δ and evaluate the result at $\delta = 0$. The corresponding infinitesimal generator is the vector field $Q = \tau\partial_t + \xi\partial_x + \eta\partial_\psi + \eta^*\partial_{\psi^*}$, where

$$\tau = \frac{\mathrm{d}\tilde{t}}{\mathrm{d}\delta}\Big|_{\delta=0}, \quad \xi = \frac{\mathrm{d}\tilde{x}}{\mathrm{d}\delta}\Big|_{\delta=0}, \quad \eta = \frac{\mathrm{d}\tilde{\psi}}{\mathrm{d}\delta}\Big|_{\delta=0},$$

and hence Q has the form given in Theorem I.4. The classifying condition (9) is derived from the equation (3c) with $\tilde{V}(\tilde{t},\tilde{x}) = V(\tilde{t},\tilde{x})$.

In order to find the kernel invariance algebra $\mathfrak{g}^\cap$ of the class (1), $\mathfrak{g}^\cap := \bigcap_V \mathfrak{g}_V$, we vary the potential V and then split the equations (8) and (9) with respect to V and its derivatives. This gives us the equations $\tau = \chi = 0$, $\eta^0 = 0$ and $\rho_t = \sigma_t = 0$.

Proposition I.1. *The kernel invariance algebra of the class* (1) *is the algebra $\mathfrak{g}^\cap = \langle M, I\rangle$.*

Consider the linear span of all vector fields from Theorem I.4 when V varies given by

$$\mathfrak{g}_{\langle\,\rangle} := \langle D(\tau), G(\chi), \sigma M, \rho I, Z(\zeta)\rangle = \textstyle\sum_V \mathfrak{g}_V.$$

Here and in the following the parameters τ, χ, σ and ρ run through the set of real-valued smooth functions of t, ζ runs through the set of complex-valued smooth functions of (t,x) and η^0 runs through the solution set of the equation $\mathcal{L}_V$ when the potential V is fixed. We have $\mathfrak{g}_{\langle\,\rangle} = \sum_V \mathfrak{g}_V$ since each vector field Q from $\mathfrak{g}_{\langle\,\rangle}$ with nonvanishing τ or χ or with jointly vanishing τ, χ, σ and ρ necessarily belongs to $\mathfrak{g}_V$ for some V.

The nonzero commutation relations between vector fields spanning $\mathfrak{g}_{\langle\,\rangle}$ are

$$[D(\tau^1), D(\tau^2)] = D(\tau^1\tau^2_t - \tau^2\tau^1_t), \quad [D(\tau), G(\chi)] = G\left(\tau\chi_t - \frac{1}{2}\tau_t\chi\right),$$

$$[D(\tau), \sigma M] = \tau\sigma_t M, \quad [D(\tau), \rho I] = \tau\rho_t I,$$
$$[D(\tau), Z(\zeta)] = Z\left(\tau\zeta_t + \frac{1}{2}\tau_t x\zeta_x - \frac{i}{8}\tau_{tt}x^2\zeta\right),$$
$$[G(\chi^1), G(\chi^2)] = \left(\chi^1\chi^2_t - \chi^2\chi^1_t\right) M, \quad [G(\chi), Z(\zeta)] = Z\left(\chi\zeta_x - \frac{i}{2}\chi_t x\zeta\right),$$
$$[\sigma M, Z(\zeta)] = Z(-i\sigma\zeta), \quad [\rho I, Z(\zeta)] = Z(-\rho\zeta).$$

The commutation relations between elements of $\mathfrak{g}_{\langle\,\rangle}$ show that $\mathfrak{g}_{\langle\,\rangle}$ itself is a Lie algebra. It is convenient to represent $\mathfrak{g}_{\langle\,\rangle}$ as a semi-direct sum,

$$\mathfrak{g}_{\langle\,\rangle} = \mathfrak{g}^{\rm ess}_{\langle\,\rangle} \mathbin{\supset\!\!\!\!+} \mathfrak{g}^{\rm lin}_{\langle\,\rangle}, \quad \text{where} \quad \mathfrak{g}^{\rm ess}_{\langle\,\rangle} := \langle D(\tau), G(\chi), \sigma M, \rho I\rangle \quad \text{and} \quad \mathfrak{g}^{\rm lin}_{\langle\,\rangle} := \langle Z(\zeta)\rangle$$

are a subalgebra and an abelian ideal of $\mathfrak{g}_{\langle\,\rangle}$, respectively. Note that the kernel invariance algebra $\mathfrak{g}^\cap$ is an ideal of $\mathfrak{g}^{\rm ess}_{\langle\,\rangle}$ and of $\mathfrak{g}_{\langle\,\rangle}$. The above representation of $\mathfrak{g}_{\langle\,\rangle}$ induces a similar representation for each $\mathfrak{g}_V$,

$$\mathfrak{g}_V = \mathfrak{g}^{\rm ess}_V \mathbin{\supset\!\!\!\!+} \mathfrak{g}^{\rm lin}_V,$$

where $\mathfrak{g}^{\rm ess}_V := \mathfrak{g}_V \cap \mathfrak{g}^{\rm ess}_{\langle\,\rangle}$ and $\mathfrak{g}^{\rm lin}_V := \mathfrak{g}_V \cap \mathfrak{g}^{\rm lin}_{\langle\,\rangle} = \langle Z(\eta^0),\, \eta^0 \in \mathcal{L}_V\rangle$ are a finite-dimensional subalgebra (see Lemma I.1 below) and an infinite-dimensional abelian ideal of $\mathfrak{g}_V$, respectively. We call $\mathfrak{g}^{\rm ess}_V$ the *essential Lie invariance algebra* of the equation $\mathcal{L}_V$ for each V. The ideal $\mathfrak{g}^{\rm lin}_V$ consists of vector fields associated with transformations of *linear superposition* and therefore it is a trivial part of $\mathfrak{g}_V$.

Definition I.2. A subalgebra $\mathfrak{s}$ of $\mathfrak{g}^{\rm ess}_{\langle\,\rangle}$ is called *appropriate* if there exists a potential V such that $\mathfrak{s} = \mathfrak{g}^{\rm ess}_V$.

The algebras $\mathfrak{g}^{\rm ess}_{\langle\,\rangle}$ and $\mathfrak{g}^\sim$ are related to each other by $\mathfrak{g}^{\rm ess}_{\langle\,\rangle} = \pi_*\mathfrak{g}^\sim$, where π is the projection of the joint space of the variables and the arbitrary element on the space of the variables only. The mapping π_* induced by π is well defined on $\mathfrak{g}^\sim$ due to the structure of elements of $\mathfrak{g}^\sim$. Note that the vector fields $\hat D(\tau)$, $\hat G(\chi)$, $\hat M(\sigma)$, $\hat I(\rho)$ spanning $\mathfrak{g}^\sim$ are mapped by π_* to the vector fields $D(\tau)$, $G(\chi)$, σM, ρI spanning $\mathfrak{g}^{\rm ess}_{\langle\,\rangle}$, respectively. The above relation is stronger than that implied by the specific semi-normalization of the class (1), $\mathfrak{g}^{\rm ess}_{\langle\,\rangle} \subseteq \pi_*\mathfrak{g}^\sim$. Since the algebra $\mathfrak{g}^{\rm ess}_{\langle\,\rangle}$ coincides with the set $\pi_*\mathfrak{g}^\sim$ of infinitesimal generators of one-parameter subgroups of the group $\pi_*G^\sim$, the structure of $\mathfrak{g}^{\rm ess}_{\langle\,\rangle}$ is compatible with the action of $\pi_*G^\sim$ on this algebra. Moreover, both $\mathfrak{g}^{\rm ess}_{\langle\,\rangle}$ and $\mathfrak{g}^{\rm lin}_{\langle\,\rangle}$ are invariant with respect to the action of the group $\pi_*G^\sim$. This is why the action of $G^\sim$ on equations from the class (1) induces the well-defined action of $\pi_*G^\sim$ on the essential Lie invariance algebras of these equations, which are subalgebras of $\mathfrak{g}^{\rm ess}_{\langle\,\rangle}$. The kernel $\mathfrak{g}^\cap$ is obviously an ideal in $\mathfrak{g}^{\rm ess}_V$ for any V.

Collecting all the above arguments, we obtain the following assertion.

Proposition I.2. *The problem of group classification of (1+1)-dimensional linear Schrödinger equations reduces to the classification of appropriate subalgebras of the algebra $\mathfrak{g}^{\rm ess}_{\langle\,\rangle}$ with respect to the equivalence relation generated by the action of $\pi_*G^\sim$.*

Equivalently, we can classify the counterparts of appropriate subalgebras in $\mathfrak{g}^\sim$ up to $G^\sim$-equivalence and then project them to the space of variables [4].

6 Group classification

To classify appropriate subalgebras of the algebra $\mathfrak{g}^{\rm ess}_{\langle\,\rangle}$, we need to compute the action of transformations from the group $\pi_*G^\sim$ on vector fields from $\mathfrak{g}^{\rm ess}_{\langle\,\rangle}$. For any transformation $\varphi\in\pi_*G^\sim$ and any vector field $Q\in\mathfrak{g}^{\rm ess}_{\langle\,\rangle}$, the pushforward action of φ on Q is given by

$$\tilde Q:=\varphi_*Q=Q(T)\partial_{\tilde t}+Q(X)\partial_{\tilde x}+Q(\Psi)\partial_{\tilde\psi}+Q(\Psi^*)\partial_{\tilde\psi^*},$$

where in each component of $\tilde Q$ we substitute the expressions of the variables without tildes in terms of the “tilded” variables, $(t,x,\psi,\psi^*)=\varphi^{-1}(\tilde t,\tilde x,\tilde\psi,\tilde\psi^*)$, and φ^{-1} denotes the inverse of φ.

For convenience, we introduce the following notation for elementary transformations from $\pi_*G^\sim$, which generate the entire group $\pi_*G^\sim$: $\mathcal D(T)$, $\mathcal G(X^0)$, $\mathcal M(\Sigma)$ and $\mathcal I(\Upsilon)$ respectively denote the transformations of the form (3a)–(3b) with $\Phi=0$ and $\varepsilon=1$, where the parameter-functions T, X^0, Σ and Υ, successively excluding one of them, are set to the values corresponding to the identity transformation, which are t for T and zeroes for X^0, Σ and Υ. The nontrivial pushforward actions of elementary transformations from $\pi_*G^\sim$ to the vector fields spanning $\mathfrak{g}^{\rm ess}_{\langle\,\rangle}$ are

$$\begin{aligned}
&\mathcal D_*(T)D(\tau)=\tilde D(T_t\tau),\quad \mathcal D_*(T)G(\chi)=\tilde G(T_t^{1/2}\chi),\\
&\mathcal D_*(T)(\sigma M)=\sigma\tilde M,\quad \mathcal D_*(T)(\rho I)=\rho\tilde I,\\
&\mathcal G_*(X^0)D(\tau)=\tilde D(\tau)+\tilde G\left(\tau X^0_t-\frac12\tau_tX^0\right)\\
&\qquad\qquad+\left(\frac18\tau_{tt}(X^0)^2-\frac14\tau_tX^0X^0_t-\frac12\tau X^0X^0_{tt}\right)\tilde M,\\
&\mathcal G_*(X^0)G(\chi)=\tilde G(\chi)+\frac12(\chi X^0_t-\chi_tX^0)\tilde M,\\
&\mathcal M_*(\Sigma)D(\tau)=\tilde D(\tau)+\tau\Sigma_t\tilde M,\quad \mathcal I_*(\Upsilon)D(\tau)=\tilde D(\tau)+\tau\Upsilon_t\tilde I,
\end{aligned}$$

where in each pushforward by $\mathcal D_*(T)$ we should substitute the expression for t given by inverting the relation $\tilde t=T(t)$; $t=\tilde t$ for the other pushforwards. Tildes over vector fields mean that these vector fields are represented in the new variables.

Lemma I.1. $\dim\mathfrak{g}^{\rm ess}_V\leqslant7$ *for any potential* V.

Proof. Since we work within the local framework, we can assume that the equation $\mathcal L_V$ is considered on a domain of the form $\Omega_0\times\Omega_1$, where Ω_0 and Ω_1 are intervals on the t- and x-axes, respectively. Then we successively evaluate the classifying condition (9) at three different points $x=x_0-\delta$, $x=x_0$ and $x=x_0+\delta$ from Ω_1 for varying t. This gives

$$\begin{aligned}
&\frac18\tau_{ttt}(x_0-\delta)^2+\frac12\chi_{tt}(x_0-\delta)-i\rho_t+\sigma_t-\frac i4\tau_{tt}=R_1,\\
&\frac18\tau_{ttt}x_0^2+\frac12\chi_{tt}x_0-i\rho_t+\sigma_t-\frac i4\tau_{tt}=R_2,
\end{aligned}$$

$$\frac{1}{8}\tau_{ttt}(x_0+\delta)^2+\frac{1}{2}\chi_{tt}(x_0+\delta)-i\rho_t+\sigma_t-\frac{i}{4}\tau_{tt}=R_3,$$

where the right hand sides R_1, R_2 and R_3 are the results of substituting the above values of x into $\tau V_t+\left(\frac{1}{2}\tau_t x+\chi\right)V_x+\tau_t V$. Combining the above equations and splitting them into real and imaginary parts, we obtain a canonical system of linear ordinary differential equations of the form

$$\tau_{ttt}=\dots,\quad \chi_{tt}=\dots,\quad \rho_t=\dots,\quad \sigma_t=\dots$$

to τ, χ, ρ and σ. The qualification "canonical" means that the system is solved with respect to the highest-order derivatives. The right hand sides of all its equations are denoted by dots since their precise form is not important for our argument. It is obvious that the solution set of the above system is a linear space and parameterized by seven arbitrary constants. □

In order to classify appropriate subalgebras of $\mathfrak{g}^{\rm ess}_{\langle\,\rangle}$, for each subalgebra $\mathfrak{s}$ of $\mathfrak{g}^{\rm ess}_{\langle\,\rangle}$ we introduce two integers

$$k_1=k_1(\mathfrak{s}):=\dim\pi^0_*\mathfrak{s},\quad k_2=k_2(\mathfrak{s}):=\dim\mathfrak{s}\cap\langle G(\chi),\sigma M,\rho I\rangle-2,$$

where π^0 denotes the projection onto the space of the variable t and $\pi^0_*\mathfrak{s}\subset\pi^0_*\mathfrak{g}^{\rm ess}_{\langle\,\rangle}=\langle\tau\partial_t\rangle$. The values of k_1 and k_2 are invariant under the action of $\pi_*G^\sim$.

Lemma I.2. *$\pi^0_*\mathfrak{g}^{\rm ess}_V$ is a Lie algebra for any potential V and $k_1=\dim\pi^0_*\mathfrak{g}^{\rm ess}_V\leqslant 3$. Further, $\pi^0_*\mathfrak{g}^{\rm ess}_V\in\{0,\langle\partial_t\rangle,\langle\partial_t,t\partial_t\rangle,\langle\partial_t,t\partial_t,t^2\partial_t\rangle\}\bmod\pi^0_*G^\sim$.*

Proof. To prove that $\pi^0_*\mathfrak{g}^{\rm ess}_V$ is a Lie algebra we show that it is a linear subspace and closed under Lie bracket of vector fields. Given $\tau^j\partial_t\in\pi^0_*\mathfrak{g}^{\rm ess}_V$, $j=1,2$, there exist $Q^j\in\mathfrak{g}^{\rm ess}_V$ such that $\pi^0_*Q^j=\tau^j\partial_t$. Then for any real constants c_1 and c_2 we have $c_1Q^1+c_2Q^2\in\mathfrak{g}^{\rm ess}_V$. Therefore, $c_1\tau^1\partial_t+c_2\tau^2\partial_t=\pi^0_*(c_1Q^1+c_2Q^2)\in\pi^0_*\mathfrak{g}^{\rm ess}_V$. Next, $[\tau^1\partial_t,\tau^2\partial_t]=(\tau^1\tau^2_t-\tau^2\tau^1_t)\partial_t=\pi^0_*[Q^1,Q^2]\in\pi^0_*\mathfrak{g}^{\rm ess}_V$. Further, $\dim\pi^0_*\mathfrak{g}^{\rm ess}_V\leqslant\dim\mathfrak{g}^{\rm ess}_V\leqslant 7$.

Thus $\pi^0_*\mathfrak{g}^{\rm ess}_V$ is a finite-dimensional subalgebra of the Lie algebra $\pi^0_*\mathfrak{g}^{\rm ess}_{\langle\,\rangle}$ of vector fields on the real line. The group $\pi^0_*G^\sim$ coincides with the entire group of local diffeomorphisms of the real line and the rest of the lemma follows from Lie's theorem on finite-dimensional Lie algebras of vector fields on the real line. □

Lemma I.3. *If a vector field Q is of the form $Q=G(\chi)+\sigma M+\rho I$ with $\chi\neq 0$, then $Q=G(1)+\tilde\rho I\bmod\pi_*G^\sim$ for another function $\tilde\rho$.*

Proof. We successively push forward the vector field Q by the transformations $\mathcal{G}(X^0)$ and $\mathcal{D}(T)$, where X^0 and T are arbitrary fixed solutions of the ordinary differential equations $\chi X^0_t-\chi_t X^0=2\sigma$ and $T_t=1/\chi^2$, respectively. This leads to a vector field of the same form, with $\chi=1$ and $\sigma=0$. □

Lemma I.4. *If $G(1)+\rho^1 I\in\mathfrak{g}^{\rm ess}_V$, then also $G(t)+\rho^2 I\in\mathfrak{g}^{\rm ess}_V$ with $\rho^2=\int t\rho^1_t\,{\rm d}t$.*

Proof. The fact that $G(1)+\rho^1 I\in\mathfrak{g}^{\rm ess}_V$ means that the values $\tau=\sigma=0$, $\chi=1$ and $\rho=\rho^1$ satisfy the classifying condition (9) with the given potential V, which gives $V_x=-i\rho_t$. Then $tV_x=-it\rho_t$ implies that the classifying condition (9) is also satisfied by $\tau=\sigma=0$, $\chi=t$ and $\rho^2=\int t\rho^1_t\,{\rm d}t$. □

Lemma I.5. *$\mathfrak{g}^{\rm ess}_V \cap \langle \sigma M, \rho I\rangle = \mathfrak{g}^\cap$ for any potential V.*

Proof. We need to show that $\mathfrak{g}^{\rm ess}_V \cap \langle \sigma M, \rho I\rangle \subset \mathfrak{g}^\cap$ and $\mathfrak{g}^\cap \subset \mathfrak{g}^{\rm ess}_V \cap \langle \sigma M, \rho I\rangle$. The first inclusion follows from the classifying condition (9) for $\tau = \chi = 0$, which implies $\sigma_t = \rho_t = 0$. The second inclusion is obvious since the kernel invariance algebra $\mathfrak{g}^\cap$ is contained in $\mathfrak{g}^{\rm ess}_V$ for any V. □

Lemma I.6. *$k_2 = \dim \mathfrak{g}^{\rm ess}_V \cap \langle G(\chi), \sigma M, \rho I\rangle - 2 \in \{0, 2\}$ for any potential V.*

Proof. Denote $\mathfrak{a}_V := \mathfrak{g}^{\rm ess}_V \cap \langle G(\chi), \sigma M, \rho I\rangle$.

If $\mathfrak{a}_V \subseteq \langle \sigma M, \rho I\rangle$, then $\mathfrak{a}_V = \mathfrak{g}^{\rm ess}_V \cap \langle \sigma M, \rho I\rangle = \mathfrak{g}^\cap$, i.e., $k_2 = \dim \mathfrak{a}_V - 2 = 0$.

If $\mathfrak{a}_V \nsubseteq \langle \sigma M, \rho I\rangle$, then there exists $Q^1 \in \mathfrak{a}_V$ such that $Q^1 \notin \langle \sigma M, \rho I\rangle$. From Lemma I.3, up to $\pi_* G^\sim$-equivalence we may assume that Q^1 is locally of the form $Q^1 = G(1) + \rho^1 I$. Then Lemma I.4 implies that $Q^2 = G(t) + \rho^2 I$ with $\rho^2 = \int t\rho^1_t\, {\rm d}t$ belongs to $\mathfrak{a}_V$. We also have $\mathfrak{a}_V \supset \mathfrak{g}^\cap$. It then follows that $\langle M, I, Q^1, Q^2\rangle \subseteq \mathfrak{a}_V$ and hence $\dim \mathfrak{a}_V \geqslant 4$. On the other hand, as follows from the proof of Lemma I.1 under the constraint $\tau = 0$, the classifying condition (9) implies in particular a canonical system of linear ordinary differential equations of the form

$$\chi_{tt} = \dots, \quad \rho_t = \dots, \quad \sigma_t = \dots$$

in the parameter-functions χ, ρ and σ, whose solution space is four-dimensional. This means that $\dim \mathfrak{a}_V \leqslant 4$. Therefore, $k_2 = \dim \mathfrak{a}_V - 2 = 2$. □

Summarizing the above results, any appropriate subalgebra of $\mathfrak{g}^{\rm ess}_{\langle\,\rangle}$ is spanned by

- the basis vector fields M and I of the kernel $\mathfrak{g}^\cap$,
- k_1 vector fields $D(\tau^j) + G(\chi^j) + \sigma^j M + \rho^j I$, where $j = 1, \dots, k_1$, $k_1 \leqslant 3$, and $\tau^1, \dots, \tau^{k_1}$ are linearly independent,
- k_2 vector fields $G(\chi^l) + \sigma^l M + \rho^l I$ where $l = 1, \dots, k_2$, $k_2 \in \{0, 2\}$ and $\chi^1, \dots, \chi^{k_2}$ are linearly independent.

Theorem I.5. *A complete list of $G^\sim$-inequivalent (and, therefore, $\mathcal{G}^\sim$-inequivalent) Lie symmetry extensions in the class* (1) *is exhausted by the cases collected in Table 1.*

Proof. We consider possible cases for the various values of k_1 and k_2.

$\boldsymbol{k_1 = k_2 = 0.}$ This is the general case with no extension, i.e., $\mathfrak{g}^{\rm ess}_V = \mathfrak{g}^\cap$ (Case 1 of Table 1).

$\boldsymbol{k_1 = 0,\ k_2 = 2.}$ Lemmas I.3 and I.4 imply that up to $G^\sim$-equivalence the algebra $\mathfrak{g}^{\rm ess}_V$ contains the vector fields $G(1) + \rho^1 I$ and $G(t) + \rho^2 I$, where ρ^1 is a smooth real-valued function of t and $\rho^2 = \int t\rho^1_t\, {\rm d}t$. Integrating the classifying condition (9) for these vector fields with respect to V gives $V = -i\rho_t x + \alpha(t) + i\beta(t)$, and $\alpha = \beta = 0 \bmod G^\sim$. Denoting $-\rho_t$ by γ, we obtain $V = i\gamma(t)x$, $\rho^1 = -\int \gamma\, {\rm d}t$ and $\rho^2 = -\int t\gamma\, {\rm d}t$, which leads to Case 2 of Table 1.

Table 1. Results of classification.

no.	k_1	k_2	V	Basis of $\mathfrak{g}_V^{\rm ess}$
1	0	0	$V(t,x)$	$M,\ I$
2	0	2	$i\gamma(t)x$	$M,\ I,\ G(1)-\left(\int\gamma(t)\,\mathrm{d}t\right)I,\ G(t)-\left(\int t\gamma(t)\,\mathrm{d}t\right)I$
3	1	0	$V(x)$	$M,\ I,\ D(1)$
4a	1	2	$\frac14x^2+ibx$	$M,\ I,\ D(1),\ G(e^t)-be^tI,\ G(e^{-t})+be^{-t}I$
4b	1	2	$-\frac14x^2+ibx$	$M,\ I,\ D(1),\ G(\cos t)+b(\sin t)I,\ G(\sin t)-b(\cos t)I$
4c	1	2	ibx	$M,\ I,\ D(1),\ G(1)-btI,\ G(t)-\frac12bt^2I$
5	3	0	cx^{-2}	$M,\ I,\ D(1),\ D(t),\ D(t^2)-\frac12tI$
6	3	2	0	$M,\ I,\ D(1),\ D(t),\ D(t^2)-\frac12tI,\ G(1),\ G(t)$

Lie symmetry extensions given in Table 1 are maximal if the parameters involved satisfy the following conditions: In Case 1, the potential V does not satisfy an equation of the form (9). In Case 2, the real-valued function γ of t is constrained by the condition $\gamma\neq c_3|c_2t^2+c_1t+c_0|^{-3/2}$ for any real constants c_0, c_1, c_2 and c_3 with c_0, c_1 and c_2 not vanishing simultaneously. In Case 3, $V\neq b_2x^2+b_1x+b_0+c(x+a)^{-2}$ for any real constants a, b_2 and for any complex constants b_1, b_0 and c with $c\,\mathrm{Im}\,b_1=0$. The real constant b in Cases 4a–4c and the complex constant c in Case 5 are nonzero. Further, $b>0 \bmod G^\sim$ in Cases 4a and 4b and $b=1 \bmod G^\sim$ in Case 4c.

Let us describe the values of γ for which the Lie symmetry extension constructed is maximal. We substitute the potential $V=i\gamma(t)x$ into the classifying condition (9) and, after splitting with respect to x, derive the system

$$\tau_{ttt}=0,\quad \chi_{tt}=0,\quad \sigma_t=0,\quad \rho_t=\chi\gamma-\frac{\tau_{tt}}{4},\quad \tau\gamma_t+\frac32\tau_t\gamma=0.$$

An additional Lie symmetry extension for such a potential may be realized only by vector fields with nonzero values of τ. Then the integration of the first and last equations of the above system yields

$$\tau=c_2t^2+c_1t+c_0,\quad \gamma=c_3|\tau|^{-3/2}=c_3|c_2t^2+c_1t+c_0|^{-3/2},$$

where c_0, c_1, c_2 and c_3 are real constants. Therefore, Case 2 presents a maximal Lie symmetry extension if $\gamma\neq c_3|c_2t^2+c_1t+c_0|^{-3/2}$ for any real constants c_0, c_1, c_2 and c_3, where the constants c_0, c_1 and c_2 do not vanish simultaneously.

$\boldsymbol{k_1=1,\ k_2=0.}$ The algebra $\mathfrak{g}_V^{\rm ess}$ necessarily contains a vector field P^0 of the form $P^0=D(\tau^0)+G(\chi^0)+\sigma^0M+\rho^0I$, where all the parameter functions are real-valued functions of t with $\tau^0\neq0$. Push-forwarding P^0 by a transformation from $\pi_*G^\sim$, we can set $\tau^0=1$ and $\chi^0=\sigma^0=\rho^0=0$. That is, up to $\pi_*G^\sim$-equivalence we can assume that $P^0=D(1)$, cf. Lemma I.2. The classifying condition (9) for the vector field P^0 gives $V_t=0$, which implies Case 3 of Table 1 with an arbitrary time-independent potential V.

We now find the condition when the Lie symmetry extension obtained is really maximal. The presence of any additional extension means that the algebra $\mathfrak{g}_V^{\rm ess}$ necessarily contains a vector field $Q=D(\tau)+G(\chi)+\sigma M+\rho I$ with $\tau_t\neq 0$ or $\chi\neq 0$. Substituting Q into the classifying condition (9) and fixing a value of t gives a linear ordinary differential equation with respect to $V=V(x)$. The general solution of any such equation is of the form $V=b_2x^2+b_1x+b_0+c(x+a)^{-2}$, where a and b_2 are real constants and b_1, b_0 and c are complex constants. Moreover, the constant b_1 is zero if $\tau_t\neq 0$ and, if $\tau_t=0$ and $\chi\neq 0$, we have $c=0$. Therefore, the Lie symmetry extension of Case 3 is maximal if and only if $V\neq\ b_2x^2+b_1x+b_0+c(x+a)^{-2}$ for any real constants a, b_2 and for any complex constant b_1, b_0 and c with $c\,{\rm Im}\,b_1=0$.

$\boldsymbol{k_1=1,\ k_2=2.}$ In this case a basis of $g_V^{\rm ess}$ consists of the vector fields M, I, $P^0=D(\tau^0)+G(\chi^0)+\sigma^0M+\rho^0I$ and $Q^p=G(\chi^p)+\sigma^pM+\rho^pI$, where all the parameters are real-valued functions of t with $\tau^0\neq 0$ and χ^1 and χ^2 being linearly independent. Here and in the following the indices p and q run from 1 to 2 and we sum over repeated indices. The vector field P^0 is reduced to $D(1)$ up to $\pi_*G^{\sim}$-equivalence, as in the previous case. The commutation relations of $\mathfrak{g}_V^{\rm ess}$

$$
\begin{aligned}
&[P^0,Q^p]=G(\chi^p_t)+\sigma^p_tM+\rho^p_tI=a_{pq}Q^q+a_{p_3}M+a_{p_4}I,\\
&[Q^1,Q^2]=(\chi^1\chi^2_t-\chi^2\chi^1_t)M=a_0M,
\end{aligned}
$$

where a_{pq}, a_{p_3}, a_{p_4} and a_0 are real constants, yield

$$
\chi^p_t=a_{pq}\chi^q,\ \ \sigma^p_t=a_{pq}\sigma^q+a_{p_3},\ \ \rho^p_t=a_{pq}\rho^q+a_{p_4},\ \ \chi^1\chi^2_t-\chi^2\chi^1_t=a_0. \qquad (10)
$$

The matrix (a_{pq}) is not zero in view of the linear independence of χ^1 and χ^2. Moreover, the consistency of the system (10) implies that the trace of (a_{pq}) is zero. Using equivalence transformations of time scaling, we can further scale the eigenvalues of the matrix (a_{pq}) with the same nonzero real values. Replacing the vector fields Q^1 and Q^2 by their independent linear combinations leads to a matrix similarity transformation of (a_{pq}). Hence, the matrix (a_{pq}) can be assumed to be of one of the following real Jordan forms:

$$
\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad \begin{pmatrix}0&-1\\1&0\end{pmatrix},\quad \begin{pmatrix}0&0\\1&0\end{pmatrix}. \qquad (11)
$$

The further consideration of each of these forms consists of a few steps: We integrate the system of differential equations (10) for the chosen form of (a_{pq}), which gives the components of the vector fields Q^1 and Q^2. From the classifying condition (9) for the basis vector fields of the algebra $\mathfrak{g}_V^{\rm ess}$ we obtain three independent equations for the potential V, including the equation $V_t=0$. These equations must be solved jointly, and their consistency leads to additional constraints for constant parameters involved in Q^1 and Q^2. The expressions for both the vector fields Q^1 and Q^2 and the potential V can be simplified by equivalence transformations and by changing the basis in the algebra $\mathfrak{g}_V^{\rm ess}$ we can obtain expressions for Q^1 and Q^2 as in Cases 4a–4c of Table 1.

Integrating the system (10) for the first Jordan form, we obtain $\chi^1 = b_{01}e^t$, $\sigma^1 = b_{11}e^t - a_{13}$, $\rho^1 = b_{12}e^t - a_{14}$, $\chi^2 = b_{02}e^{-t}$, $\sigma^2 = b_{21}e^{-t} + a_{23}$, $\rho^2 = b_{22}e^{-t} + a_{24}$, where a_{p_3}, a_{p_4}, b_{0p} and b_{pq} are real constants. Scaling the vector fields Q^1 and Q^2 and taking linear combinations of them with M and I, we can set $b_{0q} = 1$ and $a_{p3} = a_{p4} = 0$. The classifying condition (9) for the vector fields P^0, Q^1 and Q^2 leads to three independent equations in V, $V_t = 0$, $V_x = \frac{1}{2}x - ib_{12} + b_{11}$ and $V_x = \frac{1}{2}x + ib_{22} - b_{21}$. These equations are consistent only if the constant parameters involved in Q^1 and Q^2 satisfy the constraints $-b_{12} = b_{22} =: b$ and $b_{11} = -b_{21} =: -\hat{b}$. Then the potential V is of the form $V = \frac{1}{4}(x+2\hat{b})^2 + ibx + c_1 + ic_2$ for some real constants c_1 and c_2. We apply the equivalence transformation (3) with $T = t$, $X^0 = 2\hat{b}$, $\Sigma = -c_1 t$, $\Upsilon = c_2 t$, $\varepsilon = 1$ and $\Phi = 0$ and take a linear combination of the transformed vector field P^0 with M and I. This allows us to set $\hat{b} = c_1 = c_2 = 0$ and finally gives Case 4a of Table 1.

In the same way, we consider the second Jordan form from (11). After integrating the corresponding system (10), we obtain $\chi^1 = b_{01}\cos t - b_{02}\sin t$, $\sigma^1 = b_{11}\cos t - b_{12}\sin t - a_{23}$, $\rho^1 = b_{21}\cos t - b_{22}\sin t - a_{24}$, $\chi^2 = b_{01}\sin t + b_{02}\cos t$, $\sigma^2 = b_{11}\sin t + b_{12}\cos t + a_{13}$ and $\rho^2 = b_{21}\sin t + b_{22}\cos t + a_{14}$, where b_{0q} and b_{pq} are real constants. Combining the vector fields Q^1 and Q^2 with each other and with M and I, we can put $b_{01} = 1$, $b_{02} = 0$ and $a_{p3} = a_{p4} = 0$. Substituting the components of P^0, Q^1 and Q^2 that we obtain into the classifying condition (9) gives three independent equations in V,

$$
\begin{aligned}
&V_t = 0,\\
&V_x \cos t = -\frac{1}{2}x\cos t + i(b_{21}\sin t + b_{22}\cos t) - b_{11}\sin t - b_{12}\cos t,\\
&V_x \sin t = -\frac{1}{2}x\sin t - i(b_{21}\cos t - b_{22}\sin t) + b_{11}\cos t - b_{12}\sin t,
\end{aligned}
$$

with the consistency condition $b_{11} = b_{21} = 0$. We denote $b_{12} =: \hat{b}$ and $b_{22} =: b$. Any solution of the above equations for V can be written as $V = -\frac{1}{4}(x + 2\hat{b})^2 + ibx + c_1 + ic_2$ for some real constants c_1 and c_2. By applying the equivalence transformation (3) with $T = t$, $X^0 = 2\hat{b}$, $\Sigma = -c_1 t$, $\Upsilon = c_2 t$, $\varepsilon = 1$ and $\Phi = 0$ and taking a linear combination of the transformed vector field P^0 with M and I, we can put $\hat{b} = c_1 = c_2 = 0$. This yields Case 4b of Table 1.

Finally, the general solution of the system (10) for the last Jordan form of the matrix (a_{pq}) is $\chi^1 = b_{01}$, $\sigma^1 = a_{13}t + b_{11}$, $\rho^1 = a_{14}t + b_{21}$, $\chi^2 = b_{01}t + b_{02}$, $\sigma^2 = \frac{1}{2}a_{13}t^2 + (a_{23} + b_{11})t + b_{12}$, $\rho^2 = \frac{1}{2}a_{14}t^2 + (a_{24} + b_{21})t + b_{22}$, where b_{0q} and b_{pq} are real constants. The constants b_{pq} and b_{02} can be put equal to zero and $b_{01} = 1$ by taking a linear combination of the vector fields Q^1 and Q^2 with each other and with M and I. Then we successively evaluate the classifying condition (9) for the components of the vector fields P^0, Q^1 and Q^2. This gives the following equations for V:

$$
V_t = 0, \quad tV_x = -ia_{14}t + a_{13}t, \quad tV_x = -ia_{14}t + a_{13}t - ia_{24} + a_{23},
$$

which are consistent if and only if $a_{23} = a_{24} = 0$. Any solution of these equations is of the form $V = ibx + \hat{b}x + c_1 + ic_2$, where c_1 and c_2 are real constants and we denote $a_{13} =: \hat{b}$ and $a_{14} =: -b$. Next we apply the equivalence transformation (3) with $T = t$, $X^0 = -\hat{b}t^2$, $\Sigma = \frac{1}{3}\hat{b}^2t^3 - c_1t$, $\Upsilon = \frac{1}{3}b\hat{b}t^3 + c_2t$, $\varepsilon = 1$ and $\Phi = 0$ and take a linear combination of the vector fields P^0 with Q^2, M and I. In this way the constants $\hat{b}$, c_1 and c_2 are set equal to zero, which gives Case 4c of Table 1.

The Lie symmetry extensions presented in Cases 4a–4c of Table 1 are really maximal if $b \neq 0$. Moreover, up to $G^{\sim}$-equivalence we can set $b = 1$ in Case 2a and $b > 0$ in Cases 2b and 2c; cf. the proof of Theorem I.8.

$\boldsymbol{k_1 \geqslant 2,\ k_2 = 0.}$ Lemma I.2 implies that up to $\pi_*G^{\sim}$-equivalence the algebra $\mathfrak{g}^{\rm ess}_V$ contains at least two operators $P^0 = D(1)$ and $P^1 = D(t) + G(\chi^1) + \sigma^1M + \rho^1I$. Here we also annihilate the tail of P^0 by pushforwarding P^0 by a transformation from $\pi_*G^{\sim}$. As $[P^0, P^1] \in \mathfrak{g}^{\rm ess}_V$, we have

$$[P^0, P^1] = D(1) + G(\chi^1_t) + \sigma^1_tM + \rho^1_tI = P^0 + a_1M + b_1I$$

for some constants a_1 and b_1. Collecting components in the above equality gives the system $\chi^1_t = 0$, $\sigma^1_t = a_1$, $\rho^1_t = b_1$ with the general solution $\chi^1 = a_2$, $\sigma^1 = a_1t + a_0$ and $\rho^1 = b_1t + b_0$, where a_2, a_0 and b_0 are real constants of integration. Pushforwarding P^0 and P^1 with $\mathcal{G}_*(2a_2)$, $\mathcal{M}_*(-a_1t)$ and $\mathcal{I}_*(-b_1t)$ and taking a linear combination of P^0 and P^1 with M and I, we find that we can set the constants a_0, a_1, a_2, b_0 and b_1 to zero. Therefore, the basis vector field P^1 reduces to the form $P^1 = D(t)$, whereas the forms of P^0, M and I are preserved.

The classifying condition (9) for $P^0 = D(1)$ and $P^1 = D(t)$ gives two independent equations in V, $V_t = 0$ and $xV_x + 2V = 0$. Integrating these equations gives $V = cx^{-2}$, where c is a complex constant. If $c = 0$, then $k_2 > 0$, which contradicts the case assumption $k_2 = 0$. Thus, the constant c is nonzero. We find the maximal Lie invariance algebra in this case. We substitute $V = cx^{-2}$ with $c \neq 0$ into the classifying condition (9) and derive the system of differential equations for functions parameterizing vector fields from $\mathfrak{g}^{\rm ess}_V$, $\tau_{ttt} = 0$, $\chi = 0$, $\sigma_t = 0$, $\rho_t = -\frac{1}{4}\tau_{tt}$. The solution of the above system implies that the algebra $\mathfrak{g}^{\rm ess}_V$ is spanned by M, I, P^0, P^1 and one more vector field $P^2 = D(t^2) - \frac{1}{2}tI$, which gives Case 5 of Table 1.

$\boldsymbol{k_1 \geqslant 2,\ k_2 = 2.}$ In this case, the algebra $\mathfrak{g}^{\rm ess}_V$ necessarily contains the vector fields M, I, $P^l = D(\bar{\tau}^l) + G(\bar{\chi}^l) + \bar{\sigma}^lM + \bar{\rho}^l$, $l = 0, 1$ and $Q^p = G(\chi^p) + \sigma^pM + \rho^pI$, where all the parameters are real-valued smooth functions of t with $\bar{\tau}^0$ and $\bar{\tau}^1$ (resp. χ^1 and χ^2) being linearly independent. Recall that the indices p and q run from 1 to 2, and we sum over repeated indices. As in the previous case, up to $\pi_*G^{\sim}$-equivalence the vector fields P^0 and P^1 reduce to the form $P^0 = D(1)$ and $P^1 = D(t) + G(\bar{\chi}^1) + \bar{\sigma}^1M + \bar{\rho}^1I$.

Since the algebra $\mathfrak{g}^{\rm ess}_V$ is closed with respect to the Lie bracket of vector fields, we have $[P^l, Q^p] \in \mathfrak{g}^{\rm ess}_V$, i.e.,

$$[P^0, Q^p] = G(\chi^p_t) + \sigma^p_tM + \rho^p_tI = a_{pq}Q^q + a_{p_3}M + a_{p_4}I,$$
$$[P^1, Q^p] = d_{pq}Q^q + d_{p_3}M + d_{p_4}I,$$

where a_{pq}, a_{p_3}, a_{p_4}, d_{pq}, d_{p_3} and d_{p_4} are real constants. Using the above commutation relations with P^0 in the same way as in the case $k_1 = 1$, $k_2 = 2$, we derive three inequivalent cases for the vector fields Q^p depending on the Jordan forms of the matrix (a_{pq}) presented in (11). For the first and second Jordan forms, the commutators $[P^1, Q^p]$ do not belong to the linear span of P^0, P^1, Q^1, Q^2, M and I. Hence these cases are irrelevant.

For the last Jordan form from (11), up to $G^{\sim}$-equivalence and up to linear combining of the above vector fields, we can further assume that

$$Q^1 = G(1) - btI, \quad Q^2 = G(t) - \frac{1}{2}bt^2I, \quad V = ibx$$

for some real constant b. We expand the commutation relation for the vector fields P^1 and Q^1:

$$[P^1, Q^1] = -\frac{1}{2}G(1) - bte^tI - \bar{\chi}^1_t M = d_{11}Q^1 + d_{12}Q^2 + d_{13}M + d_{14}I,$$

and equating components gives $b = 0$, i.e., $V = 0$. Substituting the value $V = 0$ into the classifying condition (9) and splitting with respect to x yields the system of differential equations $\tau_{ttt} = 0$, $\chi_{tt} = 0$, $\sigma_t = 0$, $\rho_t = -\frac{1}{4}\tau_{tt}$. The solution of this system for $V = 0$ shows that the algebra $\mathfrak{g}^{\mathrm{ess}}_V$ is spanned by the vector fields presented in Case 6 of Table 1. □

Remark I.5. It might be convenient to completely describe properties of appropriate subalgebras before their classification but often such an approach is not justified. Thus, Lemma I.2 shows that the invariant k_1 is not greater than three, i.e., $k_1 \in \{0, 1, 2, 3\}$. As we proved in Theorem I.5, this invariant cannot be equal to two. The reason is that any (finite-dimensional) subalgebra $\mathfrak{s}$ of $\mathfrak{g}^{\mathrm{ess}}_{\langle\,\rangle}$ with $\dim \pi^0_*\mathfrak{s} = 2$ is not appropriate since the condition of extension maximality is not satisfied. Therefore, Lemma I.2 could be strengthened by the constraint $k_1 \in \{0, 1, 3\}$. At the same time, the proof of the condition $k_1 \neq 2$ needs realizing the major part of the group classification of the class (1).

7 Alternative proof

Here we present an alternative way of classifying Lie symmetry extensions in the class (1), in which the invariant k_2 is considered as leading. The case $k_2 = 0$, after partitioning into the subcases $k_1 = 0$, $k_1 = 1$ or $k_1 \geqslant 2$, results in the same extensions as presented in Table 1 for these values of k_1 and k_2.

Let us consider the case $k_2 = 2$ more closely. Lemmas I.3 and I.4 imply that, up to $G^{\sim}$-equivalence, the algebra $\mathfrak{g}^{\mathrm{ess}}_V$ contains the vector fields $G(1) + \rho^1 I$ and $G(t) + \rho^2 I$, where ρ^1 is a smooth real-valued function of t and $\rho^2 = \int t\rho^1_t\,\mathrm{d}t$. Integrating the classifying condition (9) for these vector fields with respect to V gives $V = -i\rho_t x + \alpha(t) + i\beta(t)$, and $\alpha = \beta = 0 \bmod G^{\sim}$. Denoting $-\rho_t$ by γ, we obtain $V = i\gamma(t)x$. Thus, we carry out the group classification of the subclass of equations from the class (1) with potentials of this form, i.e.,

$$i\psi_t + \psi_{xx} + i\gamma(t)x\psi = 0, \tag{12}$$

where $\gamma(t)$ is an arbitrary real-valued function of t, which will be taken as the arbitrary element of the subclass instead of V. For potentials of the above form, the equation (3c) splits with respect to x and gives the system of differential equations

$$2T_{ttt}T_t - 3T_{tt}^2 = 0, \quad \left(\frac{X_t^0}{T_t}\right)_t = 0, \quad \Sigma_t = \frac{(X_t^0)^2}{4T_t}, \quad \Upsilon_t = -\frac{T_{tt}}{4T_t} - \varepsilon\frac{\gamma X^0}{|T_t|^{1/2}},$$

whose general solution is

$$T = \frac{a_1t + a_0}{a_3t + a_2}, \quad X^0 = b_1T + b_0,$$
$$\Sigma = \frac{b_1^2}{4}T + c_1, \quad \Upsilon = -\frac{1}{4}\ln|T_t| - \varepsilon\int\frac{\gamma X^0}{|T_t|^{1/2}}\,\mathrm{d}t + c_0,$$

where a_i, $i = 0, \dots, 3$, b_j and c_j, $j = 0, 1$, are real constants with $a_1a_2 - a_0a_3 \neq 0$ and the integral denotes a fixed primitive function for the integrand. Since the constants a_i, $i = 0, \dots, 3$, are defined up to a nonzero constant multiplier (and thus only three of the constants are essential), we set $a_1a_2 - a_0a_3 = \operatorname{sgn} T_t := \varepsilon' = \pm 1$.

To single out the equivalence groupoid $\mathcal{G}^\sim_{(12)}$ of the subclass (12) from the equivalence groupoid $\mathcal{G}^\sim$ of the whole class (1), we substitute the above values of T, X^0, Σ and Υ into (3) and obtain the following statement:

Theorem I.6. *The equivalence groupoid $\mathcal{G}^\sim_{(12)}$ of the subclass* (12) *consists of triples of the form $(\gamma, \tilde\gamma, \varphi)$, where φ is a point transformation in the space of variables, whose components are*

$$\tilde t = T := \frac{a_1t + a_0}{a_3t + a_2}, \quad \tilde x = \varepsilon|T_t|^{1/2}x + b_1T + b_0, \tag{13a}$$

$$\begin{aligned}\tilde\psi = {}&\exp\left(\frac{i}{8}\frac{T_{tt}}{|T_t|}x^2 + \frac{i}{2}\varepsilon b_1|T_t|^{1/2}x - \varepsilon\int\gamma\frac{b_1T + b_0}{|T_t|^{1/2}}\,\mathrm{d}t - i\frac{b_1^2}{4}T\right)\\ &\frac{c}{|T_t|^{1/4}}(\hat\psi + \hat\Phi),\end{aligned} \tag{13b}$$

the transformed parameter $\tilde\gamma$ is given in terms of γ as

$$\tilde\gamma = \frac{\varepsilon\varepsilon'}{|T_t|^{3/2}}\gamma, \tag{13c}$$

a_0, a_1, a_2, a_3, b_0 and b_1 are arbitrary real constants with $a_1a_2 - a_0a_3 =: \varepsilon' = \pm 1$, c is a nonzero complex constant, $\Phi = \Phi(t, x)$ is an arbitrary solution of the initial equation, $\varepsilon = \pm 1$.

Corollary I.6. *The (usual) equivalence group $G^\sim_{(12)}$ of the subclass* (12) *consists of point transformations of the form* (13) *with $b_0 = b_1 = 0$ and $\Phi = 0$.*

Proof. We argue in a similar way to Corollary I.2: since each transformation from $G^\sim_{(12)}$ generates a family of admissible transformations in the subclass (12), it is necessarily of the form (13). There is only one common solution for the equations from the subclass (12): the zero function. Hence the independence of the transformation components for the variables on the arbitrary element γ is equivalent to the conditions $b_0 = b_1 = 0$ and $\Phi = 0$. □

Corollary I.7. *The equivalence algebra of the subclass* (12) *is the algebra*

$$\mathfrak{g}^{\sim}_{(12)} = \langle \hat{D}_{(12)}(1), \hat{D}_{(12)}(t), \hat{D}_{(12)}(t^2), \hat{M}(1), \hat{I}(1)\rangle,$$

where, as in Corollary I.3, $\hat{M}(1) = i(\psi\partial_\psi - \psi^*\partial_{\psi^*})$, $\hat{I}(1) = \psi\partial_\psi + \psi^*\partial_{\psi^*}$, *and*

$$\hat{D}_{(12)}(\tau) = \tau\partial_t + \frac{1}{2}\tau_t x\partial_x + \frac{1}{8}\tau_{tt}x^2\hat{M}(1) - \frac{1}{4}\tau_t\hat{I}(1) - \frac{3}{2}\tau_t\gamma\partial_\gamma.$$

The proof is analogous to that of Corollary I.3.

Corollary I.8. *For each* $\gamma = \gamma(t)$, *the equation* $\mathcal{L}_V$ *with* $V = i\gamma x$ *admits the group* $G^{\rm unf}_V$ *of point symmetry transformations of the form* (13a)–(13b) *with* $T = t$ *and* $\varepsilon = 1$.

Proof. The relation (13c) obviously implies that for each fixed value of the arbitrary element γ, transformations of the form (13a)–(13b) with $T = t$ and $\varepsilon = 1$ leave this value invariant. Other transformations are point symmetries of $\mathcal{L}_V$ with $V = i\gamma x$ only for some values of γ. □

Corollary I.9. *The subclass* (12) *is uniformly semi-normalized with respect to the family of uniform point symmetry groups* $\{G^{\rm unf}_V\}$ *of equations from this subclass and the subgroup* H *of* $G^{\sim}_{(12)}$ *singled out by the constraint* $c = 1$.

Proof. It is obvious that for any V the intersection of π_*H and $G^{\rm unf}_V$ consists of the identity transformation only. Consider an arbitrary admissible transformation $(\gamma,\tilde{\gamma},\varphi)$ in the subclass (12), which maps the equation $\mathcal{L}_V$ with $V = i\gamma(t)x$ to the equation $\mathcal{L}_{\tilde{V}}$ with $\tilde{V} = i\tilde{\gamma}(\tilde{t})\tilde{x}$. Then φ is of the form (13a)–(13b) and thus $G^{\rm unf}_{\tilde{V}} = \varphi G^{\rm unf}_V\varphi^{-1}$. We denote the dependence of φ on the transformation parameters appearing in (13a)–(13b) by writing $\varphi = \varphi(T,\varepsilon,b_1,b_0,c,\Phi)$. It is obvious that $\varphi = \varphi^2\varphi^0\varphi^1$, where $\varphi^1 = \varphi(t,1,0,0,1,\Phi) \in G^{\rm unf}_V$, $\varphi^2 = \varphi(t,1,b_1,b_0,c,0) \in G^{\rm unf}_{\tilde{V}}$, and the transformation $\varphi^0 = \varphi(T,\varepsilon,0,0,1,0)$, prolonged to γ according to (13c), belongs to H. □

Applying Theorem I.4 to equations from the subclass (12), we consider the classifying condition (9) for the associated form of potentials, $V = i\gamma(t)x$, and split this condition with respect to x. As a result, we obtain the following system of differential equations for the parameters of Lie symmetry vector fields:

$$\tau_{ttt} = 0, \quad \chi_{tt} = 0, \quad \sigma_t = 0, \quad \rho_t = -\gamma\chi - \frac{1}{4}\tau_{tt}, \tag{14}$$

as well as the classifying condition

$$\left(\gamma|\tau|^{3/2}\right)_t = 0. \tag{15}$$

It is then clear that the kernel invariance algebra $\mathfrak{g}^{\cap}_{(12)}$ of the subclass (12) is spanned by the vector fields M and I.

Theorem I.7. *The maximal Lie invariance algebra $\mathfrak{g}_V$ of an equation $\mathcal{L}_V$ for $V = i\gamma(t)x$ is spanned by the vector fields $D_{(12)}(\tau)$, $G(1)+\rho^1 I$, $G(t)+\rho^2 I$, M, I, $Z(\eta^0)$, where*

$$D_{(12)}(\tau) := D(\tau) - \frac{1}{4}\tau_t I = \tau\partial_t + \frac{1}{2}\tau_t x\partial_x + \frac{1}{8}\tau_{tt}x^2 M - \frac{1}{4}\tau_t I,$$

the parameter τ runs through the set $\mathfrak{P}_\gamma$ of quadratic polynomials in t that satisfy the classifying condition (15), *$\rho^1 = -\int\gamma(t)\,\mathrm{d}t$, $\rho^2 = -\int t\gamma(t)\,\mathrm{d}t$ and η^0 runs through the solution set of the equation $\mathcal{L}_V$.*

Each equation $\mathcal{L}_V$ with $V = i\gamma(t)x$, belonging to the subclass (12), is invariant with respect to the Lie algebra $\mathfrak{g}_V^{\rm unf} = \langle G(1)+\rho^1 I, G(t)+\rho^2 I, M, I, Z(\eta^0)\rangle$ of the group $G_V^{\rm unf}$, where η^0 again runs through the solution set of the equation $\mathcal{L}_V$. Such algebras have a similar structure for all equations from the subclass. The commutation relations between vector fields from $\mathfrak{g}_{\langle\,\rangle}$ imply that the essential part $\mathfrak{g}_V^{\rm ess}$ of $\mathfrak{g}_V$ admits the representation $\mathfrak{g}_V^{\rm ess} = \mathfrak{g}_V^{\rm ext} \Subset (\mathfrak{g}_V^{\rm unf} \cap \mathfrak{g}_{\langle\,\rangle}^{\rm ess})$, where $\mathfrak{g}_V^{\rm ext} = \{D_{(12)}(\tau) \mid \tau \in \mathfrak{P}_\gamma\}$ is a subalgebra of $\mathfrak{g}_V^{\rm ess}$, and $\mathfrak{g}_V^{\rm unf}$ is an ideal of $\mathfrak{g}_V^{\rm ess} \cap \mathfrak{g}_{\langle\,\rangle}^{\rm ess}$. Interpreting the above representation, we can say that the algebra $\mathfrak{g}_V^{\rm ess}$ is obtained by extending the algebra $\mathfrak{g}_V^{\rm unf} \cap \mathfrak{g}_{\langle\,\rangle}^{\rm ess}$ with elements of $\mathfrak{g}_V^{\rm ext}$.

Consider the linear span

$$\mathfrak{g}_{\langle\,\rangle}^{\rm ext} := \sum_{V=i\gamma(t)x} \mathfrak{g}_V^{\rm ext} = \langle D_{(12)}(1),\, D_{(12)}(t),\, D_{(12)}(t^2)\rangle \subset \pi_*\mathfrak{g}_{(12)}^{\sim},$$

where π is the projection of the joint space of the variables and the arbitrary element on the space of the variables only. The algebra $\mathfrak{g}_{\langle\,\rangle}^{\rm ext}$ is isomorphic to the algebra $\mathrm{sl}(2,\mathbb{R})$. The pushforwards of vector fields from $\mathfrak{g}_{\langle\,\rangle}^{\rm ext}$ by transformations from the group $\pi_*G_{(12)}^{\sim}$ constitute the inner automorphism group $\mathrm{Inn}(\mathfrak{g}_{\langle\,\rangle}^{\rm ext})$ of the algebra $\mathfrak{g}_{\langle\,\rangle}^{\rm ext}$. The action of $G_{(12)}^{\sim}$ on equations from the subclass (12) induces the action of $\mathrm{Inn}(\mathfrak{g}_{\langle\,\rangle}^{\rm ext})$ on the subalgebras of the algebra $\mathfrak{g}_{\langle\,\rangle}^{\rm ext}$. Consequently, the classification of possible Lie symmetry extensions in the subclass (12) reduces to the classification of subalgebras of the algebra $\mathrm{sl}(2,\mathbb{R})$, which is well known.

Theorem I.8. *A complete list of $G_{(12)}^{\sim}$-inequivalent (and, therefore, $\mathcal{G}_{(12)}^{\sim}$-inequivalent) Lie symmetry extensions in the subclass* (12) *is given by Table 2.*

Proof. An optimal set of subalgebras of the algebra $\mathfrak{g}_{\langle\,\rangle}^{\rm ext}$ is given by

$$\{0\},\ \langle D(1)\rangle,\ \langle D(t)\rangle,\ \langle D(t^2{+}1){-}\tfrac{1}{2}tI\rangle,\ \langle D(1), D(t)\rangle,\ \langle D(1), D(t), D(t^2{+}1){-}\tfrac{1}{2}tI\rangle.$$

The zero subalgebra gives the general case with no extension of $\mathfrak{g}_V^{\rm unf}$, which is Case 1 of Table 2.

For the one-dimensional subalgebras, we substitute the corresponding values of τ, $\tau = 1$, $\tau = t$ and $\tau = t^2 + 1$ into the classifying condition (15), integrate the resulting equations with respect to γ and obtain Cases 2a–2c of Table 2, respectively. Using equivalence transformations that do not change the form of γ, we can set $b = 1$ in Case 2a and $b > 0$ in Cases 2b and 2c.

Table 2. Results of the group classification of the subclass (12).

no.	k_1	V	Basis of $\mathfrak{g}_V^{\mathrm{ess}}$
1	0	$i\gamma(t)x$	$M,\ I,\ G(1)-\big(\int\gamma(t)\,\mathrm{d}t\big)\,I,\ G(t)-\big(\int t\gamma(t)\,\mathrm{d}t\big)\,I$
2a	1	ibx	$M,\ I,\ G(1)-btI,\ G(t)-\frac12 bt^2I,\ D(1)$
2b	1	$ib\|t\|^{-3/2}x$	$M,\ I,\ G(1)+2bt\|t\|^{-3/2}I,\ G(t)-2b\|t\|^{1/2}I,\ D(t)$
2c	1	$ib(t^2+1)^{-3/2}x$	$M,\ I,\ G(1)-bt(t^2+1)^{-1/2}I,\ G(t)+b(t^2+1)^{-1/2}I,$
			$D(t^2+1)-\frac12 tI$
3	3	0	$M,\ I,\ G(1),\ G(t),\ D(1),\ D(t),\ D(t^2)-\frac12 tI$

Lie symmetry extension given in Case 1 of Table 2 is maximal if and only if the arbitrary element γ is of the form $\gamma\neq c_3|c_2t^2+c_1t+c_0|^{-3/2}$ for any real constants c_0, c_1, c_2 and c_3 with c_0, c_1 and c_2 not vanishing simultaneously. The real constant b in Cases 2a–2c is nonzero. Moreover, $b=1 \bmod G^\sim$ in Case 2a and $b>0 \bmod G^\sim$ in Cases 2b and 2c.

Similarly, the classifying condition (15) for the two-dimensional subalgebra gives an overdetermined system of two equations with $\tau=1$ and $\tau=t$, for which the only solution is $\gamma=0$. The maximal extension of $\mathfrak{g}_V^{\mathrm{unf}}$ for $\gamma=0$ is three-dimensional and is given by the last subalgebra of the list. This gives Case 3 of Table 1. □

All cases presented in Table 2 are related to those of Table 1. In the symbol T.N, used in the following, T denotes the table number and N is the case number (in Table T). Thus, Cases 2.1, 2.2a and 2.3 coincide with Cases 1.2, 1.4c and 1.6, respectively. Some cases are connected via equivalence transformations, which are of the form (3),

$$2.2b\ \to 1.4a:\quad T=\frac{\operatorname{sgn}t}{4}\ln|t|,\quad X^0=\Sigma=\Upsilon=0,\quad \Phi=0;$$
$$2.2c\ \to 1.4b:\quad T=\arctan t,\quad X^0=\Sigma=\Upsilon=0,\quad \Phi=0.$$

Thus, the result of group classification of the class (1) can be reformulated with involving Table 2.

Corollary I.10. *A complete list of inequivalent Lie symmetry extensions in the class* (1) *is exhausted by Cases 1, 3 and 5 of Table 1 and the cases collected in Table 2.*

8 Subclass with real-valued potentials

We derive results on group analysis of the subclass $\mathrm{Sch}_\mathbb{R}$ of equations of the form (1) with real-valued potentials using those for the whole class (1). The condition that potentials are real valued leads to additional constraints for transformations and infinitesimal generators.

Theorem I.9. *The equivalence groupoid* $\mathcal{G}^{\sim}_{\mathbb{R}}$ *of the subclass* $\mathrm{Sch}_{\mathbb{R}}$ *consists of triples of the form* $(V, \tilde{V}, \varphi)$*, where* φ *is a point transformation in the space of variables, whose components are*

$$\tilde{t} = T, \quad \tilde{x} = \varepsilon |T_t|^{1/2} x + X^0, \tag{16a}$$

$$\tilde{\psi} = \frac{a}{|T_t|^{1/4}} \exp\left(\frac{i}{8}\frac{T_{tt}}{|T_t|}x^2 + \frac{i}{2}\frac{\varepsilon\varepsilon' X^0_t}{|T_t|^{1/2}}x + i\Sigma\right)(\hat{\psi} + \hat{\Phi}), \tag{16b}$$

the transformed potential $\tilde{V}$ *is expressed in terms of* V *as*

$$\tilde{V} = \frac{V}{|T_t|} + \frac{2T_{ttt}T_t - 3T_{tt}^2}{16\varepsilon' T_t^3}x^2 + \frac{\varepsilon\varepsilon'}{2|T_t|^{1/2}}\left(\frac{X^0_t}{T_t}\right)_t x - \frac{(X^0_t)^2}{4T_t^2} + \frac{\Sigma_t}{T_t}, \tag{16c}$$

$T = T(t)$*,* $X^0 = X^0(t)$ *and* $\Sigma = \Sigma(t)$ *are arbitrary smooth real-valued functions of* t *with* $T_t \neq 0$ *and* $\Phi = \Phi(t, x)$ *is an arbitrary solution of the initial equation.* a *is a nonzero real constant,* $\varepsilon = \pm 1$ *and* $\varepsilon' = \operatorname{sgn} T_t$*.*

Corollary I.11. *The subclass* $\mathrm{Sch}_{\mathbb{R}}$ *is uniformly semi-normalized with respect to linear superposition of solutions. Its equivalence group* $G^{\sim}_{\mathbb{R}}$ *consists of point transformations of the form* (16) *with* $\Phi = 0$*.*

Corollary I.12. *The equivalence algebra of the subclass* $\mathrm{Sch}_{\mathbb{R}}$ *is the algebra*

$$\mathfrak{g}^{\sim}_{\mathbb{R}} = \langle \hat{D}_{\mathbb{R}}(\tau), \hat{G}_{\mathbb{R}}(\chi), \hat{M}_{\mathbb{R}}(\sigma), \hat{I}_{\mathbb{R}} \rangle$$

where τ*,* χ *and* σ *run through the set of smooth real-valued functions of* t*. The vector fields* $\hat{D}_{\mathbb{R}}(\tau)$*,* $\hat{G}_{\mathbb{R}}(\chi)$*,* $\hat{M}_{\mathbb{R}}(\sigma)$ *and* $\hat{I}_{\mathbb{R}}$ *are given by*

$$\begin{aligned}
\hat{D}_{\mathbb{R}}(\tau) &= \tau\partial_t + \frac{1}{2}\tau_t x\partial_x + \frac{i}{8}\tau_{tt}x^2(\psi\partial_\psi - \psi^*\partial_{\psi^*}) \\
&\qquad - \frac{1}{4}\tau_t\hat{I}_{\mathbb{R}} - \left(\tau_t V - \frac{1}{8}\tau_{ttt}x^2\right)\partial_V, \\
\hat{G}_{\mathbb{R}}(\chi) &= \chi\partial_x + \frac{i}{2}\chi_t x(\psi\partial_\psi - \psi^*\partial_{\psi^*}) + \frac{\chi_{tt}}{2}x\partial_V, \\
\hat{M}_{\mathbb{R}}(\sigma) &= i\sigma(\psi\partial_\psi - \psi^*\partial_{\psi^*}) + \sigma_t\partial_V, \quad \hat{I}_{\mathbb{R}} = \psi\partial_\psi + \psi^*\partial_{\psi^*}.
\end{aligned}$$

Corollary I.13. *A (1+1)-dimensional linear Schrödinger equation of the form* (1) *with a real-valued potential* V *is equivalent to the free linear Schrödinger equation with respect to a point transformation if and only if the potential is a quadratic polynomial in* x*, i.e.,* $V = \gamma^2(t)x^2 + \gamma^1(t)x + \gamma^0(t)$ *for some smooth real-valued functions* γ^0*,* γ^1 *and* γ^2 *of* t*.*

A study of the determining equations for Lie symmetries of equations from the subclass $\mathrm{Sch}_{\mathbb{R}}$ shows that the classifying condition in this case is of the form (9) with $\rho_t = -\frac{1}{4}\tau_{tt}$,

$$\tau V_t + \left(\frac{1}{2}\tau_t x + \chi\right)V_x + \tau_t V = \frac{1}{8}\tau_{ttt}x^2 + \frac{1}{2}\chi_{tt}x + \sigma_t. \tag{17}$$

The kernel invariance algebra $\mathfrak{g}^{\cap}_{\mathbb{R}}$ of the subclass $\mathrm{Sch}_{\mathbb{R}}$ coincides with the kernel invariance algebra $\mathfrak{g}^{\cap}$ of the whole class (1), cf. Proposition 1.

Theorem I.10. *The maximal Lie invariance algebra $\mathfrak{g}_V$ of an equation $\mathcal{L}_V$ from the subclass $\mathrm{Sch}_{\mathbb{R}}$ is spanned by the vector fields $D_{\mathbb{R}}(\tau)$, $G(\chi)$, I, σM and $Z(\eta^0)$, where*

$$
\begin{aligned}
&D_{\mathbb{R}}(\tau) := D(\tau) - \frac{1}{4}\tau_t I = \tau\partial_t + \frac{1}{2}\tau_t x\partial_x + \frac{1}{8}\tau_{tt}x^2 M - \frac{1}{4}\tau_t I,\\
&G(\chi) = \chi\partial_x + \frac{1}{2}\chi_t x M, \quad M = i\psi\partial_\psi - i\psi^*\partial_{\psi^*}, \quad I = \psi\partial_\psi + \psi^*\partial_{\psi^*},\\
&Z(\eta^0) = \eta^0\partial_\psi + \eta^{0*}\partial_{\psi^*},
\end{aligned}
$$

the parameters τ, χ and σ run through the set of real-valued smooth functions of t satisfying the classifying condition (17)*, and η^0 runs through the solution set of the equation $\mathcal{L}_V$.*

It is obvious that properties of appropriate subalgebras for the subclass $\mathrm{Sch}_{\mathbb{R}}$ can be obtained by specifying the same properties of appropriate subalgebras for the whole class (1). Thus, inequivalent cases of real-valued potentials admitting Lie symmetry extensions can be singled out from the classification list presented in Table 1. We note, however, that the group classification of real-valued potentials can be easily carried out from the outset.

Theorem I.11. *A complete list of inequivalent Lie symmetry extensions in the subclass* $\mathrm{Sch}_{\mathbb{R}}$ *is given in Table 3.*

Table 3. The classification list for real-valued potentials.

no.	k_1	k_2	V	Basis of $\mathfrak{g}^{\mathrm{ess}}_V$
1	0	0	$V(t,x)$	$M,\ I$
2	1	0	$V(x)$	$M,\ I,\ D(1)$
3	3	0	cx^{-2}	$M,\ I, D(1),\ D(t),\ D(t^2)-\frac12 tI$
4	3	2	0	$M,\ I,\ D(1),\ D(t),\ D(t^2)-\frac12 tI,\ G(1),\ G(t)$

Lie symmetry extensions given in Table 3 are maximal if and only if the potential V does not satisfy an equation of the form (17) in Case 1 and $V\neq b_2x^2+b_1x+b_0+c(x+a)^{-2}$ for any real constants a, b_0, b_1, b_2 and c in Case 2. The real constant c in Case 3 is nonzero.

Proof. The proof follows the same pattern as Theorem I.5, and we sketch the proof by considering the invariants k_1 and k_2. The case $k_2=0$ is split into the three subcases $k_1=0$, $k_1=1$ and $k_1\geqslant 2$. The proof for each subcase is the same as for Theorem I.5 except that the parameter ρ in each Lie symmetry vector field satisfies the equation $\rho_t=-\frac14\tau_{tt}$. If $k_2=2$, then the algebra $\mathfrak{g}^{\mathrm{ess}}_V$ contains a vector field $Q^1 = G(\chi^1) + \sigma^1 M + \rho^1 I$, where the parameters χ^1 and σ^1 are real-valued smooth functions of t with $\chi^1 \neq 0$ and ρ^1 is a real constant. Combining Q^1 with I and using $G^{\sim}$-equivalence, we may assume that $Q^1 = G(1)$. The equation $\mathcal{L}_V$ is invariant with respect to $G(1)$ if and only if the potential V does not depend on x. Then the equation $\mathcal{L}_V$ is equivalent to the free linear Schrödinger equation. □

9 Conclusion

In this paper we have completely solved the group classification problem for (1+1)-dimensional linear Schrödinger equations with complex-valued potentials. The classification list is presented in Theorem I.5 or, equivalently, in Corollary I.10. This also gives the group classifications for the larger class of similar equations with variable mass and for the smaller class of such equations with real-valued potentials. We have introduced the notion of uniformly semi-normalized classes of differential equations and developed a special version of the algebraic method of group classification for such classes. This is, in fact, the main result of the paper. The class (1) has the specific property of uniform semi-normalization with respect to linear superposition transformations, which is quite common for classes of homogeneous linear differential equations. Within the framework of the algebraic method, the group classification problem of the class (1) reduces to the classification of appropriate low-dimensional subalgebras of the associated equivalence algebra $\mathfrak{g}^{\sim}$.

We show that the linear span $\mathfrak{g}_{\langle\,\rangle}$ of the vector fields from the maximal Lie invariance algebras of equations from the class (1) is itself a Lie algebra. For each potential V, the maximal Lie invariance algebra $\mathfrak{g}_V$ of the equation $\mathcal{L}_V$ from the class (1) is the semi-direct sum of a subalgebra $\mathfrak{g}_V^{\rm ess}$, of dimension not greater than seven, and an infinite dimensional abelian ideal $\mathfrak{g}_V^{\rm lin}$, which is the trivial part of $\mathfrak{g}_V$ and is associated with the linear superposition principle, $\mathfrak{g}_V = \mathfrak{g}_V^{\rm ess} \Subset \mathfrak{g}_V^{\rm lin}$. The above representation of $\mathfrak{g}_V$'s yields a similar representation for $\mathfrak{g}_{\langle\,\rangle} = \sum_V \mathfrak{g}_V$, $\mathfrak{g}_{\langle\,\rangle} = \mathfrak{g}_{\langle\,\rangle}^{\rm ess} \Subset \mathfrak{g}_{\langle\,\rangle}^{\rm lin}$, where $\mathfrak{g}_{\langle\,\rangle}^{\rm ess} = \sum_V \mathfrak{g}_V^{\rm ess}$ is a (finite-dimensional) subalgebra of $\mathfrak{g}_{\langle\,\rangle}$, and $\mathfrak{g}_{\langle\,\rangle}^{\rm lin} = \sum_V \mathfrak{g}_V^{\rm lin}$ is its abelian ideal. The projection of the equivalence algebra $\mathfrak{g}^{\sim}$ of the class (1) on the space of variables coincides with $\mathfrak{g}_{\langle\,\rangle}^{\rm ess}$. Thus, two objects, $\mathfrak{g}_{\langle\,\rangle}^{\rm ess}$ and $\mathfrak{g}^{\sim}$, are directly related to the class (1) and consistent with each other. This is why we classify appropriate subalgebras of $\mathfrak{g}_{\langle\,\rangle}^{\rm ess}$ up to $G^{\sim}$-equivalence, each of which coincides with $\mathfrak{g}_V^{\rm ess}$ for some V.

The partition into classification cases is provided by two nonnegative integers k_1 and k_2, which are characteristic invariants of subalgebras of $\mathfrak{g}_{\langle\,\rangle}^{\rm ess}$. This leads to two equivalent classification lists for the potential V depending on which of these invariants is assumed as the leading invariant. The list presented in Table 1 (resp. described in Corollary I.10) is constructed under the assumption that the invariant k_1 (resp. k_2) is leading. Each of the lists consists of eight $G^{\sim}$-inequivalent families of potentials. We have proved that for appropriate subalgebras the invariant k_2 can take only two values: 0 and 2, and the invariant k_1 is not greater than three. Further, the invariant k_1 cannot equal two for appropriate subalgebras due to the fact that the corresponding subalgebras cannot be maximal Lie symmetry algebras for equations from the class (1). At the same time, the proof of the condition $k_1 \neq 2$ needs realizing the major part of the group classification of the class under study.

The cases in the second list for which $k_2 = 0$ coincide with those from the first list. For $k_2 = 2$, the group classification of the class (1) reduces to the group classification of its subclass (12). This subclass is uniformly semi-normalized with respect to a larger family of point symmetry groups than the corresponding groups of linear superposition transformations, which makes the subclass (12) a useful example for group analysis of differential equations. For each equation $\mathcal{L}_V$ from the subclass, the essential part $\mathfrak{g}_V^{\rm ess}$ of its maximal Lie invariance algebra $\mathfrak{g}_V$ can be written as $\mathfrak{g}_V^{\rm ess} = \mathfrak{g}_V^{\rm ext} \Subset (\mathfrak{g}_V^{\rm unf} \cap \mathfrak{g}_{\langle\,\rangle}^{\rm ess})$, where $\mathfrak{g}_V^{\rm unf}$ is an ideal of $\mathfrak{g}_V$ and has a similar structure for all equations from the subclass, and $\mathfrak{g}_V^{\rm ext}$ is a subalgebra of $\mathfrak{g}_V^{\rm ess}$. The vector fields from all $\mathfrak{g}_V^{\rm ext}$'s of equations from the subclass (12) constitute the algebra $\mathfrak{g}_{\langle\,\rangle}^{\rm ext}$, which is contained in the projection of the equivalence algebra of the subclass (12) and is isomorphic to the algebra $\mathrm{sl}(2,\mathbb{R})$. Therefore, the classification of subalgebras of $\mathrm{sl}(2,\mathbb{R})$ (which is well known) yields the solution of the group classification problem of the subclass (12), whose result is presented in Table 2.

Since the subclass $\mathrm{Sch}_\mathbb{R}$ of (1+1)-dimensional linear Schrödinger equations with real-valued potentials is important for applications, we have given its group classification separately by singling out related results from the group classification of the class (1). Since the subclass $\mathrm{Sch}_\mathbb{R}$ is also uniformly semi-normalized with respect to linear superposition of solutions, this procedure can be realized within the framework of the algebraic approach by specifying the properties of appropriate subalgebras for the case of real-valued potentials.

Furthermore, the semi-normalization of the above classes of linear Schrödinger equations guarantees that there are no additional point equivalence transformations between classification cases listed for each of these classes.

The new version of the algebraic method that is given in Section 3 and then applied to the symmetry analysis of the class (1) can be regarded as a model for optimizing the group classification of other classes of differential equations (including higher-dimensional cases). We intend to extend our approach to multidimensional linear Schrödinger equations with complex-valued potentials. In this context, it seems that the technique used in the proof of Theorem I.5 is more useful for generalizing to the multidimensional case than the alternative proof presented in Section 7.

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Paper II

Group classification of multidimensional linear Schrödinger equations with the algebraic method

Group classification of multidimensional linear Schrödinger equations with the algebraic method

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We investigate the group classification problem for $(1+n)$-dimensional linear Schrödinger equations with complex-valued potentials using the algebraic approach for arbitrary $n \geqslant 1$. We compute the equivalence groupoid of the class by the direct method in finite form and we show that this class of equations has the property of being uniformly semi-normalized with respect to linear superposition of solutions. We also find it convenient to consider a slight weakening of this property and we introduce the notion of weakly uniformly semi-normalized classes of differential equations. These properties allow us to reduce the group classification for this type of equation to the classification of certain low-dimensional subalgebras of the associated equivalence algebra. For $n = 2$ we give a complete group classification of these equations.

1 Introduction

The theory of group classification plays a central role in the symmetry analysis of differential equations and its application to physics. This theory originated from the work of Sophus Lie on the study of infinitesimal point symmetries of ordinary and partial differential equations [20], [21]. The study of Schrödinger equations within this framework dates back to the beginning of the 1970s. Lie symmetries of linear Schrödinger equations with real-valued potentials were studied in [4], [23], [25], [26], [27], [28]. Nonlinear Schrödinger equations have also been studied from the point of view of symmetry classification in [5], [9], [10], [11], [12], [13], [14], [33], [35], [41]. However, the Lie symmetries of linear Schrödinger equations with complex potentials have not been studied although such equations have been of interest in quantum mechanics, condensed matter physics and quantum field theory [2], [8], [22], [24].

In this paper we consider the class $\mathcal{F}$ of (1+n)-dimensional ($n \geqslant 1$) linear Schrödinger equations with real- and complex-valued potentials consisting of equations of the form

$$i\psi_t + \psi_{aa} + V(t,x)\psi = 0, \tag{1}$$

where t and $x = (x_1, \dots, x_n)$ are real independent variables, ψ is a complex-valued function and $V(t,x)$ is an arbitrary smooth complex-valued potential. Here and in the following, expressions such as V_t and V_a denote differentiation with respect to t and x_a, respectively and the indices a, b, c, and d run from 1 to n. We use the summation convention for repeated indices.

The group classification of the class $\mathcal{F}$ with $n = 1$ was carried out in [18] and a complete list of inequivalent Lie symmetry algebras together with their corresponding families of potentials was obtained. The group classification of the subclass of (1+1)-dimensional linear Schrödinger equations with real-valued potentials V was also obtained using the solution of the group classification problem for the class $\mathcal{F}$. The case of the class $\mathcal{F}$ in dimension 1+2 was dealt with in [19]. The treatment in this case becomes somewhat more involved since rotations in the space-variables enter the picture.

The present paper generalizes results obtained in [18], [19] to $1+n$-dimensions for $n \geqslant 1$. We find the equivalence groupoid $\mathcal{G}^\sim$, the equivalence group $G^\sim$ and the equivalence algebra $\mathfrak{g}^\sim$ for the class $\mathcal{F}$. We analyze the system of determining equations for Lie symmetries of equations from the class $\mathcal{F}$ and study properties of subalgebras of $\mathfrak{g}^\sim$ whose projections to the space of independent and dependent variables are essential Lie invariance algebras of equations from the class $\mathcal{F}$. In particular, we prove that the dimensions of these invariance algebras is less than or equal to $n(n+3)/2+5$. For $n = 2$ we solve completely the group classification problem for the class $\mathcal{F}$. This involves developing the theory of uniformly semi-normalized classes of differential equations that was presented in [18].

The structure of this paper is the following: Basic notions and some results related to group classification of differential equations are briefly reviewed in Section 2. In Section 3 we define what is meant by a uniformly semi-normalized class and uniform semi-normalization with respect to a proper subgroup of an equivalence group. In Section 4 we introduce the new notion of a *weakly uniformly semi-normalized class*. We then study the properties of such classes and compare them with those of uniformly semi-normalized classes. In Section 5 we compute the equivalence groupoid $\mathcal{G}^\sim$ of the class $\mathcal{F}$ for an arbitrary value $n \geqslant 1$. The equivalence group $G^\sim$ and equivalence algebra $\mathfrak{g}^\sim$ of the class $\mathcal{F}$ are obtained in Section 6. It is shown that the class $\mathcal{F}$ is uniformly semi-normalized with respect to linear superposition of solutions. In Section 7, we derive and partially solve the determining equations for the Lie symmetries of equations from the class $\mathcal{F}$, thus obtaining the general form of Lie symmetry generators, the kernel invariance algebra $\mathfrak{g}^\cap$ as well as the classifying condition for Lie symmetries. Section 8 is devoted to properties of subalgebras of $\mathfrak{g}^\sim$ that are appropriate as maximal Lie invariance algebras of equations from class $\mathcal{F}$. The solution of the group classification problem of the class $\mathcal{F}$ in the case $n = 2$ is completed in Section 9.

2 Basics notions of group classification

In order to make the presentation self-contained, we briefly define the notions of a class (of systems) of differential equations, point transformations for such classes as well as equivalence groups and symmetry groups for these classes. More details can be found in [3], [34], [35]. In this and the following two sections we use a notation for independent and dependent variables that differs from that of the other sections: $x = (x_1, \ldots, x_n)$ denotes the complete n-tuple of independent variables and $u = (u^1, \ldots, u^m)$ is the m-tuple of dependent variables (that is, functions of the independent variables).

Consider a system of differential equations $\mathcal{L}_\theta$: $L(x, u_{(p)}, \theta_{(q)}(x, u_{(p)})) = 0$, which is parameterized by a tuple of arbitrary elements

$$\theta(x, u_{(p)}) = (\theta^1(x, u_{(p)}), \ldots, \theta^k(x, u_{(p)})),$$

where $u_{(p)}$ stands for the set of the dependent variables u together with all derivatives of u with respect to x up to and including order p. Here $\theta_{(q)}$ denotes the tuple of derivatives of θ with respect to x and $u_{(p)}$ up to order q. The arbitrary elements $\theta(x, u_{(p)})$ run through the set $\mathcal{S}$ of solutions of an auxiliary system of differential equations $S(x, u_{(p)}, \theta_{(q')}(x, u_{(p)})) = 0$ and differential inequalities of the form $\Sigma(x, u_{(p)}, \theta_{(q')}(x, u_{(p)})) \neq 0$, where both x and $u_{(p)}$ play the role of independent variables, and S and Σ are tuples of smooth functions depending on $x, u_{(p)}$ and $\theta_{(q')}$. Other kinds of inequalities are also possible. We denote the class of systems $\mathcal{L}_\theta$ with arbitrary elements θ running through $\mathcal{S}$ by $\mathcal{L}|_\mathcal{S}$, i.e., $\mathcal{L}|_\mathcal{S} := \{\mathcal{L}_\theta \mid \theta \in \mathcal{S}\}$.

Let $\mathcal{L}_\theta$ and $\mathcal{L}_{\tilde\theta}$ be systems from the class $\mathcal{L}|_\mathcal{S}$. The set of point transformations of the space with local coordinates (x, u) that map the system $\mathcal{L}_\theta$ to the system $\mathcal{L}_{\tilde\theta}$ is denoted by $\mathrm{T}(\theta, \tilde\theta)$. The set of admissible transformations of the class $\mathcal{L}|_\mathcal{S}$ is defined by

$$\mathcal{G}^\sim := \{(\theta, \tilde\theta, \varphi) \mid \theta, \tilde\theta \in \mathcal{S}, \varphi \in \mathrm{T}(\theta, \tilde\theta)\}$$

and it has the structure of the groupoid with respect to composition of maps: the composition $\phi \circ \psi$ of two maps $\phi \in \mathrm{T}(\theta, \tilde\theta)$, $\psi \in \mathrm{T}(\theta', \tilde\theta')$ is possible only if $\tilde\theta = \theta'$. We call $\mathcal{G}^\sim$ the *equivalence groupoid* of the class $\mathcal{L}|_\mathcal{S}$ (see [3]).

A *(usual) equivalence transformation* $\mathcal{T}$ for the class $\mathcal{L}|_\mathcal{S}$ is a point transformation of the space with local coordinates $(x, u_{(p)}, \theta)$ whose action is projectable to the space with local coordinates $(x, u_{(p')})$ for any p' with $0 \leqslant p' \leqslant p$, so that its restriction $\mathcal{T}|_{(x, u_{(p')})}$ acting on the jet space $J^{p'}(\mathbb{R}^n, \mathbb{R}^m)$ with local coordinates $(x, u_{(p')})$ is the p'th order prolongation of $\mathcal{T}|_{(x,u)}$ (to the jet space $J^{p'}(\mathbb{R}^n, \mathbb{R}^m)$ with local coordinates $(x, u_{(p')})$) and such that it maps every system from the class $\mathcal{L}|_\mathcal{S}$ to another system belonging to the same class. The collection of all such transformations is called the *(usual) equivalence group* $G^\sim$ of the class $\mathcal{L}|_\mathcal{S}$. The *equivalence algebra* $\mathfrak{g}^\sim$ of the class $\mathcal{L}|_\mathcal{S}$ is defined to be the Lie algebra consisting of the generators of one-parameter subgroups of the group $G^\sim$.

The *maximal point symmetry group* G_θ of the system $\mathcal{L}_\theta$, $\theta \in \mathcal{S}$, is defined to be the (pseudo)group of transformations G_θ acting on the space of independent and dependent variables and mapping the solution set of $\mathcal{L}_\theta$ onto itself, that is $G_\theta =$

$\mathrm{T}(\theta,\theta)$. The *kernel group* $G^\cap$ of the class $\mathcal{L}|_\mathcal{S}$ is then defined as the intersection $G^\cap := \bigcap_{\theta\in\mathcal{S}} G_\theta$ (that is, $G^\cap$ is the set of all symmetries admitted by all the individual systems $\mathcal{L}_\theta$ of the class $\mathcal{L}|_\mathcal{S}$). The algebra $\mathfrak{g}_\theta$ consisting of all the generators of one-parameter subgroups of the maximal point symmetry group G_θ is called the *maximal Lie invariance algebra* of the system $\mathcal{L}_\theta$ and the algebra $\mathfrak{g}^\cap$ consisting of all the generators of one-parameter subgroups of the kernel group $G^\cap$ is called the *kernel invariance algebra* of the class $\mathcal{L}|_\mathcal{S}$.

With these definitions we can formulate the group classification problem of the class $\mathcal{L}|_\mathcal{S}$ as follows: *Given the class of differential equations $\mathcal{L}|_\mathcal{S}$, find all $G^\sim$-inequivalent values of $\theta\in\mathcal{S}$ for which the corresponding maximal Lie invariance algebras, $\mathfrak{g}_\theta$, are larger than the kernel invariance algebra $\mathfrak{g}^\cap$.*

The study of group classification problems is based on the infinitesimal invariance criterion [30], [32]. For a system $\mathcal{L}_\theta\in\mathcal{L}|_\mathcal{S}$, a vector field $Q=\xi^i(x,u)\partial_{x_i}+\eta^a(x,u)\partial_{u^a}$ belongs to the maximal Lie invariance algebra $\mathfrak{g}_\theta$ of $\mathcal{L}_\theta$ if and only if the condition

$$Q_{(p)}L(x,u_{(p)},\theta_{(q)}(x,u_{(p)}))=0$$

holds on the manifold defined by the system $\mathcal{L}^p_\theta$, which is defined as being $\mathcal{L}_\theta$ together with all its differential consequences of order up to and including p, subject to the condition that $\mathcal{L}^p_\theta$ be locally solvable and of maximal rank (for these two notions, see [30]). $\mathcal{L}^p_\theta$ is a submanifold of the jet space $J^p(\mathbb{R}^n,\mathbb{R}^m)$. The pth prolongation of the vector field Q is the vector field

$$Q_{(p)}=Q+\sum_{0<|\alpha|\leqslant p}\Big(\mathrm{D}_1^{\alpha_1}\cdots\mathrm{D}_n^{\alpha_n}\left(\eta^a-\xi^i u^a_i\right)+\xi^i u^a_{\alpha+\delta_i}\Big)\partial_{u^a_\alpha}$$

in the jet space $J^{(p)}$, where the tuple $\alpha=(\alpha_1,\dots,\alpha_n)$ is a multi-index, $|\alpha|:=\alpha_1+\dots+\alpha_n$, $\alpha_i\in\mathbb{N}\cup 0$, and δ_i is the multi-index whose i-th entry equals 1 and whose other entries are zero. The indices i and a run from 1 to n and from 1 to m, respectively, and we assume summation with respect to repeated indices. The variable u^a_α of the jet space $J^{(p)}$ is identified with the derivative $\partial^{|\alpha|}u^\alpha/\partial x_1^{\alpha_1}\dots\partial x_n^{\alpha_n}$. $\mathrm{D}_i=\partial_i+u^a_{\alpha+\delta_i}\partial_{u^a_\alpha}$ is the total derivative operator with respect to the variable x_i.

After splitting with respect to different powers of the derivatives of u, the infinitesimal invariance criterion gives a linear (usually overdetermined) system of determining equations for the components of Lie symmetry vector fields of a system $\mathcal{L}_\theta$ from the class $\mathcal{L}|_\mathcal{S}$. The solution of this system depends of course on the structure of the class $\mathcal{L}|_\mathcal{S}$. Those determining equations that do not involve the arbitrary elements can (usually) be integrated immediately, and their general solution gives a preliminary form of the components of the Lie symmetry generators. The remained equations, containing arbitrary elements, constitute the *classifying condition* for the Lie symmetries of the systems from the class $\mathcal{L}|_\mathcal{S}$. Splitting these equations with respect to the derivatives of the arbitrary elements and solving the system that arises in this way, we obtain the kernel invariance algebra $\mathfrak{g}^\cap$. In order to classify the Lie symmetry extensions in the class $\mathcal{L}|_\mathcal{S}$, one analyzes the classifying condition up to $G^\sim$-equivalence.

One approach to analyzing the classifying condition is to use the algebraic method of group classification, but this depends on whether or not the class is *normalized*. We say that a class of differential equations $\mathcal{L}|_\mathcal{S}$ is *normalized* if its equivalence groupoid $\mathcal{G}^\sim$ is induced by elements of its equivalence group $G^\sim$,

$$\mathcal{G}^\sim = \{(\theta, \mathcal{T}\theta, \mathcal{T}|_{(x,u)}) \mid \theta \in \mathcal{S}, \mathcal{T} \in G^\sim\}.$$

The class $\mathcal{L}|_\mathcal{S}$ is said to be *semi-normalized* if each element of its equivalence groupoid is generated by an equivalence transformation and point symmetry transformations of the corresponding initial or transformed system. It is obvious that any normalized class of differential equations is semi-normalized. If the class $\mathcal{L}|_\mathcal{S}$ is normalized, its group classification reduces to classifying subalgebras of the equivalence algebra $\mathfrak{g}^\sim$ (and of the projection of this algebra to the space with local coordinates (x,u)) of the class, up to $G^\sim$-equivalence. If the class is not normalized, one may try to partition it into normalized subclasses. More details on the algebraic method of group classification can be found in [3], [35].

3 Uniformly semi-normalized classes

In [18, Section 3], we gave a definition of a *uniformly semi-normalized* class of differential equations $\mathcal{L}|_\mathcal{S}$ which was sufficient for most purposes in our previous investigation. Here we redefine the notion of a uniformly semi-normalized class in order to extend it to more general situations than:

Definition II.1. Given a class of differential equations $\mathcal{L}|_\mathcal{S}$ with equivalence groupoid $\mathcal{G}^\sim$ and (usual) equivalence group $G^\sim$, suppose that there exists a subgroup H of $G^\sim$, and for each $\theta \in \mathcal{S}$ the point symmetry group G_θ of the system $\mathcal{L}_\theta \in \mathcal{L}|_\mathcal{S}$ contains a subgroup N_θ such that the family $\mathcal{N}_\mathcal{S} = \{N_\theta \mid \theta \in \mathcal{S}\}$ of all these subgroups satisfies the following properties:

1. For any $(\theta', \theta'', \varphi) \in \mathcal{G}^\sim$ there exist $\varphi' \in N_{\theta'}$, $\varphi'' \in N_{\theta''}$ and $\mathcal{T} \in H$ such that $\theta'' = \mathcal{T}\theta'$ and $\varphi = \varphi''(\mathcal{T}|_{(x,u)})\varphi'$.

2. $N_{\mathcal{T}\theta} = \mathcal{T}|_{(x,u)} N_\theta (\mathcal{T}|_{(x,u)})^{-1}$ for any $\theta \in \mathcal{S}$ and any $\mathcal{T} \in H$.

3. $H|_{(x,u)} \cap N_\theta = \{\mathrm{id}\}$ for any $\theta \in \mathcal{S}$.

Here $\mathcal{T}|_{(x,u)}$ and $H|_{(x,u)}$ denote the restrictions of $\mathcal{T}$ and H to the space with local coordinates (x,u): $H|_{(x,u)} = \{\mathcal{T}|_{(x,u)} \mid \mathcal{T} \in H\}$, and id is the identity transformation in this space. We then say that the class of differential equations $\mathcal{L}|_\mathcal{S}$ is *uniformly semi-normalized* with respect to the subgroup H of $G^\sim$ and the symmetry-subgroup family $\mathcal{N}_\mathcal{S}$.

The case $H|_{(x,u)} = \{\mathrm{id}\}$ is degenerate and is not of interest for group analysis. However, if H coincides with the entire equivalence group $G^\sim$, we have the simplest (and commonest) type of uniformly semi-normalized classes.

The third property in Definition II.1 just says that no transformation, other than the identity transformation, in H is induced by a symmetry transformation from the family $\mathcal{N}_\mathcal{S}$.

The term "uniformly" is justified by the second property and by the fact that, in practically relevant examples of such classes, all the subgroups N_θ's are isomorphic to each other or at least have a similar structure (in particular, they are of the same dimension).

The factorization of φ in the first property implies that the triple $(\theta',\theta'',\mathcal T|_{(x,u)})$ is the composition of the admissible transformations $(\theta',\theta',(\varphi')^{-1})$, $(\theta',\theta'',\varphi)$ and $(\theta'',\theta'',(\varphi'')^{-1})$. Therefore, this triple itself is an admissible transformation in the class $\mathcal L|_{\mathcal S}$. Since the systems $\mathcal L_{\theta''}$ and $\mathcal L_{\mathcal T\theta}$ are obtained from the system $\mathcal L_\theta$ by the same point transformation $\mathcal T|_{(x,u)}$, they coincide as systems of differential equations and define equivalent systems of algebraic equations in the underlying jet space. In other words, the arbitrary-element tuples θ'' and $\mathcal T\theta$ are *gauge equivalent*, denoted symbolically as $\theta''\overset{\mathrm g}{\sim}\mathcal T\theta$. If the gauge equivalence within the class $\mathcal L|_{\mathcal S}$ is trivial, that is, each the arbitrary-element tuple is gauge equivalent to itself only, we obtain the relation $\theta''=\mathcal T\theta'$. Consequently, for classes with trivial gauge equivalence this relation can be omitted in the definition of uniform semi-normalization as a consequence of the factorization of φ.

The relation $\theta''=\mathcal T\theta'$ with $\mathcal T\in H$ implies that the systems $\mathcal L_{\theta'}$ and $\mathcal L_{\theta''}$ are H-equivalent, which gives the following obvious result:

Proposition II.1. *If the class $\mathcal L|_{\mathcal S}$ is uniformly semi-normalized with respect to a subgroup H of $G^\sim$ and a symmetry-subgroup family $\mathcal N_{\mathcal S}=\{N_\theta\mid\theta\in\mathcal S\}$, then systems from this class are similar (that is, they are related by point transformations) if and only if they are H-equivalent.*

The first property of Definition II.1 means that the entire equivalence groupoid $\mathcal G^\sim$ is generated by distinguished equivalence transformations and transformations from uniform point symmetry groups. By the second property, one of the symmetry transformations, φ' or φ'', in the corresponding representation of admissible transformations can be taken to be the identity, and then, by the third property, the other components are defined uniquely.

Proposition II.2. *If the class $\mathcal L|_{\mathcal S}$ is uniformly semi-normalized with respect to a subgroup H of $G^\sim$ and a symmetry-subgroup family $\mathcal N_{\mathcal S}=\{N_\theta\mid\theta\in\mathcal S\}$, then for any $(\theta',\theta'',\varphi)\in\mathcal G^\sim$ there exist a unique $\varphi'\in N_{\theta'}$ (resp. a unique $\varphi''\in N_{\theta''}$) and a unique $\mathcal T\in H$ such that $\theta''=\mathcal T\theta'$ and $\varphi=\mathcal T|_{(x,u)}\varphi'$ (resp. $\varphi=\varphi''(\mathcal T|_{(x,u)})$).*

Proof. We fix an arbitrary admissible transformation $(\theta',\theta'',\varphi)$ of the class $\mathcal L|_{\mathcal S}$. The first of the properties in the definition of uniform semi-normalization of $\mathcal L|_{\mathcal S}$ guarantees the existence $\hat\varphi'\in N_{\theta'}$, $\hat\varphi''\in N_{\theta''}$ and $\mathcal T\in H$ such that $\theta''=\mathcal T\theta'$ and $\varphi=\hat\varphi''(\mathcal T|_{(x,u)})\hat\varphi'$. By the second property, for a given choice of $\mathcal T$, there exist $\check\varphi'\in N_{\theta''}$ and $\check\varphi''\in N_{\theta'}$ such that $\mathcal T|_{(x,u)}\hat\varphi'=\check\varphi'(\mathcal T|_{(x,u)})$ and $\hat\varphi''(\mathcal T|_{(x,u)})=\mathcal T|_{(x,u)}\check\varphi''$. Then

$$\varphi=\mathcal T|_{(x,u)}\check\varphi''\hat\varphi'=\mathcal T|_{(x,u)}\varphi'\quad\text{and}\quad\varphi=\hat\varphi''\check\varphi'(\mathcal T|_{(x,u)})=\varphi''(\mathcal T|_{(x,u)}),$$

where $\varphi'=\check\varphi''\hat\varphi'$ and $\varphi''=\hat\varphi''\check\varphi'$. This proves the existence of φ' and φ''.

To prove uniqueness, we suppose that there exist $\tilde{\mathcal T}\in H$ and $\tilde\varphi'\in N_{\theta'}$ such that $\varphi=\mathcal T|_{(x,u)}\varphi'=\tilde{\mathcal T}|_{(x,u)}\tilde\varphi'$. This implies

$$(\tilde{\mathcal T}|_{(x,u)})^{-1}\mathcal T|_{(x,u)}=\tilde\varphi'(\varphi')^{-1}\in H|_{(x,u)}\cap N_\theta=\{\mathrm{id}\},$$

and hence $\tilde{\mathcal{T}}|_{(x,u)} = \mathcal{T}|_{(x,u)}$ and $\tilde{\varphi}' = \varphi'$. The uniqueness of the second representation for φ is proved in the same way. □

The second property can be interpreted as the equivariance of the map $\mathcal{S} \ni \theta \mapsto (\theta, N_\theta)$ under the actions of H on $\mathcal{S}$ and on $\{(\theta, K_\theta) \mid \theta \in \mathcal{S},\, K_\theta \leqslant G_\theta\}$. If the first property of Definition II.1 holds, the equivariance under the action of the subgroup H implies the equivariance under the action of the entire group $G^\sim$. Indeed, for an arbitrary $\theta \in \mathcal{S}$ and $\mathcal{T} \in G^\sim$ we have $(\theta, \mathcal{T}\theta, \mathcal{T}|_{(x,u)}) \in \mathcal{G}^\sim$, and then by Proposition II.2 the transformation $\mathcal{T}|_{(x,u)}$ admits the representation $\mathcal{T}|_{(x,u)} = \tilde{\mathcal{T}}|_{(x,u)}\varphi$ for some $\tilde{\mathcal{T}}|_{(x,u)} \in H$ and some $\varphi \in N_\theta$. Then $\mathcal{T}\theta = \tilde{\mathcal{T}}\theta$ and

$$\begin{aligned}\mathcal{T}|_{(x,u)} N_\theta (\mathcal{T}|_{(x,u)})^{-1} &= \tilde{\mathcal{T}}|_{(x,u)}\varphi N_\theta \varphi^{-1} (\tilde{\mathcal{T}}|_{(x,u)})^{-1} = \tilde{\mathcal{T}}|_{(x,u)} N_\theta (\tilde{\mathcal{T}}|_{(x,u)})^{-1} \\ &= N_{\tilde{\mathcal{T}}\theta} = N_{\mathcal{T}\theta}.\end{aligned}$$

Thus, we obtain the following:

Proposition II.3. *If the class $\mathcal{L}|_\mathcal{S}$ is uniformly semi-normalized with respect to a subgroup H of $G^\sim$ and a symmetry-subgroup family $\mathcal{N}_\mathcal{S} = \{N_\theta \mid \theta \in \mathcal{S}\}$, then $N_{\mathcal{T}\theta} = \mathcal{T}|_{(x,u)} N_\theta (\mathcal{T}|_{(x,u)})^{-1}$ for any $\theta \in \mathcal{S}$ and any $\mathcal{T} \in G^\sim$.*

Each normalized class of differential equations is uniformly semi-normalized with respect to the improper subgroup $H = G^\sim$ and the trivial family $\mathcal{N}_\mathcal{S}$, that is $\mathcal{N}_\mathcal{S}$ consists of just the identity transformation. It is also clear that each uniformly semi-normalized class is semi-normalized. However, there are semi-normalized classes that are not uniformly semi-normalized, which was illustrated in [18, Section 3] by the class ND of nonlinear diffusion equations of the form $u_t = (f(u)u_x)_x$ with $f_u \neq 0$.

The group classification of uniformly semi-normalized classes depends crucially on *the theorem on splitting of symmetry groups in uniformly semi-normalized classes*:

Theorem II.1. *Suppose that the class of differential equations $\mathcal{L}|_\mathcal{S}$ is uniformly semi-normalized with respect to a subgroup H of $G^\sim$ and a symmetry-subgroup family $\mathcal{N}_\mathcal{S} = \{N_\theta \mid \theta \in \mathcal{S}\}$. Then for each $\theta \in \mathcal{S}$ the point symmetry group G_θ of the system $\mathcal{L}_\theta \in \mathcal{L}|_\mathcal{S}$ splits over N_θ. That is, N_θ is a normal subgroup of G_θ, $G_\theta^{\rm ess} = H|_{(x,u)} \cap G_\theta$ is a subgroup of G_θ, and the group G_θ is the semidirect product of $G_\theta^{\rm ess}$ acting on N_θ, $G_\theta = G_\theta^{\rm ess} \ltimes N_\theta$.*

Proof. We fix an arbitrary $\theta \in \mathcal{S}$ and take an arbitrary $\varphi \in G_\theta$. Then $(\theta, \theta, \varphi) \in \mathcal{G}^\sim$ and, by the first property of Definition II.1, the transformation φ admits the factorization $\varphi = \mathcal{T}|_{(x,u)}\varphi^1$ for some $\mathcal{T} \in H$ and some $\varphi^1 \in N_\theta$. The element N_θ of the family $\mathcal{N}_\mathcal{S}$ is a subgroup of G_θ, $N_\theta < G_\theta$ and hence the transformation $\varphi^0 := \mathcal{T}|_{(x,u)} = \varphi(\varphi^1)^{-1}$ also belongs to G_θ, and consequently to $H|_{(x,u)} \cap G_\theta =: G_\theta^{\rm ess}$, which is a subgroup of G_θ since it is intersection of two groups. This implies that for any $\varphi \in G_\theta$ we have the representation $\varphi = \varphi^0\varphi^1$, where $\varphi^0 \in G_\theta^{\rm ess}$ and $\varphi^1 \in N_\theta$.

Since $H|_{(x,u)} \cap N_\theta = \{\mathrm{id}\}$ by the third property of Definition II.1, the intersection $G_\theta^{\rm ess} \cap N_\theta$ also contains only the identity transformation.

For an arbitrary $\varphi \in G_\theta$ and an arbitrary $\tilde\varphi \in N_\theta$, consider the composition $\varphi\tilde\varphi\varphi^{-1}$. As an element of G_θ, the transformation φ admits the factorization $\varphi = \varphi^0\varphi^1$ for some $\varphi^0 \in G^{\rm ess}_\theta$ and some $\varphi^1 \in N_\theta$. Since $G^{\rm ess}_\theta < H|_{(x,u)}$, there exists a $\mathcal T \in H$ such that $\mathcal T\theta = \theta$ and $\varphi^0 = \mathcal T|_{(x,u)}$. By the second property of Definition II.1, we obtain $N_\theta = \varphi^0 N_\theta(\varphi^0)^{-1}$. Hence we have $\varphi\tilde\varphi\varphi^{-1} = \varphi^0\varphi^1\tilde\varphi(\varphi^1)^{-1}(\varphi^0)^{-1} \in N_\theta$. Thus we have that N_θ is a normal subgroup of G_θ, $N_\theta \lhd G_\theta$, and so $G_\theta = G^{\rm ess}_\theta \ltimes N_\theta$. □

The members of the family $\mathcal N_{\mathcal S} = \{N_\theta \mid \theta \in \mathcal S\}$ are called *uniform point symmetry groups* of the systems from the class $\mathcal L|_{\mathcal S}$ with respect to the subgroup H of $G^\sim$. The subgroup $G^{\rm ess}_\theta$ is called the *essential point symmetry group* of the system $\mathcal L_\theta$ associated with the uniform point symmetry group N_θ. The existence of a family of uniform point symmetry groups trivializes them in the following sense: since G_θ splits over N_θ for each θ, then we only need to find the subgroup $G^{\rm ess}_\theta$ in order to construct G_θ.

The infinitesimal version of Theorem II.1 may be called *the theorem on splitting of invariance algebras in uniformly semi-normalized classes*. This result follows immediately from Theorem II.1 if we replace the groups with the corresponding algebras of generators of the one-parameter subgroups of these groups:

Theorem II.2. *Suppose that a class of differential equations $\mathcal L|_{\mathcal S}$ is uniformly semi-normalized with respect to a subgroup H of $G^\sim$ and a family of symmetry subgroups $\mathcal N_{\mathcal S} = \{N_\theta \mid \theta \in \mathcal S\}$, and $\mathfrak h$ is a subalgebra of the equivalence algebra $\mathfrak g^\sim$ of this class that corresponds to the subgroup H. Then for each $\theta \in \mathcal S$ the Lie algebras $\mathfrak g^{\rm ess}_\theta$ and $\mathfrak n_\theta$ that are associated with the groups $G^{\rm ess}_\theta$ and N_θ are, respectively, a subalgebra and an ideal of the maximal Lie invariance algebra $\mathfrak g_\theta$ of the system $\mathcal L_\theta \in \mathcal L|_{\mathcal S}$. Moreover, the algebra $\mathfrak g_\theta$ is the semi-direct sum $\mathfrak g_\theta = \mathfrak g^{\rm ess}_\theta \mathbin{\supset\!\!\!\!+} \mathfrak n_\theta$, and $\mathfrak g^{\rm ess}_\theta = \mathfrak h|_{(x,u)} \cap \mathfrak g_\theta$, where $\mathfrak h|_{(x,u)}$ denotes the restriction of $\mathfrak h$ to the space with local coordinates (x,u).*

In view of Proposition II.1 and Theorem II.2, the group classification problem for a uniformly semi-normalized class $\mathcal L|_{\mathcal S}$ is solved in the following way: when computing the equivalence groupoid $\mathcal G^\sim$ and analyzing its structure, we construct a subgroup H of $G^\sim$ and a family of uniform point symmetry groups $\mathcal N_{\mathcal S} = \{N_\theta \mid \theta \in \mathcal S\}$, which then establishes the uniformly semi-normalization of the class $\mathcal L|_{\mathcal S}$ and yields the corresponding uniform Lie invariance algebras $\mathfrak n_\theta$'s. The subgroup $G^{\rm ess}_\theta = H|_{(x,u)} \cap G_\theta$ and the subalgebra $\mathfrak g^{\rm ess}_\theta = \mathfrak h|_{(x,u)} \cap \mathfrak g_\theta$ which are the complements of N_θ and $\mathfrak n_\theta$ in G_θ and $\mathfrak g_\theta$, respectively, are in general not known on this step. By Theorem II.2, we have for each $\theta \in \mathcal S$ that the maximal Lie invariance algebra $\mathfrak g_\theta$ of the system $\mathcal L_\theta$ is given by the semi-direct sum $\mathfrak g_\theta = \mathfrak g^{\rm ess}_\theta \mathbin{\supset\!\!\!\!+} \mathfrak n_\theta$. The essential Lie invariance algebras are subalgebras of $\mathfrak h|_{(x,u)}$ and are pushforwards of each other under equivalence transformations from H: $\mathfrak g^{\rm ess}_{\mathcal T\theta} = (\mathcal T|_{(x,u)})_*\mathfrak g^{\rm ess}_\theta$ for $\mathcal T \in H$. We call a subalgebra of $\mathfrak h|_{(x,u)}$ *appropriate* if it coincides with the essential Lie invariance algebra of some system from the class $\mathcal L|_{\mathcal S}$. By Proposition II.1, H-equivalence can replace $G^\sim$-equivalence and even $\mathcal G^\sim$-equivalence. Thus, we have the following:

Proposition II.4. *Let the class $\mathcal{L}|_{\mathcal{S}}$ be uniformly semi-normalized with respect to a subgroup H of $G^{\sim}$ and a symmetry-subgroup family $\mathcal{N}_{\mathcal{S}}$, and let $\mathfrak{g}^{\sim}$ be the equivalence algebra of this class. Let $\mathfrak{h}$ be the subalgebra of $\mathfrak{g}^{\sim}$ corresponding to the subgroup H. Then the solution of the group classification problem for the class $\mathcal{L}|_{\mathcal{S}}$ reduces to the classification, up to H-equivalence, of appropriate subalgebras of $\mathfrak{h}|_{(x,u)}$ or, equivalently, of the algebra $\mathfrak{h}$ itself.*

Example II.1

An important case of uniformly semi-normalized classes is given by classes of homogeneous linear systems of differential equations. Consider a class $\mathcal{L}|_{\mathcal{S}}$ of homogeneous linear systems of differential equations that is uniformly semi-normalized with respect to its entire equivalence group and the family $\mathcal{N}_{\rm lin}=\{G^{\rm lin}_\theta\mid\theta\in\mathcal{S}\}$, where for each θ the group $G^{\rm lin}_\theta$ consists of the linear superposition transformations $\tilde x_j=x_j$, $\tilde u^a=u^a+h^a(x)$, with the tuple h running through the solution set of $\mathcal{L}_\theta$. We say that the class $\mathcal{L}|_{\mathcal{S}}$ is *uniformly semi-normalized with respect to linear superposition of solutions*. See [18] for details.

There are classes of differential equations that are uniformly semi-normalized only with respect to proper subgroups of the corresponding equivalence groups.

Example II.2

The first example of such a class was constructed in [18] in the course of the group classification of (1+1)-dimensional linear Schrödinger equations with complex-valued potentials depending on (t,x). More specifically, let $n=1$. Consider the subclass of the class $\mathcal{F}$, where the potentials are pure imaginary and linear with respect to x. This subclass can be reparameterized to the class of equations of the form

$$\mathcal{E}_\gamma\colon\quad i\psi_t+\psi_{xx}+i\gamma(t)x\psi=0, \tag{2}$$

where γ is an arbitrary smooth real-valued function of t, which is the only arbitrary element. We shall use the notation $\hat\beta$ for complex numbers β where $\hat\beta$ is defined as follows: $\hat\beta=\beta$ if $T_t>0$ and $\hat\beta=\beta^*$ if $T_t<0$. The equivalence groupoid $\mathcal{G}^{\sim}_{(2)}$ of the class (2) consists of triples of the form $(\gamma,\tilde\gamma,\varphi)$, where φ is a point transformation in the space of variables with components

$$\tilde t=T:=\frac{a_1t+a_0}{a_3t+a_2},\quad \tilde x=\varepsilon|T_t|^{1/2}x+b_1T+b_0, \tag{3a}$$

$$\begin{aligned}\tilde\psi=&\exp\left(\frac i8\frac{T_{tt}}{|T_t|}x^2+\frac i2\varepsilon b_1|T_t|^{1/2}x-\varepsilon\int\gamma\frac{b_1T+b_0}{|T_t|^{1/2}}\,{\rm d}t-i\frac{b_1^2}{4}T\right)\\ &\frac{c}{|T_t|^{1/4}}(\hat\psi+\hat\Phi),\end{aligned} \tag{3b}$$

the relation between γ and $\tilde\gamma$ is given by

$$\tilde\gamma=\frac{\varepsilon\varepsilon'}{|T_t|^{3/2}}\gamma, \tag{3c}$$

a_0, a_1, a_2, a_3, b_0 and b_1 are arbitrary real constants with $a_1a_2 - a_0a_3 =: \varepsilon' = \pm 1$, c is a nonzero complex constant, $\Phi = \Phi(t, x)$ is an arbitrary solution of the initial equation $\mathcal{E}_\gamma$, and $\varepsilon = \pm 1$. The (usual) equivalence group $G^\sim_{(2)}$ of the class (2) consists of point transformations in the extended space with local coordinates $(t, x, \psi, \psi^*, \gamma)$ with components of the form (3), where $b_0 = b_1 = 0$ and $\Phi = 0$. For each γ, the equation $\mathcal{E}_\gamma$ admits the group $G^{\rm unf}_\gamma$ of point symmetry transformations of the form (3a)–(3b) with $T = t$ and $\varepsilon = 1$. Elements of the kernel symmetry group $G^\cap_{(2)}$ of the class (2) satisfy additionally the constraints $b_0 = b_1 = 0$ and $\Phi = 0$, that is, they consist of the 'scalings' of ψ by nonzero complex numbers, φ_c: $\tilde t = t$, $\tilde x = x$, $\tilde\psi = c\psi$. The class (2) is uniformly semi-normalized with respect to the family $\mathcal{N} = \{G^{\rm unf}_\gamma\}$ and the subgroup H of $G^\sim_{(2)}$ singled out from $G^\sim_{(2)}$ by the constraint $c = 1$. Admissible transformations related to the parameters b_0 and b_1 have no counterparts among the equivalence transformations and thus, for a proper interpretation within the framework of uniform semi-normalization, their transformational parts must be included in uniform symmetry groups for the corresponding values of γ. The transformations φ_c with $|c| = 1$, are compositions of transformational parts of this type for any value of γ and thus necessarily belongs to any possible uniform symmetry groups of the class (2). Since the transformations φ_c belongs to the kernel symmetry group of the class (2), which can be embedded to the equivalence group $G^\sim_{(2)}$ via trivial prolongation of its transformations to the arbitrary element γ, the class (2) is not uniformly semi-normalized with respect to the entire equivalence group $G^\sim_{(2)}$.

4 Weakly uniformly semi-normalized classes

From the results of Section 3, we see that the group classification problem can be solved by the algebraic method for classes that possess at least the first and second properties of Definition II.1.

Definition II.2. If a class of differential equations $\mathcal{L}|_\mathcal{S}$ satisfies the first two conditions of Definition II.1, we say that $\mathcal{L}|_\mathcal{S}$ is *weakly uniformly semi-normalized* with respect to the subgroup H of $G^\sim$ and the symmetry-subgroup family $\mathcal{N}_\mathcal{S}$.

It is obvious that uniformly semi-normalized classes form a special case of weakly uniformly semi-normalized classes. Proposition II.1 is still true for weakly uniformly semi-normalized classes. In Proposition II.2 we just lose the uniqueness of the representation components. Proposition II.3 also holds for weakly uniformly semi-normalized classes. However, Theorem II.1 has to be reformulated for this case since the splitting of G_θ over N_θ is then not guaranteed:

Theorem II.3. *Suppose that the class of differential equations $\mathcal{L}|_\mathcal{S}$ is weakly uniformly semi-normalized with respect to a subgroup H of $G^\sim$ and a symmetry-subgroup family $\mathcal{N}_\mathcal{S} = \{N_\theta \mid \theta \in \mathcal{S}\}$. Then for each $\theta \in \mathcal{S}$, N_θ is a normal subgroup of G_θ, $G^{\rm ess}_\theta = H|_{(x,u)} \cap G_\theta$ is a subgroup of G_θ, and the group G_θ is the Frobenius product of $G^{\rm ess}_\theta$ and N_θ, $G_\theta = G^{\rm ess}_\theta N_\theta$.*

Proof. As in the proof of Theorem II.1, we find that: $H|_{(x,u)} \cap G_\theta =: G_\theta^{\rm ess}$ is a subgroup of G_θ as it is the intersection of two groups; any $\varphi \in G_\theta$ admits the representation $\varphi = \varphi^0\varphi^1$, where $\varphi^0 \in G_\theta^{\rm ess}$ and $\varphi^1 \in N_\theta$, i.e., $G_\theta \subseteq G_\theta^{\rm ess}N_\theta$, and since $G_\theta \supseteq G_\theta^{\rm ess}N_\theta$, then $G_\theta = G_\theta^{\rm ess}N_\theta$; N_θ is a normal subgroup of G_θ. □

We should emphasize that the product of $G_\theta^{\rm ess}$ and N_θ in Theorem II.3 is in general a product of subgroups (with nontrivial intersection) as a product of subsets of groups but not a semidirect product of subgroups since the the third property of Definition II.1 does not necessarily hold, and so the condition $G_\theta^{\rm ess} \cap N_\theta = \{\mathrm{id}\}$ may not hold.

Theorem II.2 needs minimal modifications:

Theorem II.4. *Suppose that a class of differential equations $\mathcal{L}|_\mathcal{S}$ is weakly uniformly semi-normalized with respect to a subgroup H of $G^\sim$ and a family of symmetry subgroups $\mathcal{N}_\mathcal{S} = \{N_\theta \mid \theta \in \mathcal{S}\}$, and $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}^\sim$ that corresponds to the subgroup H. Then for each $\theta \in \mathcal{S}$ the Lie algebras $\mathfrak{g}_\theta^{\rm ess}$ and $\mathfrak{n}_\theta$ that are associated with the groups $G_\theta^{\rm ess}$ and N_θ are, respectively, a subalgebra and an ideal of the maximal Lie invariance algebra $\mathfrak{g}_\theta$ of the system $\mathcal{L}_\theta \in \mathcal{L}|_\mathcal{S}$. The algebra $\mathfrak{g}_\theta$ is the sum $\mathfrak{g}_\theta = \mathfrak{g}_\theta^{\rm ess} + \mathfrak{n}_\theta$, where $\mathfrak{g}_\theta^{\rm ess} = \mathfrak{h}|_{(x,u)} \cap \mathfrak{g}_\theta$, and $\mathfrak{h}|_{(x,u)}$ denotes the restriction of $\mathfrak{h}$ to the space with local coordinates (x, u).*

Proposition II.4 holds for weakly uniformly semi-normalized classes. The only difference in implementing the algebraic method for group classification of weakly uniformly semi-normalized classes is that in general the essential invariance algebra $\mathfrak{g}_\theta^{\rm ess}$ has a nontrivial intersection with the uniform invariance algebra $\mathfrak{n}_\theta$, and hence a natural basis of the maximal invariance algebra $\mathfrak{g}_\theta$ is given by a basis of the intersection $\mathfrak{g}_\theta^{\rm ess} \cap \mathfrak{n}_\theta$ and basis vectors in the complements of $\mathfrak{g}_\theta^{\rm ess} \cap \mathfrak{n}_\theta$ in $\mathfrak{g}_\theta^{\rm ess}$ and of $\mathfrak{n}_\theta$.

Remark II.1. The framework of uniformly semi-normalized classes can be adapted by changing the terminology and exchanging the roles of Definitions II.1 and II.2. Although the third property of Definition II.1 allowed us to obtain stronger (and nicer) results, this property is not essential for carrying out the group classification of a class of differential equations by the algebraic method. Thus we may assume Definition II.2 as being basic and omit the attribute "weakly" in the term introduced there. Then classes satisfying Definition II.1 can be considered as special uniformly semi-normalized classes with the additional property of satisfying Property 3 of Definition II.1.

A typical case of nontrivial intersection of $H|_{(x,u)}$ with elements of the symmetry-subgroup family $\mathcal{N}_\mathcal{S}$ is given by $H|_{(x,u)} \cap N_\theta \supseteq G^\cap$. This is to be expected since the kernel symmetry group $G^\cap$ is a normal subgroup of $G^\sim|_{(x,u)}$ and is a subgroup of G_θ for any θ, and hence it can always be added as a subgroup to uniform symmetry groups N_θ, which are normal subgroups of the corresponding G_θ. Thus, if the class of differential equations $\mathcal{L}|_\mathcal{S}$ is (weakly) uniformly semi-normalized with respect to a subgroup H of $G^\sim$ and a symmetry-subgroup family $\mathcal{N}_\mathcal{S} = \{N_\theta \mid \theta \in \mathcal{S}\}$, then it is weakly uniformly semi-normalized with respect to the same subgroup H of $G^\sim$ and the symmetry-subgroup family $\tilde{\mathcal{N}}_\mathcal{S} = \{\tilde N_\theta = G^\cap N_\theta \mid \theta \in \mathcal{S}\}$

with $H|_{(x,u)}\cap\tilde N_\theta=G^\cap$ (resp. $H|_{(x,u)}\cap\tilde N_\theta=G^\cap(H|_{(x,u)}\cap N_\theta)\supseteq G^\cap$) for any $\theta\in\mathcal S$. Indeed, by Theorem II.1 the definition of $G^\cap$, the uniform symmetry group N_θ is a normal subgroup of G_θ and the kernel symmetry group $G^\cap$ is a subgroup of G_θ for each $\theta\in\mathcal S$. Hence, the subgroups N_θ and $G^\cap$ of G_θ permute, that is, their Frobenius products coincide: $N_\theta G^\cap=G^\cap N_\theta$. Denoting this product by $\tilde N_\theta$ we find that $\tilde N_\theta$ is a subgroup of G_θ generated by $G^\cap$ and N_θ. The statement on the intersection $H|_{(x,u)}\cap\tilde N_\theta$ in the case of uniform semi-normalization follows from the third property of Definition II.1, and it is obvious for the case of weak uniform semi-normalization.

Example II.3

The class of equations (2) is weakly uniformly semi-normalized with respect to the entire equivalence group $G^\sim_{(2)}$ and the family $\mathcal N=\{G^{\rm unf}_\gamma\}$, and for any γ we have

$$G^\sim_{(2)}\big|_{(t,x,\psi,\psi^*)}\cap G^{\rm unf}_\gamma=G^\cap_{(2)}\neq\{\rm id\},$$

where id denotes the identity transformation of (t,x,ψ,ψ^*). However, the interpretation of the class (2) as uniformly semi-normalized with respect to the family $\mathcal N=\{G^{\rm unf}_\gamma\}$ and the proper subgroup H of $G^\sim_{(2)}$ is more convenient and effective for the group classification of this class since then one can apply Theorems II.1 and II.2, which are stronger than Theorems II.3 and II.4, and, by Proposition II.4, one needs only to deal with the smaller group H.

There exist weakly uniformly semi-normalized classes of differential equations that are not uniformly semi-normalized.

Example II.4

An example of this is given by the class $\mathcal F_{{\rm l}x_1}$ of (1+2)-dimensional linear Schrödinger equations with potentials of the form (19). The only possible intersection of the equivalence group of the class $\mathcal F_{{\rm l}x_1}$ with its uniform symmetry groups is its kernel symmetry group. Weak uniform semi-normalization of the class $\mathcal F_{{\rm l}x_1}$ is exploited in the proof of Theorem II.8 below.

5 Equivalence groupoid

We find the equivalence groupoid $\mathcal G^\sim$ of the class $\mathcal F$ by using the direct method. Let $\mathcal L_V$ denote an equation from $\mathcal F$ with a potential V. We seek for all invertible point transformations of the form

$$\begin{aligned}\varphi\colon\ &\tilde t=T(t,x,\psi,\psi^*),\quad \tilde x_a=X^a(t,x,\psi,\psi^*),\\ &\tilde\psi=\Psi(t,x,\psi,\psi^*),\quad \tilde\psi^*=\Psi^*(t,x,\psi,\psi^*),\end{aligned}\tag{4}$$

where $\det\partial(T,X,\Psi,\Psi^*)/\partial(t,x,\psi,\psi^*)\neq0$ with $X=(X^1,\dots,X^n)$, that map a fixed equation $\mathcal L_V$ from the class $\mathcal F$ to an equation $\mathcal L_{\tilde V}$: $i\tilde\psi_{\tilde t}+\tilde\psi_{\tilde x_a\tilde x_a}+\tilde V(\tilde t,\tilde x)\tilde\psi=0$

of the same class. In the following, we use the notation $|y|^2$ for y_ay_a for an n-tuple $y=(y_1,\dots,y_n)$. We also use the notation

$$\hat\beta=\beta \quad\text{if}\quad T_t>0 \quad\text{and}\quad \hat\beta=\beta^* \quad\text{if}\quad T_t<0$$

for complex numbers β. The following result can be proved as [35, Lemma 1].

Lemma II.1. *Any point transformation φ connecting two equations from the class $\mathcal F$ satisfies the conditions*

$$\begin{aligned}&T_a=T_\psi=T_{\psi^*}=0,\quad X^a_\psi=X^a_{\psi^*}=0,\\ &\Psi_\psi=0 \quad\textit{if}\quad T_t<0 \quad\textit{and}\quad \Psi_{\psi^*}=0 \quad\textit{if}\quad T_t>0.\end{aligned}\tag{5}$$

Theorem II.5. *The equivalence groupoid $\mathcal G^\sim$ of the class $\mathcal F$ consists of triples of the form $(V,\tilde V,\varphi)$, where φ is a point transformation in the space of variables, whose components are*

$$\tilde t=T,\quad \tilde x_a=|T_t|^{1/2}O^{ab}x_b+\mathcal X^a,\tag{6a}$$

$$\tilde\psi=\exp\left(\frac i8\frac{T_{tt}}{|T_t|}\,|x|^2+\frac i2\frac{\varepsilon'\mathcal X^b_t}{|T_t|^{1/2}}\,O^{ba}x_a+i\Sigma+\Upsilon\right)(\hat\psi+\hat\Phi),\tag{6b}$$

and the transformed potential $\tilde V$ is expressed via V as

$$\begin{aligned}\tilde V=&\frac{\hat V}{|T_t|}+\frac{2T_{ttt}T_t-3{T_{tt}}^2}{16\varepsilon'{T_t}^3}|x|^2+\frac{\varepsilon'}{2|T_t|^{1/2}}\left(\frac{\mathcal X^b_t}{T_t}\right)_tO^{ba}x_a\\ &+\frac{\Sigma_t-i\Upsilon_t}{T_t}-\frac{\mathcal X^a_t\mathcal X^a_t+inT_{tt}}{4{T_t}^2}.\end{aligned}\tag{6c}$$

Here T, $\mathcal X^a$, Σ and Υ are arbitrary smooth real-valued functions of t with $T_t\neq0$, $\varepsilon'=\operatorname{sgn}T_t$, $O=(O^{ab})$ is an arbitrary constant $n\times n$ orthogonal matrix, and $\Phi=\Phi(t,x)$ is an arbitrary solution of the initial equation.

Proof. Let φ be a point transformation mapping the equation $\mathcal L_V$ to the $\mathcal L_{\tilde V}$ in the class $\mathcal F$. Lemma II.1 implies that $T=T(t)$, $X^a=X^a(t,x)$ and $\Psi=\Psi(t,x,\hat\psi)$ with $T_t\det(\partial X/\partial x)\Psi_{\hat\psi}\neq0$. Applying total derivatives $\mathrm D_t$ and $\mathrm D_a$'s to the equality $\tilde\psi(\tilde t,\tilde x)=\Psi(t,x,\hat\psi)$, we derive

$$\begin{aligned}&\mathrm D_t\tilde\psi(\tilde t,\tilde x)=\tilde\psi_{\tilde t}T_t+\tilde\psi_{\tilde x_b}X^b_t=\mathrm D_t\Psi,\quad \mathrm D_a\tilde\psi(\tilde t,\tilde x)=\tilde\psi_{\tilde x_c}X^c_a=\mathrm D_a\Psi,\\ &\mathrm D_b\mathrm D_a\tilde\psi(\tilde t,\tilde x)=\tilde\psi_{\tilde x_c\tilde x_d}X^c_bX^d_a+\tilde\psi_{\tilde x_d}X^d_{ab}=\mathrm D_b\mathrm D_a\Psi,\end{aligned}$$

where $\mathrm D_t$ and $\mathrm D_a$ are the total derivative operators with respect to t and x_a, respectively. The above equations are equivalent to

$$\begin{aligned}&\tilde\psi_{\tilde t}=\frac{\mathrm D_t\Psi-Y^a_bX^b_t\mathrm D_a\Psi}{T_t},\quad \tilde\psi_{\tilde x_c}=Y^a_c\mathrm D_a\Psi,\\ &\tilde\psi_{\tilde x_c\tilde x_d}=Y^a_cY^b_d(\mathrm D_b\mathrm D_a\Psi-Y^d_cX^c_{ab}\mathrm D_d\Psi),\end{aligned}\tag{7}$$

where $X^a_c Y^c_b = Y^a_c X^c_b = \delta_{ab}$, and δ_{ab} is the Kronecker delta. Here, the vector $Y = (Y^1, \dots, Y^n)$ denotes the inverse of the transformation $X = (X^1, \dots, X^n)$. We substitute the expressions (4) and (7) for $\tilde\psi$, $\tilde\psi_{\tilde t}$ and $\tilde\psi_{\tilde x_c \tilde x_d}$ and then put the expression for $\hat\psi_t$, $\hat\psi_t = i\varepsilon'(\hat\psi_{aa} + \hat V\hat\psi)$, into the equation $\mathcal{L}_{\tilde V}$. We obtain the equation

$$\frac{i}{T_t}\Big(\Psi_t + \Psi_{\hat\psi}(i\varepsilon'\hat\psi_{aa} + i\varepsilon'\hat V\hat\psi) - Y^a_b(\Psi_a + \Psi_{\hat\psi}\hat\psi_a)X^b_t\Big)$$
$$+Y^a_c Y^b_c\Big(\Psi_{\hat\psi\hat\psi}\hat\psi_b\hat\psi_a + \Psi_{\hat\psi}\hat\psi_{ab} - Y^d_c(\Psi_d + \Psi_{\hat\psi}\hat\psi_d)X^c_{ab}\Big)$$
$$+Y^a_c Y^b_c\Big(\Psi_{ab} + \Psi_{a\hat\psi}\hat\psi_b + \Psi_{\hat\psi b}\hat\psi_a\Big) + \tilde V\Psi = 0$$

Then splitting this equation with respect to various derivatives of $\hat\psi$ and additionally arranging lead to the system

$$Y^a_c Y^b_c = \frac{\delta_{ab}}{|T_t|}, \quad \Psi_{\hat\psi\hat\psi} = 0, \tag{8}$$
$$\frac{2}{|T_t|}\Psi_{a\hat\psi} - \frac{i}{T_t}Y^a_b\Psi_{\hat\psi}X^b_t - \frac{1}{|T_t|}Y^a_c\Psi_{\hat\psi}X^c_{aa} = 0, \tag{9}$$
$$\begin{aligned}&\frac{i}{T_t}\Psi_t - \frac{1}{|T_t|}\hat V\Psi_{\hat\psi}\hat\psi - \frac{i}{T_t}Y^a_b\Psi_a X^b_t + \frac{1}{|T_t|}\Psi_{aa}\\ &\quad - \frac{1}{|T_t|}Y^d_c\Psi_d X^c_{aa} + \tilde V\Psi = 0.\end{aligned} \tag{10}$$

The first equation in (8) together with the condition $Y^a_c X^c_b = \delta_{ab}$ imply that $X^b_a = |T_t|Y^a_b$. Therefore, $X^c_a X^c_b = |T_t|\delta_{ab}$, i.e., $X_a \cdot X_b = |T_t|\delta_{ab}$ in terms of the tuple X. Differentiating this equation with respect to x_c and permuting the indices a, b, c, gives the equations

$$X_{ac}\cdot X_b + X_a\cdot X_{bc} = 0, \quad X_{ab}\cdot X_c + X_b\cdot X_{ac} = 0, \quad X_{bc}\cdot X_a + X_c\cdot X_{ab} = 0$$

and this in turn yields $X_a \cdot X_{bc} = 0$ for all values of (a, b, c). Since for each (t, x) the vectors $X_1, \dots, X_n$ form a basis of $\mathbb{R}^n$, we must have $X_{bc} = 0$ and hence X is affine in x with coefficients depending on t. The equations $X_a \cdot X_b = |T_t|\delta_{ab}$ then give us

$$X^a = |T_t|^{1/2}O^{ab}(t)x_b + \mathcal{X}^a(t),$$

where $O = (O^{ab})$ is a time-dependent orthogonal matrix and $\mathcal{X}^a$ is a time-dependent vector.

The general solution of the second equation in (8) is $\Psi = \Psi^1(t,x)\hat\psi + \Psi^0(t,x)$, where Ψ^0 and Ψ^1 are smooth complex-valued functions of t and x with $\Psi^1 \neq 0$.

We substitute the above expressions for X^a and Ψ into (9) to get

$$\frac{\Psi^1_a}{\Psi^1} = \frac{i}{2T_t}X^b_a X^b_t. \tag{11}$$

Since $\partial_b(\Psi^1_a/\Psi^1) = \partial_a(\Psi^1_b/\Psi^1)$ equation (11) gives $\partial_c(X^b_a X^b_t) = \partial_a(X^b_c X^b_t)$. Hence $O^{ba}O^{bc}_t = O^{bc}O^{ba}_t$, or $O^\mathsf{T}O_t - O^\mathsf{T}_t O = 0$ when written as a matrix equation, where O^T is the transpose of O. Differentiating the orthogonality condition $O^\mathsf{T}O = E$, where E is the $n \times n$ identity matrix, with respect to t gives $O^\mathsf{T}O_t + O^\mathsf{T}_t O = 0$. However, $O^\mathsf{T}O_t = O^\mathsf{T}_t O$, so we have $O^\mathsf{T}O_t = 0$, and thus $O_t = 0$. Hence O is a constant orthogonal matrix.

Integrating the system (11) we obtain:

$$\Psi^1 = \exp\left(\frac{i}{8}\frac{T_{tt}}{|T_t|}\,|x|^2 + \frac{i}{2}\frac{\varepsilon'\mathcal{X}^b_t}{|T_t|^{1/2}}\,O^{ba}x_a + i\Sigma + \Upsilon\right), \tag{12}$$

where Σ and Υ are arbitrary smooth real-valued functions of t. Finally, putting these results into equation (10), we find the equation

$$\frac{i}{T_t}\Psi_t - \frac{1}{|T_t|}\hat{V}\Psi_{\hat{\psi}}\hat{\psi} - \frac{i}{T_t}Y^a_b X^b_t \Psi_a + \frac{1}{|T_t|}\Psi_{aa} + \tilde{V}\Psi = 0.$$

Splitting this equation with respect to $\hat{\psi}$ we obtain the two equations

$$\tilde{V} = \frac{\hat{V}}{|T_t|} - \frac{i}{T_t\Psi^1}\left(\Psi^1_t - \frac{X^b_a X^b_t}{|T_t|}\Psi^1_a\right) - \frac{1}{|T_t|}\frac{\Psi^1_{aa}}{\Psi^1}, \tag{13}$$

$$i\varepsilon'\Psi^0_t - \frac{i}{T_t}X^b_a X^b_t \Psi^0_a + \Psi^0_{aa} + |T_t|\tilde{V}\Psi^0 = 0. \tag{14}$$

We introduce the function $\Phi = \hat{\Psi}^0/\hat{\Psi}^1$, (or $\Psi^0 = \Psi^1\hat{\Phi}$). The equation (14) is then equivalent to the initial linear Schrödinger equation in terms of Φ. After substituting the expression (12) for Ψ^1 into (13) then collecting coefficients of x, we obtain the expression for $\tilde{V}$ as given in the theorem. □

Corollary II.1. *A (1+n)-dimensional linear Schrödinger equation of the form* (1) *is reduced to the free linear Schrödinger equation by a point transformation if and only if*

$$V = \theta(t)|x|^2 + \theta^a(t)x_a + \theta^0(t) + i\tilde{\theta}^0(t)$$

for some real-valued smooth functions θ, θ^a, θ^0 and $\tilde{\theta}^0$ of t.

6 Equivalence group and equivalence algebra

The notion of equivalence group as developed by Ovsiannikov [32] plays a central role in the group classification of classes of differential equations. Although the equivalence group of a class of differential equations can be also be computed using either the infinitesimal or the direct method, it is much easier to find it from the equivalence groupoid of the class once this groupoid is known. The following results follow from Theorem II.5 using arguments similar to those in [18] in proving that [18, Corollary 8] and [18, Corollary 11] were consequences of [18, Theorem 6].

Corollary II.2. *The (usual) equivalence group $G^\sim$ of the class $\mathcal{F}$ consists of point transformations in the space of independent and dependent variables and arbitrary element that are of the form* (6) *with* $\Phi=0$.

Remark II.2. The identity component of $G^\sim$ consists of transformations of the form (6), where $\Phi=0$, $\det O=1$ and $T_t>0$, i.e., $\varepsilon'=1$. The entire equivalence group $G^\sim$ is generated by the transformations of its identity component together with two discrete transformations: the space reflection for a fixed a ($\tilde t=t$, $\tilde x_a=-x_a$, $\tilde x_b=x_b$, $b\neq a$, $\tilde\psi=\psi$, $\tilde V=V$) and the Wigner time reflection ($\tilde t=-t$, $\tilde x=x$, $\tilde\psi=\psi^*$, $\tilde V=V^*$).

Theorem II.6. *The equivalence algebra of the class $\mathcal{F}$ is the algebra*

$$\mathfrak{g}^\sim=\langle \hat D(\tau),\ \hat J_{ab},\ a<b,\ \hat G(\chi),\ \hat M(\sigma),\ \hat I(\rho)\rangle,$$

where τ, $\chi=(\chi^1,\dots,\chi^n)$, σ and ρ run through the set of smooth real-valued functions of t. The vector fields spanning $\mathfrak{g}^\sim$ are defined by

$$\begin{aligned}
\hat D(\tau)&=\tau\partial_t+\frac12\tau_tx_a\partial_a+\frac18\tau_{tt}|x|^2\left(i\psi\partial_\psi-i\psi^*\partial_{\psi^*}\right)\\
&\qquad-\left(\tau_tV-\frac18\tau_{ttt}|x|^2+i\frac{\tau_{tt}}{4}\right)\partial_V-\left(\tau_tV^*-\frac18\tau_{ttt}|x|^2-i\frac{\tau_{tt}}{4}\right)\partial_{V^*},\\
\hat J_{ab}&=x_a\partial_b-x_b\partial_a,\quad a\neq b,\\
\hat G(\chi)&=\chi^a\partial_a+\frac i2\chi^a_tx_a\left(\psi\partial_\psi-\psi^*\partial_{\psi^*}\right)+\frac12\chi^a_{tt}x_a\left(\partial_V+\partial_{V^*}\right),\\
\hat M(\sigma)&=i\sigma(\psi\partial_\psi-\psi^*\partial_{\psi^*})+\sigma_t(\partial_V+\partial_V^*),\\
\hat I(\rho)&=\rho(\psi\partial_\psi+\psi^*\partial_{\psi^*})-i\rho_t(\partial_V+\partial_V^*).
\end{aligned}$$

Proof. The equivalence algebra $\mathfrak{g}^\sim$ of the class $\mathcal{F}$ is obtained using the knowledge of the identity component of the equivalence group $G^\sim$. Searching for infinitesimal generators of one-parameter subgroups of $G^\sim$, we represent the parameter function Σ in the form $\Sigma=\frac14\mathcal{X}^a\mathcal{X}^a_t+\bar\Sigma$, where $\bar\Sigma$ is a function of t, for better consistency of the group parameterization with the one-parameter subgroup structure of $G^\sim$. Then we successively assume one of the transformation parameters T, O, $\mathcal{X}^a$, $\bar\Sigma$ and Υ to depend on a continuous parameter δ and set the other transformation parameters to the trivial values corresponding to the identity transformation, which are t for T, E for O and zeroes for $\mathcal{X}^a$, $\bar\Sigma$ and Υ, where E is the $n\times n$ identity matrix. The components of the associated infinitesimal generator $Q=\tau\partial_t+\xi^a\partial_a+\eta\partial_\psi+\eta^*\partial_{\psi^*}+\theta\partial_V+\theta^*\partial_{V^*}$ are given by

$$\tau=\frac{\mathrm{d}\tilde t}{\mathrm{d}\delta}\Big|_{\delta=0},\quad \xi^a=\frac{\mathrm{d}\tilde x}{\mathrm{d}\delta}\Big|_{\delta=0},\quad \eta=\frac{\mathrm{d}\tilde\psi}{\mathrm{d}\delta}\Big|_{\delta=0},\quad \theta=\frac{\mathrm{d}\tilde V}{\mathrm{d}\delta}\Big|_{\delta=0}.$$

The above procedure results in the set of vector fields spanning the algebra $\mathfrak{g}^\sim$. □

Corollary II.3. *The class $\mathcal{F}$ is uniformly semi-normalized with respect to linear superposition of solutions.*

The uniform semi-normalization of the class $\mathcal{F}$ guarantees a specific factorization of point symmetry groups of all equations from this class. For any potential V, each element φ of the point symmetry group G_V of an equation $\mathcal{L}_V$ generates an admissible point transformation (V,V,φ) of $\mathcal{F}$. Therefore, the transformation φ necessarily has the form (6a)–(6b), where the transformation parameters additionally satisfy the equation (6c) with $\tilde V(\tilde t,\tilde x)=V(\tilde t,\tilde x)$. The symmetry transformations associated with the linear superposition of solutions to the equation $\mathcal{L}_V$ which are of the above form with $T=t$, $O=E$ and $\mathcal{X}^a=\Sigma=\Upsilon=0$, form a normal subgroup of the group G_V, which we denote by $G_V^{\rm lin}$ and we call the *trivial part* of G_V. In view of the discussion on the uniform semi-normalization with respect to linear superposition of solutions in [18, Section 3], Corollary II.3 implies that the group G_V splits over $G_V^{\rm lin}$, $G_V=G_V^{\rm ess}\ltimes G_V^{\rm lin}$, the subgroup $G_V^{\rm ess}$ of G_V is singled out from G_V by the constraint $\Phi=0$ and will be considered as the only essential part of G_V.

7 Analysis of determining equations for Lie symmetries

Using the infinitesimal criterion for Lie symmetries, for each potential V we can derive the determining equations which are satisfied by the components of vector fields from the maximal Lie invariance algebra $\mathfrak{g}_V$ of the equation $\mathcal{L}_V$ from the class $\mathcal{F}$. In this section we integrate the determining equations (as far as this is possible) and then analyze the properties of the algebras $\mathfrak{g}_V$ for $\mathcal{L}_V\in\mathcal{F}$.

The infinitesimal criterion states that a vector field $Q=\tau\partial_t+\xi^a\partial_a+\eta\partial_\psi+\eta^*\partial_{\psi^*}$, where the components τ, ξ^a and η are smooth functions of (t,x,ψ,ψ^*), and η^* is the complex conjugate of η, belongs to the algebra $\mathfrak{g}_V$ if an only if

$$Q_{(2)}\big(i\psi_t+\psi_{aa}+V(t,x)\psi\big)\big|_{\mathcal{L}_V}=0$$

with $Q_{(2)}$ being the second prolongation of the vector field Q, (see Section 2). Expanding this expression, we obtain

$$i\eta^t+\eta^{aa}+(\tau V_t+\xi^aV_a)\psi+V\eta=0,\tag{15}$$

where $\eta^t=\mathrm{D}_t\left(\eta-\tau\psi_t-\xi^a\psi_a\right)+\tau\psi_{tt}+\xi^a\psi_{ta}$ and $\eta^{ab}=\mathrm{D}_a\mathrm{D}_b\left(\eta-\tau\psi_t-\xi^c\psi_c\right)+\tau\psi_{tab}+\xi^c\psi_{abc}$. We recall that D_t and D_a are the operators of total derivatives with respect to t and x_a, respectively. Substituting $\psi_t=i\psi_{aa}+iV\psi$ and $\psi^*_t=-i\psi^*_{aa}-iV^*\psi^*$ into (15) and splitting with respect to the various derivatives of ψ and ψ^* leads to the following overdetermined linear system of determining equations for the coefficients of Q:

$$\begin{gathered}\tau_\psi=\tau_{\psi^*}=\tau_a=0,\quad \xi^a_\psi=\xi^a_{\psi^*}=0,\\ \tau_t=2\xi^1_1=\dots=2\xi^n_n,\quad \xi^a_b+\xi^b_a=0,\ a\neq b,\end{gathered}\tag{16a}$$

$$\eta_{\psi^*}=\eta_{\psi\psi}=0,\quad 2\eta_{\psi a}=i\xi^a_t,\tag{16b}$$

$$i\eta_t+\eta_{aa}+\tau V_t\psi+\xi^aV_a\psi+V\eta-(\eta_\psi-\tau_t)V\psi=0.\tag{16c}$$

Equations (16a) and (16b), do not contain the potential V and can be integrated immediately to give:

$$\tau=\tau(t),\quad \xi^a=\frac{1}{2}\tau_t x_a+\kappa^{ab}x_b+\chi^a,$$
$$\eta=\left(\frac{i}{8}\tau_{tt}|x|^2+\frac{i}{2}\chi^a_t x_a+\rho+i\sigma\right)\psi+\eta^0(t,x),$$

where τ, χ^a, ρ and σ are smooth real-valued functions of t, η^0 is a complex-valued function of t and x, and (κ^{ab}) is a constant skew-symmetric matrix. Then splitting (16c) with respect to ψ, we obtain the equations

$$i\eta^0_t+\eta^0_{aa}+V\eta^0=0, \tag{17}$$

$$\begin{aligned}&\tau V_t+\left(\frac{1}{2}\tau_t x_a+\kappa^{ab}x_b+\chi^a\right)V_a+\tau_t V\\ &\quad=\frac{1}{8}\tau_{ttt}|x|^2+\frac{1}{2}\chi^a_{tt}x_a+\sigma_t-i\rho_t-i\frac{n}{4}\tau_{tt}.\end{aligned} \tag{18}$$

Both these equations involve the potential V. The first of these equations shows that the parameter-function η^0 is an arbitrary solution of the equation $\mathcal{L}_V$, and so the second equation is the only useful *classifying condition* for Lie symmetry generators of equations from the class $\mathcal{F}$ depending on the potential V.

Theorem II.7. *The maximal Lie invariance algebra $\mathfrak{g}_V$ of an equation $\mathcal{L}_V$ from $\mathcal{F}$ consists of vector fields of the form $D(\tau)+\sum_{a<b}\kappa^{ab}J_{ab}+G(\chi)+\sigma M+\rho I+Z(\eta^0)$, where*

$$D(\tau)=\tau\partial_t+\frac{1}{2}\tau_t x_a\partial_a+\frac{1}{8}\tau_{tt}|x|^2M,\quad J_{ab}=x_a\partial_b-x_b\partial_a,\quad a\neq b,$$
$$G(\chi)=\chi^a\partial_a+\frac{1}{2}\chi^a_t x_a M,\quad M=i\psi\partial_\psi-i\psi^*\partial_{\psi^*},\quad I=\psi\partial_\psi+\psi^*\partial_{\psi^*},$$
$$Z(\eta^0)=\eta^0\partial_\psi+\eta^{0*}\partial_{\psi^*}.$$

The parameters τ, $\chi=(\chi^1,\dots,\chi^n)$, ρ and σ are real-valued smooth functions of t and the matrix (κ^{ab}) is an arbitrary constant skew-symmetric matrix. They satisfy the classifying condition (18). *The parameter $\eta^0=\eta^0(t,x)$ runs through the solutions of the equation $\mathcal{L}_V$.*

The kernel invariance algebra $\mathfrak{g}^\cap$ of the class $\mathcal{F}$, $\mathfrak{g}^\cap:=\bigcap_V\mathfrak{g}_V$, is obtained by splitting the conditions (17) and (18) with respect to the potential V and its derivatives. This gives $\tau=\chi^a=0$, $\kappa^{ab}=0$, $\eta^0=0$ and $\sigma_t=\rho_t=0$ for vector fields in the kernel algebra.

Proposition II.5. *The kernel invariance algebra of the class $\mathcal{F}$ is $\mathfrak{g}^\cap=\langle M,I\rangle$.*

As in [18], denote by $\mathfrak{g}_{\langle\,\rangle}$ the linear span of all vector fields given in Theorem II.7 when the potential V varies, i.e.,

$$\mathfrak{g}_{\langle\,\rangle}:=\left\{D(\tau)+\sum_{a<b}\kappa^{ab}J_{ab}+G(\chi)+\sigma M+\rho I+Z(\zeta)\right\}=\sum_V\mathfrak{g}_V.$$

Here and the following, the parameters τ, χ^a, σ and ρ run through the set of real-valued smooth functions of t, ζ runs through the set of complex-valued smooth functions of (t,x) and η^0 runs through the solution set of the equation $\mathcal{L}_V$ when the potential V is fixed. We have $\mathfrak{g}_{\langle\,\rangle}=\sum_V\mathfrak{g}_V$ since each vector field Q from $\mathfrak{g}_{\langle\,\rangle}$ either with nonvanishing τ or (κ^{ab}) or χ or with jointly vanishing τ, κ^{ab}, χ, σ and ρ necessarily belongs to $\mathfrak{g}_V$ for some V. The nonzero commutation relations between vector fields spanning $\mathfrak{g}_{\langle\,\rangle}$ are

$$
\begin{aligned}
&[D(\tau^1),D(\tau^2)]=D(\tau^1\tau^2_t-\tau^2\tau^1_t),\quad [D(\tau),G(\chi)]=G\left(\tau\chi_t-\frac12\tau_t\chi\right),\\
&[D(\tau),\sigma M]=\tau\sigma_tM,\quad [D(\tau),\rho I]=\tau\rho_tI,\\
&[D(\tau),Z(\zeta)]=Z\left(\tau\zeta_t+\frac12\tau_tx_a\zeta_a-\frac{i}{8}\tau_{tt}|x|^2\zeta\right),\\
&[J_{ab},J_{bc}]=J_{ac},\quad a\neq b\neq c\neq a,\\
&[J_{ab},G(\chi)]=G(\hat\chi)\quad\text{with}\quad \hat\chi^a=\chi^b,\ \hat\chi^b=-\chi^a,\ \chi^c=0,\ a\neq b\neq c\neq a,\\
&[J_{ab},Z(\zeta)]=Z(x_a\zeta_b-x_b\zeta_a),\quad a\neq b,\\
&[G(\chi),G(\tilde\chi)]=\frac12\left(\chi^a\tilde\chi^a_t-\tilde\chi^a\chi^a_t\right)M,\\
&[G(\chi),Z(\zeta)]=Z\left(\chi^a\zeta_a-\frac{i}{2}\chi^a_tx_a\zeta\right),\\
&[\sigma M,Z(\zeta)]=-Z(i\sigma\zeta),\quad [\rho I,Z(\zeta)]=-Z(\rho\zeta).
\end{aligned}
$$

As can be seen from these commutation relations, the span $\mathfrak{g}_{\langle\,\rangle}$ is closed under the Lie bracket of vector fields and so it is a Lie algebra. The algebra $\mathfrak{g}_{\langle\,\rangle}$ can be represented as a semi-direct sum of the subalgebra $\mathfrak{g}^{\rm ess}_{\langle\,\rangle}:=\langle D(\tau),J_{ab},G(\chi),\sigma M,\rho I\rangle$ and the ideal $\mathfrak{g}^{\rm lin}_{\langle\,\rangle}:=\langle Z(\zeta)\rangle$, $\mathfrak{g}_{\langle\,\rangle}=\mathfrak{g}^{\rm ess}_{\langle\,\rangle}\Subset\mathfrak{g}^{\rm lin}_{\langle\,\rangle}$. The kernel invariance algebra $\mathfrak{g}^\cap$ is an ideal in $\mathfrak{g}^{\rm ess}_{\langle\,\rangle}$ and in $\mathfrak{g}_{\langle\,\rangle}$. The above representation for $\mathfrak{g}_{\langle\,\rangle}$ is inherited by each $\mathfrak{g}_V$:

$$\mathfrak{g}_V=\mathfrak{g}^{\rm ess}_V\Subset\mathfrak{g}^{\rm lin}_V,$$

where $\mathfrak{g}^{\rm ess}_V:=\mathfrak{g}_V\cap\mathfrak{g}^{\rm ess}_{\langle\,\rangle}$ and $\mathfrak{g}^{\rm lin}_V:=\mathfrak{g}_V\cap\mathfrak{g}^{\rm lin}_{\langle\,\rangle}=\langle Z(\eta^0),\,\eta^0\in\mathcal{L}_V\rangle$. $\mathfrak{g}^{\rm ess}_V$ is a finite-dimensional subalgebra (see Lemma II.2 below) whereas $\mathfrak{g}^{\rm lin}_V$ is an infinite-dimensional abelian ideal of $\mathfrak{g}_V$. We call $\mathfrak{g}^{\rm ess}_V$ the *essential Lie invariance algebra* of the corresponding equation $\mathcal{L}_V$. The ideal $\mathfrak{g}^{\rm lin}_V$ consists of vector fields associated with transformations of *linear superposition* on the solution set of the equation $\mathcal{L}_V$: it is the trivial part of $\mathfrak{g}_V$.

Definition II.3. A subalgebra $\mathfrak{s}$ of $\mathfrak{g}^{\rm ess}_V$ is called *appropriate* if there exists a potential V such that $\mathfrak{s}=\mathfrak{g}^{\rm ess}_V$.

The relation between the equivalence algebra $\mathfrak{g}^\sim$ and $\mathfrak{g}^{\rm ess}_{\langle\,\rangle}$ is given by $\mathfrak{g}^{\rm ess}_{\langle\,\rangle}=\pi_*\mathfrak{g}^\sim$, where π_* is the mapping from $\mathfrak{g}^\sim$ onto $\mathfrak{g}^{\rm ess}_{\langle\,\rangle}$ that is induced by the projection π of the joint space of the variables and the arbitrary element onto the space of the variables only. The differential mapping π_* maps the vector fields $\hat D(\tau)$, $\hat J_{ab}$,

$\hat G(\chi)$, $\hat M(\sigma)$, $\hat I(\rho)$ that span $\mathfrak g^\sim$ to the vector fields $D(\tau)$, J_{ab}, $G(\chi)$, σM, ρI that span $\mathfrak g^{\rm ess}_{\langle\,\rangle}$. The above relation is stronger than the inclusion $\mathfrak g^{\rm ess}_{\langle\,\rangle}\subseteq\pi_*\mathfrak g^\sim$ obtained from the uniform semi-normalization of the class $\mathcal F$.

We denote by $\pi_*G^\sim$ the restrictions of the transformations of $G^\sim$ to the space with local coordinates (t,x,ψ,ψ^*). Then the generators of the one-parameter subgroups of $\pi_*G^\sim$ are just $\pi_*\mathfrak g^\sim$ and $\pi_*G^\sim$ has a natural induced action on $\pi_*\mathfrak g^\sim$ (given by its adjoint action). Since $\mathfrak g^{\rm ess}_{\langle\,\rangle}=\pi_*\mathfrak g^\sim$, then $\pi_*G^\sim$ leaves $\mathfrak g^{\rm ess}_{\langle\,\rangle}$ in variant and acts as a group of Lie algebra automorphisms. The kernel $\mathfrak g^\cap$ is obviously an ideal in $\mathfrak g^{\rm ess}_V$ for any V.

Proposition II.6. *The problem of group classification of (1+n)-dimensional linear Schrödinger equations reduces to the classification of appropriate subalgebras of the algebra $\mathfrak g^{\rm ess}_{\langle\,\rangle}$ with respect to the equivalence relation generated by the action of $\pi_*G^\sim$.*

8 Further properties of Lie symmetry algebras

In order to classify all appropriate subalgebras of $\mathfrak g^{\rm ess}_{\langle\,\rangle}$, we start by describing the action of the transformations of $\pi_*G^\sim$ on the vector fields of $\mathfrak g^{\rm ess}_{\langle\,\rangle}$. For this purpose, we compute pushforwards of the vector fields of $\mathfrak g^{\rm ess}_{\langle\,\rangle}$ by elementary transformations from $\pi_*G^\sim$, that is the images of the vector fields of $\mathfrak g^{\rm ess}_{\langle\,\rangle}$ under the tangent maps of the elements of $\pi_*G^\sim$.

Given $\varphi\in\pi_*G^\sim$ and $Q\in\mathfrak g^{\rm ess}_{\langle\,\rangle}$, the pushforward Q by φ is

$$\tilde Q:=\varphi_*Q=Q(T)\partial_{\tilde t}+Q(X^a)\partial_{\tilde a}+Q(\Psi)\partial_{\tilde\psi}+Q(\Psi^*)\partial_{\tilde\psi^*},$$

where we express the coefficients of $\tilde Q$ in terms of the tilded variables by putting

$(t,x_a,\psi,\psi^*)=\varphi^{-1}(\tilde t,\tilde x_a,\tilde\psi,\tilde\psi^*)$. $\mathcal D(T)$, $\mathcal J(O)$, $\mathcal G(\mathcal X)$, $\mathcal M(\Sigma)$ and $\mathcal I(\Upsilon)$ denote the transformations of the form (6a)–(6b) with $\Phi=0$ and $\varepsilon=1$, for each of the individual parameter functions T, $\mathcal X$, Σ, Υ and O. We take these individual transformations as the elementary transformations that generate the whole of $\pi_*G^\sim$. We record here the results of those push-forward actions by the elementary transformations on the vector fields spanning $\mathfrak g^{\rm ess}_{\langle\,\rangle}$ that do not coincide with the identity transformation:

$$\begin{aligned}
&\mathcal D_*(T)D(\tau)=D(\tilde\tau),\quad \mathcal D_*(T)G(\chi)=G(\tilde\chi),\\
&\mathcal D_*(T)(\sigma M)=\tilde\sigma\tilde M,\quad \mathcal D_*(T)(\rho I)=\tilde\rho\tilde I,\\
&\mathcal J_*(O)G(\chi)=\tilde G(O\chi),\\
&\mathcal G_*(\mathcal X)D(\tau)=\tilde D(\tau)+\tilde G\left(\tau\mathcal X_t-\frac12\tau_t\mathcal X\right)\\
&\qquad\qquad+\left(\frac18\tau_{tt}|\mathcal X|^2-\frac14\tau_t\mathcal X^a\mathcal X^a_t-\frac12\tau\mathcal X^a\mathcal X^a_{tt}\right)\tilde M,\\
&\mathcal G_*(\mathcal X)J_{ab}=\tilde J_{ab}+G(\hat{\mathcal X})-\frac12(\mathcal X^a\mathcal X^b_t-\mathcal X^b\mathcal X^a_t)\tilde M,
\end{aligned}$$

$$\mathcal{G}_*(\mathcal{X})G(\chi) = \tilde{G}(\chi) + \frac{1}{2}(\chi^a\mathcal{X}^a_t - \chi^a_t\mathcal{X}^a)\tilde{M},$$
$$\mathcal{M}_*(\Sigma)D(\tau) = \tilde{D}(\tau) + \tau\Sigma_t\tilde{M}, \quad \mathcal{I}_*(\Upsilon)D(\tau) = \tilde{D}(\tau) + \tau\Upsilon_t\tilde{I},$$

where $\tilde{\tau}(\tilde{t}) = (T_t\tau)(T^{-1}(\tilde{t}))$, $\tilde{\chi}(\tilde{t}) = \chi T_t^{1/2}|_{t=T^{-1}(\tilde{t})}$, $\hat{\mathcal{X}}^a = \mathcal{X}^b$, $\hat{\mathcal{X}}^b = -\mathcal{X}^a$, $\hat{\mathcal{X}}^c = 0$, $c \neq a, b$, $\tilde{\sigma} = \sigma(T^{-1}(\tilde{t}))$, $\tilde{\rho} = \rho(T^{-1}(\tilde{t}))$ and in each push-forward by $\mathcal{D}_*(T)$ we should replace t with its expression in terms of $\tilde{t}$ given by inverting the relation $\tilde{t} = T(t)$. Note that $t = \tilde{t}$ for each of the other push-forward actions. Tildes over vector fields mean that these vector fields are represented in the new variables.

Lemma II.2. $\dim \mathfrak{g}_V^{\rm ess} \leqslant \dfrac{n(n+3)}{2} + 5$ *for any potential* V*, and this upper bound is the least upper bound.*

Proof. Fix a potential V. Then, as in the proof of Lemma 13 in [18], the classifying condition (18) gives a system of linear ordinary differential equations in normal form:

$$\tau_{ttt} = \gamma^{00}\tau_t + \gamma^{01}\tau + \gamma^{0,a+1}\chi^a + \theta^{0ab}\kappa^{ab},$$
$$\chi^c_{tt} = \gamma^{c0}\tau_t + \gamma^{c1}\tau + \gamma^{c,a+1}\chi^a + \theta^{cab}\kappa^{ab},$$
$$\sigma_t = \gamma^{n+1,0}\tau_t + \gamma^{n+1,1}\tau + \gamma^{n+1,a+1}\chi^a + \theta^{n+1,ab}\kappa^{ab},$$
$$\rho_t = -\frac{n}{4}\tau_{tt} + \gamma^{n+2,0}\tau_t + \gamma^{n+2,1}\tau + \gamma^{n+2,a+1}\chi^a + \theta^{n+2,ab}\kappa^{ab},$$

where the coefficients γ^{pq} and θ^{pab}, $p = 0, \dots, n+2$, $q = 0, \dots, n+1$, $a < b$, are functions of t. Thus, $\dim \mathfrak{g}_V^{\rm ess}$ is no greater than the sum of the number of pairs (a, b) with $a < b$ (which come from the rotations) and the number of arbitrary constants in the general solution of the above system. The generators of rotations form a Lie algebra of dimension $n(n-1)/2$ and the number of arbitrary constants that come from the above system is $2n+5$. Thus we have $\dim \mathfrak{g}_V^{\rm ess} \leqslant \dfrac{n(n+3)}{2} + 5$ for any potential V. This upper bound is minimal since $\dim \mathfrak{g}_V^{\rm ess} = \dfrac{n(n+3)}{2} + 5$ for the potential $V = 0$, that is, for the free Schrödinger equation. □

Corollary II.4. $\dim \mathfrak{g}_V^{\rm ess} \cap \langle G(\chi), \sigma M, \rho I\rangle \leqslant 2n + 2$.

Proof. We omit the first equation in the system given in Lemma II.2 and then put $\tau = 0$ and $\kappa^{ab} = 0$ since this corresponds to the absence of the rotations and the transformation $T(t)$. This then gives $2n + 2$ constants. □

Corollary II.5. $\mathfrak{g}_V^{\rm ess} \cap \langle \sigma M, \rho I\rangle = \mathfrak{g}^\cap$ *for any potential* V.

Proof. Similarly, we keep only the last two equations of the system from the proof of Lemma II.2 and set $\tau = \chi^a = 0$ and $\kappa^{ab} = 0$. □

Lemma II.3. *For all* V, $\pi^0_*\mathfrak{g}_V^{\rm ess}$ *is a Lie algebra and* $\dim \pi^0_*\mathfrak{g}_V^{\rm ess} \leqslant 3$*. Moreover,*

$$\pi^0_*\mathfrak{g}_V^{\rm ess} \in \{0, \langle\partial_t\rangle, \langle\partial_t, t\partial_t\rangle, \langle\partial_t, t\partial_t, t^2\partial_t\rangle\} \bmod \pi^0_*G^\sim,$$

where π^0 *denotes the projection on the space of the variable* t *and* $\pi^0_*\mathfrak{s} \subset \pi^0_*\mathfrak{g}^{\rm ess}_{\langle\,\rangle} = \langle\tau\partial_t\rangle$.

Proof. The proof is similar to the one given in [18]: this is just the classification of Lie algebras of vector fields on the real line, as given by Lie. □

9 Group classification of (1+2)-dimensional linear Schrödinger equations

We give a complete group classification of linear Schrödinger equations for the case $n = 2$. The analysis becomes more complicated for larger values of n: for $n = 3$ the computations are much more cumbersome and for greater values the number of subalgebras grows dramatically. We specialize the results of Section 7 to the case $n = 2$ and we obtain that for any V

$$\dim \mathfrak{g}_V^{\rm ess} \leqslant 10, \quad \mathfrak{g}_V^{\rm ess} \cap \langle \sigma M, \rho I\rangle = \langle M, I\rangle = \mathfrak{g}^{\cap},$$
$$2 \leqslant \dim \mathfrak{g}_V^{\rm ess} \cap \langle G(\chi), \sigma M, \rho I\rangle \leqslant 6, \quad \dim \pi^0_* \mathfrak{g}_V^{\rm ess} \leqslant 3.$$

Just as in [18] we introduce for each appropriate subalgebra $\mathfrak{s}$ of $\mathfrak{g}^{\rm ess}_{\langle\,\rangle}$, five $\pi_* G^{\sim}$-invariant integers in order to carry out the group classification (these integers help us in the 'book-keeping' for the calculations):

$$k_0 := \dim \mathfrak{s} \cap \langle \sigma M, \rho I\rangle = \dim \mathfrak{g}^{\cap} = 2,$$
$$k_1 := \dim \mathfrak{s} \cap \langle G(\chi), \sigma M, \rho I\rangle - k_0 \in \{0, 1, 2, 3, 4\},$$
$$k_2 := \dim \mathfrak{s} \cap \langle J,\, G(\chi),\, \sigma M,\, \rho I\rangle - k_1 - k_0 \in \{0, 1\},$$
$$k_3 := \dim \mathfrak{s} - k_2 - k_1 - k_0 = \dim \pi^0_* \mathfrak{s} \in \{0, 1, 2, 3\},$$
$$r_0 := \operatorname{rank}\{\chi \mid \exists\, \sigma, \rho\colon G(\chi) + \sigma M + \rho I \in \mathfrak{s}\} \in \{0, 1, 2\},$$

where π^0 denotes the projection onto the space with local coordinate t. The parameters k_2, k_3 and r_0 are used for labeling the different cases. Note that

$$\dim \mathfrak{s} = k_0 + k_1 + k_2 + k_3 \leqslant 10.$$

Lemma II.4. *A vector field $G(\chi)+\sigma M+\rho I$, where the tuple χ is not proportional to a constant tuple, reduces, up to $\pi_* G^{\sim}$-equivalence, to $G(h\cos t, h\sin t)+\tilde\rho I$ with nonzero $h = h(t)$.*

Proof. We can write the tuple χ as $\chi = (h\cos T, h\sin T)$, where h and T are smooth functions of t with $T_t > 0$. Thus we can apply the equivalence transformation $\tilde t = T$, $\tilde x = T_t^{1/2} x$, $\tilde\psi = exp\left(\frac{i}{8}\frac{T_{tt}}{T_t}|x|^2\right)\psi$. All the vector fields of $\mathfrak{g}^{\rm ess}_{\langle\,\rangle}$ are mapped under $\mathcal{D}_*(T)$ to new vector fields having exactly the same structure as before (this can be seen from the list given in Section 8). In particular, the vector field $G(\chi)+\sigma M+\rho I$ is mapped to $G(\tilde\chi)+\tilde\sigma\tilde M+\tilde\rho\tilde I$, where $\tilde\chi = (\tilde h(\tilde t)\cos\tilde t, \tilde h(\tilde t)\sin\tilde t)$, and $\tilde h(\tilde t) = (T_t^{1/2} h)(T^{-1}(\tilde t))$. Thus we obtain the reduced form $G(\chi) + \sigma M + \rho I$ for this vector field with $\chi = (h(t)\cos t, h(t)\sin t)$. Then we apply the equivalence transformation $\tilde t = t$, $\tilde x = x + \mathcal{X}$, $\tilde\psi = \exp\left(\frac{i}{2}\mathcal{X}_t x\right)\psi$. This gives the push-forward map $\mathcal{G}_*(\mathcal{X})$, where $\mathcal{X} = (\mathcal{X}^1, \mathcal{X}^2)$ is a function of t, which gives

$$\mathcal{G}_*(\mathcal{X})(G(\chi) + \sigma M + \rho I) = \tilde G(\chi) + \frac{1}{2}(\chi^a \mathcal{X}^a_t - \chi^a_t \mathcal{X}^a)\tilde M + \sigma\tilde M + \rho\tilde I.$$

We choose $\mathcal{X}$ so that $\sigma + \frac{1}{2}(\chi^a \mathcal{X}^a_t - \chi^a_t \mathcal{X}^a) = 0$ and we obtain the required form $G(h\cos t, h\sin t) + \rho I$. $\square$

Lemma II.5. *If $G(1,0) + \rho^1 I \in \mathfrak{g}^{\rm ess}_V$, then the vector field $G(t,0) + \rho^2 I$ with $\rho^2 = \int t\rho^1_t\,{\rm d}t$ also belongs to $\mathfrak{g}^{\rm ess}_V$.*

Proof. Suppose that $G(1,0) + \rho^1 I \in \mathfrak{g}^{\rm ess}_V$. We substitute the components of this vector field into the classifying condition (18), which gives $V_1 = -i\rho^1_t$. This equation coincides with the one obtained by evaluating the classifying condition (18) at $\tau = \sigma = 0$, $\chi = t$ and $\rho^2 = \int t\rho^1_t\,{\rm d}t$. $\square$

Summarizing the above yields: any appropriate subalgebra of $\mathfrak{g}^{\rm ess}_{\langle\,\rangle}$ is spanned by

- the basis vector fields M and I of the kernel $\mathfrak{g}^\cap$,
- k_1 vector fields $G(\chi^{p1}, \chi^{p2}) + \sigma^p M + \rho^p I$, $p = 1, \dots, k_1$, with linearly independent tuples $\chi^1, \dots, \chi^{k_1}$,
- k_2 vector fields $J + G(\chi^{01}, \chi^{02}) + \sigma^0 M + \rho^0 I$,
- k_3 vector fields $D(\tau^q) + \kappa^q J + G(\chi^{q1}, \chi^{q2}) + \sigma^q M + \rho^q I$, $q = k_1+1, \dots, k_1+k_3$, with linearly independent $\tau^{k_1+1}, \dots, \tau^{k_1+k_3}$.

In the following we use the notation

$$|x| = \sqrt{x_1^2 + x_2^2}, \quad \phi = \arctan x_2/x_1,$$
$$\omega_1 = x_1\cos t + x_2 \sin t, \quad \omega_2 = -x_1 \sin t + x_2\cos t.$$

Theorem II.8. *A complete list of inequivalent Lie symmetry extensions in the class of $(1+2)$-dimensional linear Schrödinger equations with complex-valued potentials is given below, where U is an arbitrary complex-valued smooth function of its arguments or an arbitrary complex constant that satisfies constraints indicated in the corresponding cases, and the other functions and constants take real values.*

0. $V = V(t,x)$: $\mathfrak{g}^{\rm ess}_V = \mathfrak{g}^\cap = \langle M,\, I\rangle$.
1. $V = U(x_1, x_2)$: $\mathfrak{g}^{\rm ess}_V = \langle M,\, I,\, D(1)\rangle$.
2. $V = U(\omega_1, \omega_2)$: $\mathfrak{g}^{\rm ess}_V = \langle M,\, I,\, D(1) + J\rangle$.
3. $V = |x|^{-2}U(\zeta)$, $\zeta = \phi - 2\beta\ln|x|$, $\beta > 0$, $U_\zeta \neq 0$: $\mathfrak{g}^{\rm ess}_V = \langle M,\, I,\, D(1),\, D(t) + \beta J\rangle$.
4. $V = |x|^{-2}U(\phi)$, $U_\phi \neq 0$: $\mathfrak{g}^{\rm ess}_V = \langle M,\, I,\, D(1),\, D(t),\, D(t^2) - tI\rangle$.
5. $V = U(t,|x|) + (\sigma_t - i\rho_t)\phi$, $\sigma \in \{0, t\} \bmod G^\sim$ and, if $\sigma = 0$, $\rho \in \{0,t\} \bmod G^\sim$: $\mathfrak{g}^{\rm ess}_V = \langle M,\, I,\, J + \sigma M + \rho I\rangle$.
6. $V = U(|x|) + (\alpha_1 - i\beta_1)\phi$: $\mathfrak{g}^{\rm ess}_V = \langle M,\, I,\, J + \alpha_1 tM + \beta_1 tI,\, D(1)\rangle$.
7. $V = |x|^{-2}U$, $U \neq 0$: $\mathfrak{g}^{\rm ess}_V = \langle M,\, I,\, J,\, D(1),\, D(t),\, D(t^2) - tI\rangle$.

8. $V = U(t, x_2) + i\gamma(t)x_1$:
$\mathfrak{g}_V^{\rm ess} = \langle M,\, I,\, G(1,0) - \left(\int \gamma(t)\,\mathrm{d}t\right) I,\, G(t,0) - \left(\int t\gamma(t)\,\mathrm{d}t\right) I\rangle$.

9. $V = U(\zeta) + i\beta x_1$, $\zeta = x_2$:
$\mathfrak{g}_V^{\rm ess} = \langle M,\, I,\, G(1,0) - \beta tI,\, G(t,0) - \frac{\beta}{2}t^2 I,\, D(1)\rangle$.

10. $V = t^{-1}U(\zeta) + i\beta|t|^{-3/2}x_1$, $\zeta = |t|^{-1/2}x_2$:
$\mathfrak{g}_V^{\rm ess} = \langle M,\, I,\, G(1,0) - 2\beta t|t|^{-3/2}I,\, G(t,0) - 2\beta t|t|^{-1/2}I,\, D(t)\rangle$.

11. $V = (t^2+1)^{-1}U(\zeta) + i\beta(t^2+1)^{-3/2}x_1$, $\zeta = (t^2+1)^{-1/2}x_2$:
$\mathfrak{g}_V^{\rm ess} = \langle M,\, I,\, G(1,0) - \beta t(t^2+1)^{-1/2}I,\, G(t,0) + \beta(t^2+1)^{-1/2}I,\, D(t^2+1) - tI\rangle$.

12. $V = Ux_2^{-2}$, $U \neq 0$: $\mathfrak{g}_V^{\rm ess} = \langle M,\, I,\, G(1,0),\, G(t,0),\, D(1),\, D(t),\, D(t^2) - tI\rangle$.

13. $V = U(t,\omega_2) + \frac{1}{4}(h^{-1}h_{tt} - 1)\omega_1^2 + h_t h^{-1}\omega_1\omega_2 - i\rho_t h^{-1}\omega_1$, $h = h(t) \neq 0$:
$\mathfrak{g}_V^{\rm ess} = \langle M,\, I,\, G(h\cos t, h\sin t) + \rho I\rangle$.

14. $V = U(\omega_2) + \frac{1}{4}(\beta - 1)\omega_1^2 - \beta\omega_1\omega_2 - i\alpha\beta\omega_1$, $\beta \neq 0$:
$\mathfrak{g}_V^{\rm ess} = \langle M,\, I,\, G(e^{\beta t}\cos t, e^{\beta t}\sin t) + \alpha e^{\beta t}I,\, D(1) + J\rangle$.

15. $V = U(\omega_2) - \frac{1}{4}\omega_1^2 + (\tilde\alpha - i\alpha)\omega_1$: $\mathfrak{g}_V^{\rm ess} = \langle M,\, I,\, G(\cos t, \sin t) + \tilde\alpha tM + \alpha tI,\, D(1) + J\rangle$.

16. $V = \frac{1}{4}h^{ab}(t)x_a x_b + ih^{0a}(t)x_a$, $h^{12} = h^{21}$:
$\mathfrak{g}_V^{\rm ess} = \langle M,\, I,\, G(\chi^{p1}, \chi^{p2}) + \rho^p I,\, p = 1,\dots,4\rangle$, where $\{(\chi^{p1}(t), \chi^{p2}(t))\}$ is a fundamental set of solutions of the system $\chi^a_{tt} = h^{ab}\chi^b$, and $\rho^p = -\int h^{0a}\chi^{pa}\mathrm{d}t$.

17. $V = \frac{1}{4}\alpha x_1^2 + \frac{1}{4}\beta x_2^2 + i\nu_a x_a$, $\alpha \neq \beta$ or $(\nu_1,\nu_2) \neq (0,0)$:
$\mathfrak{g}_V^{\rm ess} = \langle M,\, I,\, G(\chi^{11}, 0) + \rho^1 I,\, G(\chi^{21}, 0) + \rho^2 I,\, G(0, \chi^{32}) + \rho^3 I,\, G(0, \chi^{42}) + \rho^4 I,\, D(1)\rangle$, where $\{\chi^{11}(t), \chi^{21}(t)\}$ is a fundamental set of solutions of the equation $\chi^1_{tt} = \alpha\chi^1$, $\rho^p = -\nu_1\int \chi^{p1}\mathrm{d}t$, $p = 1,2$, and $\{\chi^{32}(t), \chi^{42}(t)\}$ is a fundamental set of solutions of the equation $\chi^2_{tt} = \beta\chi^2$, $\rho^p = -\nu_2\int \chi^{p2}\mathrm{d}t$, $p = 3,4$.

18. $V = \frac{1}{4}\alpha\omega_1^2 + \frac{1}{4}\beta\omega_2^2 + i\nu_a\omega_a$, $\alpha \neq \beta$ or $(\nu_1,\nu_2) \neq (0,0)$:
$\mathfrak{g}_V^{\rm ess} = \langle M,\, I,\, G(\theta^{p1}\cos t - \theta^{p2}\sin t, \theta^{p1}\sin t + \theta^{p2}\cos t) + \rho^p I,\, D(1) + J,\, p = 1,\dots,4\rangle$, where $(\theta^{p1}(t), \theta^{p2}(t))$ are linearly independent solutions of the system $\theta^1_{tt} - 2\theta^2_t = (1+\alpha)\theta^1$, $\theta^2_{tt} + 2\theta^2_t = (1+\beta)\theta^2$, and $\rho^p = -\int \chi^{pa}h^{0a}\mathrm{d}t$, $p = 1,\dots,4$.

19. $V = 0$:
$\mathfrak{g}_V^{\rm ess} = \langle M,\, I,\, G(1,0),\, G(t,0),\, G(0,1),\, G(0,t),\, J,\, D(1),\, D(t), D(t^2) - tI\rangle$.

Remark II.3. The Lie invariance algebras listed in Theorem II.8 are really maximal for the corresponding potentials if they are $G^{\sim}$-inequivalent to other potentials with larger Lie invariance algebras. We have presented simple necessary and sufficient conditions that provide such inequivalence for some cases. In other cases these conditions are not so obvious. For example, in Cases 9–11 the condition of maximal Lie symmetry extension is ($\beta \neq 0$ or $(\zeta^2 U)_\zeta \neq 0$) and ($U_{\zeta\zeta\zeta} \neq 0$ or

$\operatorname{Im} U_{\zeta\zeta} \neq 0$), which excludes those V that are $G^{\sim}$-equivalent to those in Cases 12 and 16–19. Case 8 is more complicated since in this case to the conditions $\gamma \neq 0$ or $(x_2^2 U)_2 \neq 0$) and $U_{222} \neq 0$ or $\operatorname{Im} U_{22} \neq 0$ we need to add conditions that exclude potentials that are $G^{\sim}$-equivalent to those in Cases 9–11. Similarly, potentials in Cases 13–15 are $G^{\sim}$-inequivalent to those in Cases 16–19 if and only if $U_{\omega_2\omega_2\omega_2} \neq 0$ or $\operatorname{Im} U_{\omega_2\omega_2} \neq 0$. This condition is necessary and sufficient for the maximality of the Lie symmetry extensions given in Cases 14 and 15, and for Case 13 it should be extended to guarantee the exclusion of potentials related to Cases 14 and 15. Schrödinger equations related to Case 16 are inequivalent to the free Schrödinger equation presented in Case 19 if and only if the parameter-functions h's satisfy at least one of the conditions $h^{12} \neq h^{21}$, $h^{12} = h^{21} \neq 0$, $h^{11} \neq h^{22}$, $h^{01} \neq 0$ and $h^{02} \neq 0$. We need additional constraints on h's to exclude, up to $G^{\sim}$-equivalence, potentials of Cases 17 and 18.

Proof. We single out different cases using the values that can be taken by the integers k_3, k_2, r_0. Basis vector fields of a Lie symmetry extension are denoted by

$$Q^s = D(\tau^s) + \kappa^s J + G(\chi^{s1}, \chi^{s2}) + \sigma^s M + \rho^s I,$$

where $0 \leq s \leq \dim \mathfrak{g}_V^{\mathrm{ess}} - 2$, the κ's are real constants, and all the other parameters are real-valued functions of t.

$\boldsymbol{k_2 = r_0 = 0}$. The corresponding Lie symmetry extensions are spanned by $\{Q^s\}$ with linearly independent τ^s. By Lemma II.3, $\langle \pi_*^0 Q^s \rangle$ is an algebra isomorphic to a subalgebra of the algebra $\mathrm{sl}(2,\mathbb{R})$. Hence the group classification in this case reduces to the classification of subalgebras of the algebra $\mathrm{sl}(2,\mathbb{R})$. Varying k_3 we single out the following four subcases.

$k_3 = 0$. This is the general case with no extension, i.e., $\mathfrak{g}_V^{\mathrm{ess}} = \mathfrak{g}^{\cap}$ (Case 0).

$k_3 = 1$. The algebra $\mathfrak{g}_V^{\mathrm{ess}}$ contains a single linearly independent vector field Q^1 with $\tau^1 \neq 0$. Acting on Q^1 by a (push-forward of a) transformation from $\pi_* G^{\sim}$, we can set $\tau^1 = 1$, $\kappa^1 \in \{0, 1\}$ and $\chi^{1a} = \sigma^1 = \rho^1 = 0$, so that the vector field Q^1 reduces to the form $Q^1 = D(1) + \kappa^1 J$. Using the classifying condition (18) on Q^1 with $\kappa^1 = 0$ and $\kappa^1 = 1$ gives differential equations for V whose general solutions are given in Cases 1 and 2, respectively.

$k_3 = 2$. The Lie symmetry extension is given by vector fields Q^1 and Q^2 with linearly independent τ^1 and τ^2. Modulo $\pi_* G^{\sim}$-equivalence, we can set $\tau^1 = 1$ and $\tau^2 = t$ (cf. Lemma II.3) and, as in the previous case, $\chi^{1a} = \sigma^1 = \rho^1 = 0$. Then the condition $[Q^1, Q^2] \in \mathfrak{g}_V^{\mathrm{ess}}$ implies that $D(1) + G(\chi_t^{21}, \chi_t^{22}) + \sigma_t^2 M + \rho_t^2 I = Q^1 + \alpha_1 M + \beta_1 I$, where α_1 and β_1 are real constants. This equation splits into the equations $\kappa^1 = 0$, $\chi_t^{2a} = 0$, $\sigma_t^2 = \alpha_1$, $\rho_t^2 = \beta_1$. Taking these into account, we apply transformations of $\pi_* G^{\sim}$ to Q^1 and Q^2 and take linear combinations of them with M and I so as to be able to assume that $\chi^{2a} = \sigma^2 = \rho^2 = 0$ and $\beta := \kappa^2 \geqslant 0$ while Q^1 is unchanged. Putting the coefficients of Q^1, Q^2 into the classifying condition (18) gives equations for V which then yield Case 3. For the extension to be maximal, we need $\beta \neq 0$ (otherwise we get Case 4 with $k_3 = 3$) and $U_\zeta \neq 0$ (otherwise we get Case 7).

$k_3 = 3$. The algebra $\mathfrak{g}_V^{\rm ess}$ is spanned by the basis elements of the kernel algebra and vector fields Q^q with linearly independent τ^q, $q = 1, 2, 3$. By Lemma II.3, we can assume that $\tau^1 = 1$, $\tau^2 = t$ and $\tau^3 = t^2$. The commutation relations of $\mathfrak{g}_V^{\rm ess}$ imply $\kappa^q = 0$. As in the previous case, we can also set $\chi^{qa} = \sigma^q = \rho^q = 0$, $q = 1, 2$. Since $[Q^3, Q^1] \in \mathfrak{g}_V^{\rm ess}$ and $[Q^3, Q^2] \in \mathfrak{g}_V^{\rm ess}$ we find that $\chi^{3a} = 0$, $\sigma^3_t = \alpha_1$ and $\rho^3_t = \alpha_2$, where α_1 and α_2 are real constants. Up to linear combinations with M and I, the vector field Q^3 reduces to $Q^3 = D(t^2) + \alpha_1 tM + \alpha_2 tI$. The classifying condition (18) applied to the coefficients of Q^1, Q^2 and Q^3 yields the system

$$V_t = 0, \quad tV_t + \frac{1}{2}x_aV_a + V = 0, \quad t^2V_t + tx_aV_a + 2tV = \alpha_1 - i\alpha_2 - i,$$

and the compatibility of these two equations then leads to $\alpha_1 = 0$ and $\alpha_2 = -1$. This gives Case 4 and the extension is maximal if and only if $U_\phi \neq 0$ since otherwise we would obtain Case 7.

$\boldsymbol{k_2 = 1, r_0 = 0}$. The algebra $\mathfrak{g}_V^{\rm ess}$ contains a vector field Q^0 with $\tau^0 = 0$ and $\kappa^0 = 1$, and additional extensions are given by vector fields Q^q with linearly independent τ^q. Since $[Q^0, Q^q] \in \mathfrak{g}_V^{\rm ess}$ we find that $\chi^{qa} = 0$. Applying $\mathcal{G}_*(\mathcal{X})$ with $\mathcal{X}^1 = \chi^{02}$ and $\mathcal{X}^2 = -\chi^{01}$ to $\mathfrak{g}_V^{\rm ess}$, we find that we may put $\chi^{0a} = 0$ so that $Q^0 = J + \sigma^0 M + \rho^0 I$. As in the case $k_2 = r_0 = 0$, we partition Lie symmetry extensions into three subcases depending on the values of k_3.

$k_3 = 0$. If the function σ^0 is constant, then we may assume it to be zero by taking suitable linearly combinations of Q^0 with M; if ρ^0 is constant, then we may assume it to be zero by taking suitable linear combinations with I. If one of these functions is not constant, up to $\pi_*G^\sim$-equivalence it can be taken to be equal to t. Then the classifying condition (18) gives us an equation in V whose general solution is presented in Case 5.

$k_3 = 1$. The vector field Q^1 is reduced to $Q^1 = D(1)$ by $\pi_*G^\sim$-equivalence. As the Lie algebra $\mathfrak{g}_V^{\rm ess}$ is closed with respect to Lie bracket of vector fields, we obtain

$$[Q^1, Q^0] = \sigma^0_t M + \rho^0_t I = \alpha_1 M + \beta_1 I,$$

where α_1 and β_1 are real constants. Equating the components of the vector fields on both the sides of the last equality and then solving the equations that arise, we obtain $\sigma^0 = \alpha_1 t + \alpha_0$ and $\rho^0 = \beta_1 t + \beta_0$, where α_0 and β_0 are real constants, which can be set to zero by taking suitable linear combinations with M and I. The vector field Q^0 is then reduced to $Q^0 = J + \alpha_1 tM + \beta_1 t$. Putting the components of Q^0 and Q^1 into the classifying condition (18) yields two independent equations for V: $V_t = 0$, $x_1V_2 - x_2V_1 = \alpha_1 - i\beta_1$, and these give Case 6.

$k_3 \geqslant 2$. The algebra $\mathfrak{g}_V^{\rm ess}$ necessarily contains the vector fields M and I from $\mathfrak{g}^\cap$ and vector fields Q^0 and Q^q, $q = 1, 2$, with linearly independent τ^1 and τ^2. Up to $\pi_*G^\sim$-equivalence the vector fields Q^1 and Q^2 are reduced to the form $Q^1 = D(1)$ and $Q^2 = D(t)$. As the commutators $[Q^q, Q^0]$ belong to $\mathfrak{g}_V^{\rm ess}$, we have

$$[Q^1, Q^0] = \sigma^0_t M + \rho^0_t I = \alpha_1 M + \beta_1 I,$$
$$[Q^2, Q^0] = t\sigma^0_t M + t\rho^0_t I = \alpha_2 M + \beta_2 I,$$

where α_j and β_j, $j=1,2$, are real constants. The above commutation relations give the system $\sigma^0_t=\alpha_1$, $t\sigma^0_t=\alpha_2$, $\rho^0_t=\beta_1$, $t\rho^0_t=\beta_2$, which implies that σ^0 and ρ^0 are constants. Therefore, by taking linear combinations with M and I, they can be set to be equal to zero. Put the coefficients of Q^0, Q^1 and Q^2 into the classifying condition (18) and we obtain the system $V_t=0$, $x_1V_2-x_2V_1=0$, $x_aV_a+2V=0$, whose general solution is $V=|x|^{-2}U$. However, any such potential admits the Lie symmetry vector field $Q^3=D(t^2)-tI$. This means that in fact $k_3=3$. For the Lie symmetry extension to be maximal, we require $U\neq0$ and we obtain Case 7.

$\boldsymbol{r_0=1}$. Suppose that $k_1=0$. In this case we have at least one vector field $Q^1=G(\chi^{11},\chi^{12})+\sigma^1M+\rho^1I$ in $\mathfrak g^{\rm ess}_V$ with a nonzero tuple (χ^{11},χ^{12}). A further Lie symmetry can be another vector field of the same form or a vector field Q^q, $q=k_1+1,\dots,k_1+k_3$, with linearly independent τ^q. We have two cases: either (χ^{11},χ^{12}) is proportional to a constant tuple or it is not proportional to a constant tuple.

1. Let (χ^{11},χ^{12}) be proportional to a constant tuple. Up to $\pi_*G^\sim$-equivalence we can set $(\chi^{11},\chi^{12})=(1,0)$, $\sigma^1=0$ and thus Q^1 reduces to the form $G(1,0)+\rho^1I$. Lemma II.5 implies that the algebra $\mathfrak g^{\rm ess}_V$ then also contains the vector field $G(t,0)+\rho^2I$ with $\rho^2=\int t\,\rho^1_t{\rm d}t$. Then, substituting the components of the vector fields into the classifying condition (18) yields two equations for V: $V_1=-i\rho^1_t$ and $tV_1=-it\rho^1_t$, whose general solution is

$$V=U(t,x_2)+i\gamma(t)x_1, \tag{19}$$

where U is a complex-valued smooth function of (t,x) and $\gamma=-\rho^1_t$ is a real-valued smooth function of t. Thus, any equation from the class $\mathcal F$ with potential of the form (19) is invariant with respect to the vector fields $Q^1_\gamma=G(1,0)-\big(\int\gamma(t)\,{\rm d}t\big)\,I$ and $Q^2_\gamma=G(t,0)-\big(\int t\gamma(t)\,{\rm d}t\big)\,I$. The function U satisfies the conditions $U_{222}\neq0$ or $\operatorname{Im}U_{22}\neq0$ since otherwise $r_0=2$ (see the case $r_0=2$ below).

Consider the subclass $\mathcal F_{{\rm l}x_1}$ of (1+2)-dimensional linear Schrödinger equations with potentials of the form (19) constrained by the above conditions for U. We can reparameterize the class $\mathcal F_{{\rm l}x_1}$, assuming the parameter functions U and γ as arbitrary elements instead of V. The equivalence groupoid $\mathcal G^\sim_{{\rm l}x_1}$ of $\mathcal F_{{\rm l}x_1}$ can be singled out from the equivalence groupoid $\mathcal G^\sim$ for the entire class $\mathcal F$, as given in Theorem II.5. A point transformation connects two equations $\mathcal L_V$ and $\mathcal L_{\tilde V}$ from the class $\mathcal F_{{\rm l}x_1}$ if and only if it is of the form (6a)–(6b), where T is a fractionally linear function of t, the matrix O is diagonal with $O^{11},O^{22}=\pm1$, and $\mathcal X^1=\nu_1T+\nu_0$ with arbitrary real constants ν_1 and ν_0. The corresponding tuples of arbitrary elements are related in the following way:

$$\begin{aligned}\tilde U={}&\frac{\hat U}{|T_t|}+\frac{\varepsilon'O^{22}}{2|T_t|^{1/2}}\left(\frac{\mathcal X^2_t}{T_t}\right)_t x_2+\frac{\Sigma_t-i\Upsilon_t}{T_t}\\&-\frac{(\mathcal X^2_t)^2+2iT_{tt}}{4{T_t}^2}-\frac{{\nu_1}^2}{4}-i\tilde\gamma\mathcal X^1,\\\tilde\gamma={}&\frac{\varepsilon'O^{11}}{|T_t|^{3/2}}\gamma.\end{aligned}\tag{20}$$

The equivalence group $G^{\sim}_{\mathrm{l}x_1}$ of the class $\mathcal{F}_{\mathrm{l}x_1}$ consists of the point transformations of the form (6a)–(6b) prolonged to the arbitrary elements according to (20), where the parameters satisfy, additionally to all the above conditions for elements of $\mathcal{G}^{\sim}_{\mathrm{l}x_1}$, the constraint $\Phi = 0$. For each equation $\mathcal{L}_V$ from the class $\mathcal{F}_{\mathrm{l}x_1}$, we consider the group G^{unf}_V of its point symmetry transformations of the form (6a)–(6b), where $T = t$, O is the identity matrix, $\mathcal{X}^1 = \nu_1 t + \nu_0$, $\mathcal{X}^2 = 0$, $\Sigma = \frac{1}{4}{\nu_1}^2 t + \mu_0$, $\Upsilon_t = \gamma\mathcal{X}^1$, μ_0, ν_1 and ν_0 are arbitrary constants, and $\Phi = \Phi(t,x)$ is an arbitrary solution of the initial equation. The corresponding Lie algebra $\mathfrak{g}^{\mathrm{unf}}_V$ is spanned by the vector fields M, I, Q^1_γ, Q^2_γ and $Z(\eta^0)$ with η^0 running through the solution set of $\mathcal{L}_V$. Since all the conditions of Definition II.2 are satisfied, the class $\mathcal{F}_{\mathrm{l}x_1}$ is weakly uniformly semi-normalized with respect to the entire equivalence group $G^{\sim}_{\mathrm{l}x_1}$ and the family $\mathcal{N} = \{G^{\mathrm{unf}}_V\}$. Further, this is the only way of treating the class $\mathcal{F}_{\mathrm{l}x_1}$ within the framework of uniform semi-normalization, and for any equation $\mathcal{L}_V$ from this class we have $G^{\sim}_{\mathrm{l}x_1}|_{(t,x,\psi,\psi^*)} \cap G^{\mathrm{unf}}_V = G^{\cap}_{\mathrm{l}x_1} = G^{\cap} \neq \{\mathrm{id}\}$. In view of the Proposition II.4, the group classification of the class $\mathcal{F}_{\mathrm{l}x_1}$ reduces to the classification of $\pi_*G^{\sim}_{\mathrm{l}x_1}$-inequivalent appropriate subalgebras of the algebra $\pi_*\mathfrak{g}^{\sim}_{\mathrm{l}x_1} = \langle D(1), D(t), D(t^2), G(0,\chi^2), \sigma M, \rho I\rangle$. Put $\mathfrak{g}^{\mathrm{ext}}_V := \mathfrak{g}^{\mathrm{ess}}_V \cap \pi_*\mathfrak{g}^{\sim}_{\mathrm{l}x_1}$.

Since $r_0 = 1$, any Lie symmetry extension in the subclass $\mathcal{F}_{\mathrm{l}x_1}$ is spanned by vector fields of the form $D(\tau) + G(0,\chi^2) + \sigma M + \rho I$, where τ runs through a set of linearly independent quadratic polynomial in t. This shows that the computation of inequivalent Lie symmetry extensions reduces to the classification of subalgebras of the algebra $\mathrm{sl}(2,\mathbb{R})$. We have the following subcases in terms of k_3:

$k_3 = 0$. There is no additional extension, so we have Case 8.

$k_3 = 1$. The algebra $\mathfrak{g}^{\mathrm{ext}}_V$ necessarily contains a vector field Q^3 with nonzero τ^3. Up to $\pi_*G^{\sim}_{\mathrm{l}x_1}$-equivalence, Q^3 can be taken to be one of the following vector fields: $D(1)$, $D(t)$ and $D(t^2+1) - tI$. With V given as in (19), we put the coefficients of Q^3 into the classifying condition (18) for each of the possible choices of Q^3, and this yields equations for U and γ. The solutions of these equations give us Cases 9, 10 and 11, respectively.

$k_3 \geqslant 2$. Apart from the vector fields M and I, the algebra $\mathfrak{g}^{\mathrm{ext}}_V$ also contains at least two other vector fields Q^q, $q = 3, 4$, with linearly independent τ^3 and τ^4. Up to $\pi_*G^{\sim}_{\mathrm{l}x_1}$-equivalence and linear combinations with M and I, the vector fields Q^q reduce to $Q^3 = D(1)$ and $Q^4 = D(t)$. Putting the coefficients of these vector fields into the classifying condition (18) for the potential (19) of V, we obtain $U_t = 0$, $x_2U_2 + 2U = 0$ and $\gamma = 0$, and thus $V = \tilde{U}x_2^{-2}$ where $\tilde{U}$ is a nonzero constant. However, the equation $\mathcal{L}_V$ with this V possesses one more Lie symmetry vector field $Q^5 = D(t^2) - tI$. This shows that the Lie symmetry extension with $k_3 = 2$ is not maximal. We obtain Case 12.

2. Suppose now that the tuple (χ^{11}, χ^{12}) is not proportional to a tuple of constants.

If we have no additional extension, then the algebra $\mathfrak{g}^{\mathrm{ess}}_V$ is spanned by the vector fields M, I and $Q^1 = G(\chi^{11}, \chi^{12}) + \sigma^1 M + \rho^1 I$.[1] By Lemma II.4 the vector

[1]In contrast to the previous case, where the tuple (χ^{11}, χ^{12}) is proportional to a tuple of constants, in the present case under the algebra $\mathfrak{g}^{\mathrm{ess}}_V$ contains, up to linear dependence, only one vector field of the form as Q^1. Indeed, suppose that this algebra contains one more vector

field Q^1 can be reduced to the form $Q^1 = G(h\cos t, h\sin t) + \rho I$, where h and ρ are smooth functions of t with $h \neq 0$. Substituting the components of Q^1 into the classifying condition (18) yields the equation

$$V_1\cos t + V_2\sin t = \frac{1}{2}(h^{-1}h_{tt} - 1)\omega_1 + h^{-1}h_t\omega_2 - ih^{-1}\rho_t.$$

The solution of this equation gives Case 13.

If there is an extension, the algebra $\mathfrak{g}^{\rm ess}_V$ also contains a vector field Q^2 with nonzero τ^2, which takes, up to $\pi_*G^{\sim}$-equivalence, the form $Q^2 = D(1) + \kappa^2 J$, $\kappa^2 \in \{0,1\}$. The condition $[Q^2, Q^1] \in \mathfrak{g}^{\rm ess}_V$ implies that

$$G(\chi^{11}_t, \chi^{12}_t) + \kappa^2 G(\chi^{12}, -\chi^{11}) + \sigma^1_t M + \rho^1_t = \beta_1 Q^1 + \beta_2 M + \beta_3 I,$$

where $\beta_j,\ j = 1,2,3$ are real constants. Equating the corresponding components of the vector fields on both the sides of this equation, we obtain the system

$$\begin{aligned} &\chi^{11}_t + \kappa^2\chi^{12} = \beta_1\chi^{11}, \quad \chi^{12}_t - \kappa^2\chi^{11} = \beta_1\chi^{12},\\ &\sigma^1_t = \beta_1\sigma^1 + \beta_2, \quad \rho^1_t = \beta_1\rho^1 + \beta_3. \end{aligned} \tag{21}$$

$\kappa^2 = 0$ is not possible for otherwise the tuple (χ^{11}, χ^{12}) would be proportional to a constant tuple, which is a contradiction. Hence $\kappa^2 = 1$ and $Q^2 = D(1) + J$. Integrating the systems (21), rearranging and taking suitable linear combinations with elements from the kernel, we obtain the Q^1, which now depends on β_1.

If $\beta := \beta_1 \neq 0$, then

$$Q^1 = G(e^{\beta t}\cos t, e^{\beta t}\sin t) + \tilde\alpha e^{\beta t}M + \alpha e^{\beta t}I$$

with real constants $\tilde\alpha$ and α. Acting on vector fields from $\mathfrak{g}^{\rm ess}_V$ by transformations of $\mathcal{G}_*(\nu\sin t, \nu\cos t)$ with $\nu = 2\tilde\alpha/\beta$, we find that we may put $\tilde\alpha = 0$. Substituting the components of the vector fields Q^1 and Q^2 into the classifying condition (18) gives the system

$$V_t + x_1V_2 - x_2V_1 = 0, \quad V_1\cos t + V_2\sin t = \frac{1}{2}(\beta^2 - 1)\omega_1 + \beta\omega_2 - i\beta\alpha.$$

The solution of this system gives Case 14.

If $\beta_1 = 0$, then the solution of the system (21) gives, after taking suitable linear combinations with M and I, $Q^1 = G(\cos t, \sin t) + \beta_2 tM + \beta_3 tI$. In this case no further simplifications are possible. Renaming β_2 and β_3 as $\beta_2 := \tilde\alpha$

field $Q^2 = G(\chi^{21}, \chi^{22}) + \sigma^2 M + \rho^2 I$, where the tuple (χ^{21}, χ^{22}) is not linearly dependent with (χ^{11}, χ^{12}). Then the case condition $r_0 = 1$ implies that $\chi^{2a} = \lambda\chi^{1a}$ where λ is a nonconstant function of t. Successively evaluating the classifying condition (18) at the vector fields Q^1 and Q^2, we derive two equations for V, for which the difference of the second equation and the first equation multiplied by λ leads, in view of the proportionality of the tuples, to the condition $(\lambda_t\chi^{1a}_t + \frac{1}{2}\lambda_{tt}\chi^{1a})x_a + \sigma^2_t - \lambda\sigma^1_t - i(\rho^2_t - \lambda\rho^1_t) = 0$. Splitting it with respect to x_a and integrating the obtained equations, we derive $\chi^{1a} = c_a|\lambda_t|^{-1/2}$, where c_a are real constants, which gives a contradiction.

and $\beta_3 := \alpha$, we obtain, on substituting the coefficients of Q^1 and Q^2 into the classifying condition (18), the following two equations for V:

$$V_t + x_2V_1 - x_1V_2 = 0, \quad V_1\cos t + V_2\sin t = -\frac{1}{2}\omega_1 + \tilde\alpha - i\alpha.$$

The solution of these equations gives Case 15.

Next if $k_2 = 1$ then then the algebra $\mathfrak{g}^{\rm ess}_V$ necessarily contains vector fields of the form $Q^0 = J + G(\chi^{01},\chi^{02}) + \sigma^0M + \rho^0I$ and $Q^1 = G(\chi^{11},\chi^{12}) + \sigma^1M + \rho^1I$, where $(\chi^{11},\chi^{12}) \neq (0,0)$. The commutator of these vector fields

$$[Q^0,Q^1] = G(\chi^{12},-\chi^{11}) + \frac{1}{2}(\chi^{01}\chi^{11}_t + \chi^{02}\chi^{12}_t - \chi^{01}_t\chi^{11} - \chi^{02}_t\chi^{12})M$$

should belong to $\mathfrak{g}^{\rm ess}_V$ and hence $r_0 = 2$. Therefore, the case $k_2 = r_0 = 1$ is impossible.

Further classification. The case not considered so far is $r_0 = 2$.

Since $r_0 = 2$, the algebra $\mathfrak{g}^{\rm ess}_V$ contains at least two vector fields of the form $Q^a = G(\chi^{a1},\chi^{a2}) + \sigma^aM + \rho^aI$, $a = 1, 2$, where $\chi^{11}\chi^{22} - \chi^{12}\chi^{21} \neq 0$. Substituting the components of the vector fields Q^a into the classifying condition (18) leads to the following system for V:

$$\chi^{ab}V_b = \frac{1}{2}\chi^{ab}_{tt}x_b + \sigma^a_t - i\rho^a_t.$$

This system can be written as $V_a = \frac{1}{2}h^{ab}(t)x_b + \tilde h^{0a}(t) + ih^{0a}(t)$, where all h's are real-valued functions of t that satisfy the conditions

$$\chi^{ab}_{tt} = \chi^{ac}h^{cb}, \quad \sigma^a_t = \chi^{ac}\tilde h^{0c}, \quad \rho^a_t = -\chi^{ac}h^{0c}.$$

Since $V_{12} = V_{21}$, the matrix (h^{ab}) is symmetric. Thus the potential V is a quadratic polynomial in x_1 and x_2 with coefficients depending on t,

$$V = \frac{1}{4}h^{ab}(t)x_ax_b + \tilde h^{0b}(t)x_b + ih^{0b}(t)x_b + \tilde h^{00}(t) + ih^{00}(t), \tag{22}$$

where the functions $\tilde h^{0b}$, $\tilde h^{00}$ and h^{00} can be chosen to be zero, by $G^{\sim}$-equivalence.

Next consider the subclass $\mathcal{F}_{\rm q}$ of equations from the class $\mathcal{F}$ with potentials of the form (22). Theorem II.5 implies that the subclass $\mathcal{F}_{\rm q}$ is uniformly semi-normalized with respect to linearly superposition of solutions, as in the case of the entire class $\mathcal{F}$, and its equivalence group coincides with the equivalence group $G^{\sim}$ of $\mathcal{F}$. Further group classification of the subclass $\mathcal{F}_{\rm q}$ depends on whether $k_2 = 0$ or $k_2 = 1$. The values allowed for k_2 partition the subclass $\mathcal{F}_{\rm q}$ into two subclasses: $\mathcal{F}^0_{\rm q}$ and $\mathcal{F}^1_{\rm q}$, the first corresponding to $k_2 = 0$ and the second to $k_2 = 1$. These subclasses are uniformly semi-normalized with respect to the equivalence group $G^{\sim}$ of the whole class $\mathcal{F}$. The classifying condition (18) implies that the subclass $\mathcal{F}^1_{\rm q}$ is singled out from the class $\mathcal{F}_{\rm q}$ by the conditions $h^{12} = h^{21} = 0$, $h^{11} = h^{22}$, $h^{01} = 0$ and $h^{02} = 0$, and the subclass $\mathcal{F}^0_{\rm q}$ is then singled out by requiring that one of these conditions fails to hold. By Corollary II.1 the above condition means that each equation in the subclass $\mathcal{F}^1_{\rm q}$ is $G^\sim$-equivalent to the (1+2)-dimensional free Schrödinger equation (with $V=0$). In other words, the subclass $\mathcal{F}^1_{\rm q}$ is the $G^\sim$-orbit of the free Schrödinger equation, which gives Case 19.

Finally, we come to the subclass $\mathcal{F}^0_{\rm q}$, for which $k_2=0$ and $r_0=2$. Up to $G^\sim$-equivalence, the canonical form of potentials for equations from the subclass $\mathcal{F}^0_{\rm q}$ is

$$V=\frac14 h^{ab}(t)x_ax_b+ih^{0b}(t)x_b. \tag{23}$$

Substituting this V into the classifying condition (18) with $\tau=0$ and $\kappa=0$, splitting with respect to x_a and taking real and imaginary parts, we obtain the following system for $(\chi^1,\chi^2,\sigma,\rho)$:

$$\chi^a_{tt}=h^{ab}\chi^b,\quad \sigma_t=0,\quad \rho_t=-h^{0b}\chi^b. \tag{24}$$

This system admits a fundamental set of solutions $(\chi^{p1},\chi^{p2},0,\rho^p)$, $p=1,\dots,4$, $(0,0,1,0)$ and $(0,0,0,1)$, where the tuples (χ^{p1},χ^{p2}) are linearly independent, and $\rho^p=-\int h^{0b}\chi^{pb}{\rm d}t$. Thus, the algebra $\mathfrak{g}^{\rm ess}_V$ contains the four vector fields $Q^p=G(\chi^{p1},\chi^{p2})+\rho^pI$. The last two solutions correspond to the vector fields M and I from the kernel. If there are no other Lie symmetry extensions, we obtain Case 16.

If there are further extensions, the algebra $\mathfrak{g}^{\rm ess}_V$ must contain a vector field Q^5 with nonzero τ^5. Up to $G^\sim$-equivalence, the parameter-function τ^5 may be taken to be $\tau^5=1$ and $\kappa^5\in\{0,1\}$, and the canonical form (23) for V is the same. We substitute the components of Q^5 together with the form (23) for V into the classifying condition (18) and split with respect to different powers of x_a. We find that the tuple $(\chi^{51},\chi^{52},\sigma^5,\rho^5)$ satisfies the system (24), and the parameter-functions h^{ab} and h^{0b} satisfy the following equations:

$$\begin{gathered}h^{11}_t+2\kappa^5h^{12}=0,\quad h^{12}_t+\kappa^5(h^{22}-h^{11})=0,\quad h^{22}_t-2\kappa^5h^{12}=0,\\ h^{01}_t+\kappa^5h^{02}=0,\quad h^{02}_t-\kappa^5h^{01}=0.\end{gathered} \tag{25}$$

Therefore, up to linear combinations of Q^5 with Q^p, M and I, we can assume that $\chi^{5a}=\sigma^5=\rho^5=0$. This reduces Q^5 to $D(1)+\kappa^5J$.

If $\kappa^5=0$, the system (25) implies that all h^{ab} and h^{0a} are constants. Up to rotations, we can reduce the matrix (h^{ab}) to a diagonal matrix $\operatorname{diag}(\alpha,\beta)$. The maximality of the Lie symmetry extension requires $\alpha\neq\beta$ or $(\nu_1,\nu_2)\neq(0,0)$ with $\nu_a:=h^{0a}$, and this gives Case 17.

If $\kappa^5\neq0$, up to time-translations, the general solution of the system (25) is $h^{11}=\alpha\cos^2t+\beta\sin^2t$, $h^{12}=h^{21}=(\alpha-\beta)\cos t\sin t$, $h^{22}=\alpha\sin^2t+\beta\cos^2t$, $h^{01}=\nu_1\cos t-\nu_2\sin t$ and $h^{02}=\nu_1\sin t+\nu_2\cos t$, where α, β, ν_1 and ν_2 are real constants. We have a maximal Lie symmetry extension if these constants satisfy the conditions $\alpha\neq\beta$ or $(\nu_1,\nu_2)\neq(0,0)$. Rewriting the potential V in terms of ω_a leads to Case 18.

We now show that the dimension of further Lie symmetry extensions cannot exceed one. Suppose that this is not the case. Then the algebra $\mathfrak{g}^{\rm ess}_V$ contains a two-dimensional subalgebra spanned by vector fields Q^5 and Q^6 with linearly independent τ^5 and τ^6. Up to $\pi_*G^{\sim}$-equivalence, we can assume that $\tau^5 = 1$ and $\tau^6 = t$. Arguing as above, the vector field Q^5 takes the canonical form $Q^5 = D(1) + \kappa^5 J$. The condition $[Q^5,\ Q^6] \in \mathfrak{g}^{\rm ess}_V$ implies that $\kappa^5 = 0$. Putting the coefficients of Q^5 and Q^6 into the classifying condition (18) for the potential V of the form (23) and splitting with respect to different powers of x_a we obtain the equations $h^{ab} = h^{0b} = 0$, which contradicts the condition singling out the subclass $\mathcal{F}^0_{\rm q}$ from the class $\mathcal{F}_{\rm q}$. □

10 Conclusion

The equivalence groupoid $\mathcal{G}^{\sim}$ of the class $\mathcal{F}$, computed with direct method, has the property of being uniformly semi-normalized. This motivates the use of the algebraic method of group classification. It allows us to single out the equivalence group $G^{\sim}$ of the class $\mathcal{F}$ from $\mathcal{G}^{\sim}$ and to obtain the associated equivalence algebra as the set of infinitesimal generators of one-parameter subgroups of the group $G^{\sim}$. Working within the framework of the algebraic method, we reduce the group classification of equations from the class $\mathcal{F}$ to the classification of certain low-dimensional subalgebras of the equivalence algebra.

After the analysis of the determining equations for Lie symmetries, we construct the linear span $\mathfrak{g}_{\langle\,\rangle}$ of the vector fields from the maximal Lie invariance algebras of equations from the class $\mathcal{F}$. This linear span can be represented as the semi-direct sum $\mathfrak{g}_{\langle\,\rangle} = \mathfrak{g}^{\rm ess}_{\langle\,\rangle} \mathbin{\ltimes} \mathfrak{g}^{\rm lin}_{\langle\,\rangle}$ of the so-called essential subalgebra $\mathfrak{g}^{\rm ess}_{\langle\,\rangle}$ and an ideal $\mathfrak{g}^{\rm lin}_{\langle\,\rangle}$, related to the transformations defined by the linear superposition of solutions,

For each equation $\mathcal{L}_V$ from the class $\mathcal{F}$ the representation in terms of vector fields for $\mathfrak{g}_{\langle\,\rangle}$ induces a similar representation for the maximal Lie invariance algebra $\mathfrak{g}_V$ of $\mathcal{L}_V$, $\mathfrak{g}_V = \mathfrak{g}^{\rm ess}_V \mathbin{\ltimes} \mathfrak{g}^{\rm lin}_V$.

We have carried out the complete group classification of the class of (1+2)-dimensional linear Schrödinger equations with complex-valued potentials. All inequivalent families of potentials possessing proper Lie symmetry extensions are given in the list in Theorem 16.

Unlike the case of a single space variable ($n = 1$), the vector field of infinitesimal rotations in two dimensions is also a symmetry. One also has to deal with algebras of greater dimension. These two phenomena complicate the analysis of the problem. In our approach we have introduced three integers $k_3 \in \{0, 1, 2, 3\}$, $k_2 \in \{0, 1\}$ and $r_0 \in \{0, 1, 2\}$ to help with labeling the various cases that arise. These integers characterize the dimensions of parts of the corresponding essential subalgebra that are related to generalized scalings, rotations and generalized Galilean boosts and are invariant with respect to the action of the equivalence transformations on the vector fields of the symmetry algebra. Note that not all values of the tuple (k_3, k_2, r_0) are allowed since the corresponding subalgebras may not give a maximal Lie symmetry extension or may not exist at all.

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Paper III

Group classification of multidimensional nonlinear Schrödinger equations

Group classification of multidimensional nonlinear Schrödinger equations

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We compute the equivalence groupoid and the equivalence group of the class $\mathcal{N}$ of generalized multidimensional Schrödinger equations with variable mass and show that this class is not normalized. We then partition this class into two disjoint normalized subclasses and derive their corresponding equivalence groups. Restricting to the case of constant mass equal to one, we characterize the point transformations of the subclasses of the class $\mathcal{N}$ with respect to specific values of the arbitrary elements, in particular we do this for the class $\mathcal{V}$ of multidimensional nonlinear Schrödinger equations with potentials and modular nonlinearities. This class also turns out not to be normalized. We partition it into three normalized subclasses, and this allows us to apply the algebraic method and solve each subclass completely for space dimension two. The group classification in each involves three integers that are invariant with respect to the adjoint action of the equivalence transformations. As a result, a full list of Lie symmetry extensions together with their corresponding families of potentials in the class $\mathcal{V}$ is presented.

1 Introduction

In this paper we look at and give a group-theoretic classification of certain types of nonlinear Schrödinger equations. Such equations appear in many contexts where they describe models of nonlinear physical systems such as in hydrodynamics, nonlinear optics, nonlinear acoustics, quantum condensates, heat pulses in solids, plasma physics, quantum mechanics and biomolecular dynamics, to name just a few areas (see for instance [1], [2], [4], [18]).

The Lie symmetries of linear Schrödinger equations were studied in the early 1970's by Niederer ([31], [32]): the free Schrödinger equation, the harmonic oscillator and later the group classification problem for linear Schrödinger equations with arbitrary real-valued potential $V = V(t, x)$ in several dimensions [33]. The Lie symmetry analysis of Schrödinger equations with arbitrary real-valued time independent potential $V = V(x)$ in three dimensions was studied by Boyer in [5], where he obtained a generalization of Niederer's results. In 1977, Miller ([29]) considered the symmetry properties and separation of variables for the class of (1+1)-dimensional linear Schrödinger equations. Linear

Schrödinger equations were taken up recently in ([26], [27]), where a full classification of linear Schrödinger equations with potential (both real and complex) was obtained for $(1{+}n)$-dimensions with $n \leqslant 2$.

Nonlinear Schrödinger equations of the form $i\psi_t + \psi_{aa} + F(t, x, \psi, \psi^*) = 0$ were treated in [6] and all equations that invariant under subalgebras of the Lie symmetry algebra of the (1+1)-dimensional free Schödinger equations were constructed. In [14] some exact similarity solutions for nonlinear Schrödinger equations of the form $i\psi_t + \frac{1}{m}\psi_{aa} + F(\psi) = 0$, where m is nonzero constant, were obtained in three spatial dimensions. In the papers [15], [16], [17], [18] the symmetry groups of nonlinear Schrödinger equations with variable coefficients were computed, and exact solutions and group invariant solutions for this class were obtained. The conditional symmetry of equations of the form $i\psi_t + \lambda\psi_{aa} + F(|\psi|)\psi = 0$ were systematically analyzed in [9], and in [11], [12], [13],

Lie symmetries of systems of nonlinear Schrödinger equations were computed (using computer software for solving overdetermined systems of PDE's) in [44], [45], [46]. Exact solutions of Schrödinger equations with inhomogeneous nonlinearities were investigated in [2] using Lie symmetry method.

In the framework of group classification, nonlinear Schrödinger equations with harmonic oscillator type potential by the algebraic method were treated in [19], [20]. Using same approach, Schrödinger equations with potentials and power nonlinearities in (1+1)-dimensions were studied by Popovych *et al* in [42]. Admissible point transformations for the class of $(1+1)$-dimensional nonlinear Schrödinger equations with potentials and modular nonlinearities were studied in [40] as well as for the class of $(1+1)$-dimensional cubic Schrödinger equation with potential [41]. In addition to the above results, the complete group classification of multidimensional nonlinear Schrödinger equations of the form $i\psi_t + \Delta\psi + F(\psi, \psi^*) = 0$ was considered in [34]. The algebraic method for group classification has since been enhanced and become a powerful and efficient approach for the complete group classification of differential equations. The effectiveness of this method relies on its applicability to the group classification of normalized and non-normalized classes of differential equations [40], [41], [42], [38].

In this paper we build on the results presented in [38] to the several space dimensions, applying and extending the algebraic technique and generalizing the results. The concepts and basic notions of the group analysis of differential equations used in this paper, such as class of differential equations, point transformations between systems of differential equations, equivalence groupoid, equivalence group, equivalence algebra and normalization properties of a class of differential equations can be found in [25], [26], [38].

We begin with the study of point transformations for the class $\mathcal{N}$ of generalized $(1{+}n)$-dimensional ($n \in \mathbb{N}$) Schrödinger equations with variable mass of the form

$$i\psi_t + G(t, x, \psi, \psi^*, \nabla\psi, \nabla\psi^*)\psi_{aa} + F(t, x, \psi, \psi^*, \nabla\psi, \nabla\psi^*) = 0, \qquad (1)$$

where t and $x = (x_1, \ldots, x_n)$ are real independent variables, ψ is the unknown complex-valued function of t and x, G and F are smooth complex-valued functions of their arguments with $G \neq 0$. Throughout the paper, the notation β^* means the conjugate of a complex value β. Subscripts of functions denote differentiation with respect to the corresponding variables. The indices a and b run from 1 to n, the indices μ and ν run from 0 to n, $x_0 := t$, and we sum over repeated indices. The total derivative operator D_μ is defined as $\mathrm{D}_\mu = \partial_\mu + \psi_\mu\partial_\psi + \psi^*_\mu\partial_{\psi^*} + \ldots$.

Choosing specific forms of G and F we single out different subclasses from the class $\mathcal{N}$. An important subclass is the class $\mathcal{F}$ singled out from $\mathcal{N}$ by the condition $G = 1$, consisting of equations of the form

$$i\psi_t + \psi_{aa} + F(t, x, \psi, \psi^*, \nabla\psi, \nabla\psi^*) = 0. \qquad (2)$$

The set of admissible transformations of the class $\mathcal{F}$ is derived from the knowledge of the admissible transformations for the superclass $\mathcal{N}$. We then constrain the arbitrary elements G and F to $G=1$ and $F_{\psi_a}=F_{\psi_a^*}=0$, and obtain the class $\mathcal{F}^1$ whose equations are of the general form

$$i\psi_t+\psi_{aa}+F(t,x,\psi,\psi^*)=0. \tag{3}$$

A description of the point transformations of this class is deduced from the point transformations for the larger classes containing it.

Further restricting F to be $F=S(t,x,\rho)\psi$ with $\rho:=|\psi|$, we obtain the class $\mathcal{S}$ of multidimensional nonlinear Schrödinger equations of the form

$$i\psi_t+\psi_{aa}+S(t,x,\rho)\psi=0,\quad S_\rho\neq 0, \tag{4}$$

where S is an arbitrary smooth complex-valued function of its arguments. After a preliminary study of the Lie symmetry properties of the class $\mathcal{S}$, we further constrain the arbitrary function S by putting $S(t,x,\rho)=f(\rho)+V(t,x)$, $f_\rho\neq 0$. This gives a new class: the class $\mathcal{V}$ of multidimensional nonlinear Schrödinger equations with potentials and modular nonlinearities of the form

$$i\psi_t+\psi_{aa}+f(\rho)\psi+V(t,x)\psi=0,\quad f_\rho\neq 0, \tag{5}$$

where V is an arbitrary smooth complex-valued potential depending on t and x, and f is an arbitrary complex-valued nonlinearity depending only on ρ. We solve completely the group classification problem for this class when $n=2$. We note that the case $f=\rho^2$ was investigated in [38].

The structure of this paper is organized as follows: In Section 2 we compute the equivalence groupoid $\mathcal{G}^\sim_\mathcal{N}$ of the class $\mathcal{N}$. In Section 3 we derive the equivalence group $G^\sim_\mathcal{N}$ of the class $\mathcal{N}$ and the equivalence groups $G^\sim_{\mathcal{N}_0}$ and $G^\sim_{\bar{\mathcal{N}}_0}$ of the subclasses $\mathcal{N}_0$ and $\bar{\mathcal{N}}_0$ singled out within the class $\mathcal{N}$ by the constraints $G^*\neq -G$ and $G^*=-G$, respectively. The equivalence groups $G^\sim_\mathcal{F}$, $G^\sim_{\mathcal{F}^1}$ and $G^\sim_\mathcal{S}$ of the subclasses $\mathcal{F}$, $\mathcal{F}^1$ and $\mathcal{S}$, and the normalization property within these classes are then computed. For the class $\mathcal{S}$, we additionally derive its equivalence algebra $\mathfrak{g}^\sim_\mathcal{S}$. Section 4 is devoted to the analysis of the determining equations for Lie symmetries of an arbitrary equation from the class $\mathcal{S}$ and we derive the maximal Lie invariance algebra $\mathfrak{g}_S$ for this equation, the kernel invariance algebra and the classifying condition. Additionally, we also characterize the Lie symmetry properties of the algebra $\mathfrak{g}_S$. Section 5 deals with the preliminary scheme of group classification for the class $\mathcal{V}$. We find its equivalence group and show that this class is not normalized. We then partition the class $\mathcal{V}$ into the normalized subclasses $\mathcal{V}'$, $\mathcal{P}_0$ and $\mathcal{P}_\lambda$ and derive their Lie symmetry properties separately. The complete group classifications of these subclasses in (1+2)-dimensional case is presented in Section 6.1–6.3, respectively. In this way, we obtain a complete list of inequivalent Lie symmetry extensions in the class $\mathcal{V}$. The conclusion and suggestions for further directions of investigations are collected in Section 7.

2 Equivalence groupoid

In this section, we compute the equivalence groupoid $\mathcal{G}^\sim_\mathcal{N}$ for the class $\mathcal{N}$ which consists of equations of the form (1) using the direct method. We find all point transformations of the form

$$\begin{aligned}\varphi\colon\ &\tilde t=T(t,x,\psi,\psi^*),\quad \tilde x_a=X^a(t,x,\psi,\psi^*),\\ &\tilde\psi=\Psi(t,x,\psi,\psi^*),\quad \tilde\psi^*=\Psi^*(t,x,\psi,\psi^*)\end{aligned} \tag{6}$$

with $dT\wedge dX\wedge d\Psi\wedge d\Psi^*\neq 0$, $X=(X^1,\dots,X^n)$, that map a fixed equation $\mathcal{L}_{GF}$ from the class $\mathcal{N}$ to an equation $\mathcal{L}_{\tilde G\tilde F}$:

$$i\tilde\psi_{\tilde t}+\tilde G(\tilde t,\tilde x,\tilde\psi,\tilde\psi^*,\tilde\nabla\tilde\psi,\tilde\nabla\tilde\psi^*)\tilde\psi_{\tilde x_a\tilde x_a}+\tilde F(\tilde t,\tilde x,\tilde\psi,\tilde\psi^*,\tilde\nabla\tilde\psi,\tilde\nabla\tilde\psi^*)=0,$$

of the same class.

Theorem III.1. *The equivalence groupoid $\mathcal{G}^{\sim}_{\mathcal{N}}$ of the class $\mathcal{N}$ consists of triples of the form $((G,F),(\tilde G,\tilde F),\varphi)$, where φ is a point transformation of the form (6) that satisfies the equations*

$$T_\psi=T_{\psi^*}=0,\quad T_a=0,\quad X^a_\psi=X^a_{\psi^*}=0,\quad X^a_bX^a_c=H\delta_{bc}, \tag{7a}$$

for some smooth real-valued function H of (t,x), where δ_{bc} is the Kronecker delta, and,

$$\textit{if}\quad G^*\neq -G,\quad \Psi_\psi\Psi_{\psi^*}=0;$$

the transformed arbitrary elements $\tilde G$ and $\tilde F$ are given by

$$\tilde G=\frac{H}{T_t}\begin{cases}G & \textit{if}\quad \Psi_\psi\neq 0,\\ (-G^*) & \textit{if}\quad \Psi_{\psi^*}\neq 0,\end{cases} \tag{7b}$$

$$\begin{aligned}\tilde F=&\frac{\Psi_\psi}{T_t}F-\frac{\Psi_{\psi^*}}{T_t}F^*+\left(\hat\Delta\Psi+\frac{n-2}{2}\frac{H_a}{H}\mathrm{D}_a\Psi\right)\frac{\tilde G}{H}\\ &-\frac{i}{T_t}\left(\Psi_t-X^b_t\frac{X^b_a}{H}\mathrm{D}_a\Psi\right).\end{aligned} \tag{7c}$$

Here $\hat\Delta:=\partial_{aa}+2\psi_a\partial_{a\psi}+2\psi^*_a\partial_{a\psi^*}+\psi_a\psi_a\partial_{\psi\psi}+2\psi_a\psi^*_a\partial_{\psi\psi^*}+\psi^*_a\psi^*_a\partial_{\psi^*\psi^*}$. The transformations defined by X belong to the conformal group.

Proof. Let φ be a point transformation of the form (6) connecting two equations $\mathcal{L}_{GF}$ and $\mathcal{L}_{\tilde G\tilde F}$ from the class $\mathcal{N}$. Applying the total derivatives D_μ's to $\tilde\psi(\tilde t,\tilde x)=\Psi(t,x,\psi,\psi^*)$ and $\tilde\psi^*(\tilde t,\tilde x)=\Psi^*(t,x,\psi,\psi^*)$, we obtain the equations

$$\tilde\psi_{\tilde x_\nu}\mathrm{D}_\mu X^\nu=\mathrm{D}_\mu\Psi,\quad \tilde\psi^*_{\tilde x_\nu}\mathrm{D}_\mu X^\nu=\mathrm{D}_\mu\Psi^*,$$

where we have put $X^0=T$, and $x_0=t$. After rearranging terms, we obtain the equations

$$\begin{aligned}&(\Psi_\psi-\tilde\psi_{\tilde x_\nu}X^\nu_\psi)\psi_\mu+(\Psi_{\psi^*}-\tilde\psi_{\tilde x_\nu}X^\nu_{\psi^*})\psi^*_\mu=-(\Psi_\mu-\tilde\psi_{\tilde x_\nu}X^\nu_\mu),\\ &(\Psi^*_\psi-\tilde\psi^*_{\tilde x_\nu}X^\nu_\psi)\psi_\mu+(\Psi^*_{\psi^*}-\tilde\psi^*_{\tilde x_\nu}X^\nu_{\psi^*})\psi^*_\mu=-(\Psi^*_\mu-\tilde\psi^*_{\tilde x_\nu}X^\nu_\mu).\end{aligned}$$

Putting $W=W(t,x,\psi,\psi^*,\tilde\nabla\tilde\psi):=\Psi-\tilde\psi_{\tilde x_\nu}X^\nu$ in the above equations and solving for ψ_μ and ψ^*_μ yields

$$\psi_\mu=-\frac{W_\mu W^*_{\psi^*}-W^*_\mu W_{\psi^*}}{Y},\quad \psi^*_\mu=-\frac{W_\psi W^*_\mu-W^*_\psi W_\mu}{Y}, \tag{8}$$

where

$$\begin{aligned}Y:&=W_\psi W^*_{\psi^*}-W^*_\psi W_{\psi^*}\\ &=\begin{vmatrix}\Psi_\psi & \Psi_{\psi^*}\\ \Psi^*_\psi & \Psi^*_{\psi^*}\end{vmatrix}-\begin{vmatrix}X^\nu_\psi & X^\nu_{\psi^*}\\ \Psi^*_\psi & \Psi^*_{\psi^*}\end{vmatrix}\tilde\psi_{\tilde x_\nu}-\begin{vmatrix}\Psi_\psi & \Psi_{\psi^*}\\ X^\mu_\psi & X^\mu_{\psi^*}\end{vmatrix}\tilde\psi^*_{\tilde x_\mu}\\ &\quad+\begin{vmatrix}X^\nu_\psi & X^\nu_{\psi^*}\\ X^\mu_\psi & X^\mu_{\psi^*}\end{vmatrix}\tilde\psi_{\tilde x_\nu}\tilde\psi^*_{\tilde x_\mu}\neq 0.\end{aligned} \tag{9}$$

Note that $Y \neq 0$: if $Y = 0$ then each of the determinants in equation (9) vanishes and this then implies that the columns $(T_\psi, X_\psi, \Psi_\psi, \Psi^*_\psi)^\mathsf{T}$ and $(T_{\psi^*}, X_{\psi^*}, \Psi_{\psi^*}, \Psi^*_{\psi^*})^\mathsf{T}$ are linearly dependent, which contradicts the invertibility of the point transformation. Using the relations (8) we compute $\mathrm{D}_a\psi_a$ for each of the equations and then rearrange and we obtain the following expression for ψ_{aa}:

$$\begin{aligned}\psi_{aa} &= \frac{1}{Y}\Big(\mathrm{D}_a(-W_a W^*_{\psi^*} + W^*_a W_{\psi^*}) - \psi_a \mathrm{D}_a W\Big)\\ &= \psi_a \tilde{\psi}_{\tilde{x}_\mu}\mathrm{D}_a\tilde{\psi}_{\tilde{x}_\mu} + \psi_a\tilde{\psi}^*_{\tilde{x}_\mu}\mathrm{D}_a\tilde{\psi}^*_{\tilde{x}_\mu} + R,\end{aligned}$$

where R consists of terms not containing second derivatives of ψ and ψ^*. The above equality is equivalent to

$$\psi_{aa} = \frac{1}{Y}(\mathrm{D}_a X^\mu)(\mathrm{D}_a X^\nu)(-\tilde{\psi}_{\tilde{x}_\mu\tilde{x}_\nu}W^*_{\psi^*} + \tilde{\psi}^*_{\tilde{x}_a\tilde{x}_a}W_{\psi^*}) + R.$$

Using $G\psi_{aa} = -i\psi_t - F$, $G^*\psi^*_{aa} = +i\psi^*_t - F^*$ together with the condition from the invertibility of the point transformations, $(T_{\psi^*}, X_{\psi^*}, \Psi_{\psi^*}, \Psi^*_{\psi^*}) \neq (0,0,0,0)$, we collect the coefficients of $\tilde{\psi}_{\tilde{x}_0\tilde{x}_0}$ and $\tilde{\psi}^*_{\tilde{x}_0\tilde{x}_0}$. As a result we obtain $W^*_{\psi^*}(\mathrm{D}_aT)(\mathrm{D}_aT) = 0$ and $W_{\psi^*}(\mathrm{D}_aT)(\mathrm{D}_aT) = 0$, which implies that $T_\psi = T_{\psi^*} = 0$ and $T_a = 0$. Further, collecting the coefficients of $\tilde{\psi}_{\tilde{x}_b\tilde{x}_c}$ and $\tilde{\psi}^*_{\tilde{x}_b\tilde{x}_c}$ yields $X^a_a X^b_c = H\delta_{ac}$ for all a, b and c. Using these results for T and X^a in (8) we obtain

$$\begin{aligned}\psi_\mu &= \frac{1}{Y}\Big(X^\nu_\mu \Psi^*_{\psi^*}\tilde{\psi}_{\tilde{x}_\nu} + X^\nu_\mu\Psi_\psi\tilde{\psi}^*_{\tilde{x}_\nu} - \Psi_\mu\Psi^*_{\psi^*} - \Psi^*_\mu\Psi_{\psi^*}\Big),\\ \psi^*_\mu &= \frac{1}{Y}\Big(-X^\nu_\mu\Psi^*_\psi\tilde{\psi}_{\tilde{x}_\nu} + X^\nu_\mu\Psi_\psi\tilde{\psi}^*_{\tilde{x}_\nu} - \Psi_\psi\Psi^*_\mu - \Psi^*_\psi\Psi_\mu\Big),\end{aligned}$$

and the expression for ψ_{aa} becomes:

$$\begin{aligned}\psi_{aa} =&\frac{X^b_a X^b_a}{Y}\Big(\Psi^*_{\psi^*}\tilde{\psi}_{\tilde{x}_b\tilde{x}_b} - \Psi_{\psi^*}\tilde{\psi}^*_{\tilde{x}_b\tilde{x}_b}\Big) + \tilde{\psi}_{\tilde{x}_b}\mathrm{D}_a\frac{X^b_a\Psi^*_{\psi^*}}{Y}\\ &-\tilde{\psi}^*_{\tilde{x}_b}\mathrm{D}_a\frac{X^b_a\Psi_{\psi^*}}{Y} - \mathrm{D}_a\frac{\Psi_a\Psi^*_{\psi^*} - \Psi^*_a}{Y},\end{aligned}$$

and the expression for ψ^*_{aa} can be obtained from the complex-conjugate of the expression for ψ_{aa}. Substituting the above expressions for ψ_t (resp. ψ^*_t) and ψ_{aa} (resp. ψ^*_{aa}) into $i\psi_t + G\psi_{aa} + F = 0$ (resp. $-i\psi^*_t + G^*\psi^*_{aa} + F^* = 0$) and using $i\tilde{\psi}_{\tilde{t}} = -\tilde{G}\tilde{\psi}_{\tilde{x}_b\tilde{x}_b} - \tilde{F}$ (resp. $-i\tilde{\psi}^*_{\tilde{t}} = -\tilde{G}^*\tilde{\psi}^*_{\tilde{x}_b\tilde{x}_b} - \tilde{F}^*$), and then collecting the coefficients of $\tilde{\psi}_{\tilde{x}_b\tilde{x}_b}$ and $\tilde{\psi}^*_{\tilde{x}_b\tilde{x}_b}$, we obtain

$$(HG - T_t\tilde{G})\Psi^*_{\psi^*} = 0, \quad (HG + T_t\tilde{G}^*)\Psi_{\psi^*} = 0.$$

Now, we have either $\Psi^*_{\psi^*}\Psi_{\psi^*} \neq 0$ or $\Psi^*_{\psi^*}\Psi_{\psi^*} = 0$. Observe that we have $\Psi^*_{\psi^*} = (\Psi_\psi)^*$. If $\Psi^*_{\psi^*}\Psi_{\psi^*} \neq 0$ then $\tilde{G} = -\tilde{G}^*$, i.e., $G = -G^*$. Thus G is purely imaginary smooth function. Otherwise we have $\Psi^*_{\psi^*}\Psi_{\psi^*} = 0$, which implies that either $\Psi_\psi \neq 0$, in which case $\tilde{G} = \dfrac{H}{T_t}G$, or $\Psi_{\psi^*} \neq 0$, in which case $\tilde{G} = -\dfrac{H}{T_t}G^*$. Finally, collecting the remain terms gives the transformation for F. □

3 Equivalence group and equivalence algebra

Here we derive the equivalence group $G^\sim_{\mathcal{N}}$ of the class $\mathcal{N}$ and then we infer the equivalence groups of its subclasses (that are singled out by constraints on the arbitrary functions) from the equivalence groupoid $\mathcal{G}^{\sim}_{\mathcal{N}}$. We also study the normalization properties of these classes and find the equivalence algebra $\mathfrak{g}^{\sim}_{\mathcal{S}}$ of the class $\mathcal{S}$.

Corollary III.1. *The class $\mathcal{N}$ is not normalized. Its equivalence group $G^{\sim}_{\mathcal{N}}$ consists of the point transformations in the space of $(t,x,\psi_{(2)},\psi^*_{(2)},G,G^*,F,F^*)$, where the components for t, x and ψ are of the form* (6) *with T, X^a and Ψ satisfying the equations* (7a) *and $\Psi_\psi\Psi_{\psi^*}=0$, and the components for G and F are of the form* (7b) *and* (7c).

Since the class $\mathcal{N}$ is not normalized, we partition it into two disjoint subclasses $\mathcal{N}_0$ and $\bar{\mathcal{N}}_0$ defined by the constraints $G^*\neq -G$ and $G^*=-G$, respectively. There are clearly no point transformations that map equations from the class $\mathcal{N}_0$ to equations from the class $\bar{\mathcal{N}}_0$. We can obtain their corresponding equivalence groupoids $\mathcal{G}^{\sim}_{\mathcal{N}_0}$ and $\mathcal{G}^{\sim}_{\bar{\mathcal{N}}_0}$, and as well as their equivalence groups $G^{\sim}_{\mathcal{N}_0}$ and $G^{\sim}_{\bar{\mathcal{N}}_0}$ from Theorem III.1 and Corollary III.1, respectively.

Corollary III.2. *The class $\mathcal{N}$ is partitioned into the two disjoint normalized subclasses $\mathcal{N}_0$ and $\bar{\mathcal{N}}_0$ with respect to the constraints $G^*\neq -G$ and $G^*=-G$, respectively. Both the subclasses are normalized in the usual sense. The usual equivalence group $G^{\sim}_{\mathcal{N}_0}$ coincides with the equivalence group $G^{\sim}_{\mathcal{N}}$ of the entire class $\mathcal{N}$ and the usual equivalence group $G^{\sim}_{\bar{\mathcal{N}}_0}$ consists of point transformations that form the group $G^{\sim}_{\mathcal{N}}$ except the condition $\Psi_\psi\Psi_{\psi^*}=0$.*

For the case $G\neq 0,1$, we look at point transformations of a subclass $\mathcal{K}$ of the class $\mathcal{N}$ defined by certain restrictions on G provided that they preserve the form of the parameter function F. The cases $G\neq 0,1$ are excluded from this consideration: for $G=0$ we have no Schrödinger equation; for $G=1$ we have an equation from the class $\mathcal{F}$ of equations (2).

Theorem III.2. *The equivalence group $G^{\sim}_{\mathcal{K}}$ of any subclass $\mathcal{K}$ of the class $\mathcal{N}$ singled out by specific forms of the arbitrary element G, $G\neq 0,1$ and that preserves the form of the arbitrary element F as given in* (1)*, is a subgroup of either the equivalence group $G^{\sim}_{\bar{\mathcal{N}}_0}$ or the equivalence group $G^{\sim}_{\mathcal{N}_0}$ of the classes $\bar{\mathcal{N}}$ and $\mathcal{N}_0$, respectively.*

Proof. The proof follows from Theorem III.1. Any given form of G satisfying the restrictions in the above Theorem does not change the transformations (7a)–(7c). Therefore, any subclass $\mathcal{K}$ consisting of these transformations will not be normalized. Consequently, the same partition in terms of the conditions $G^*\neq -G$ and $G^*=-G$ is made for the subclass $\mathcal{K}$ as in the larger class $\mathcal{N}$. As this partition induces the similar partition of the equivalence groupoids of the corresponding subclasses, it is obvious that any equivalence group $G^{\sim}_{\mathcal{K}}$ is a subgroup of either the equivalence group $G^{\sim}_{\bar{\mathcal{N}}_0}$ or the equivalence group $G^{\sim}_{\mathcal{N}_0}$. □

Restricting the class $\mathcal{N}$ by putting $G=1$ and preserve keeping the arbitrary function F, we obtain the class $\mathcal{F}$ whose equations are of the form (2). It is easy to derive its equivalence groupoid $\mathcal{G}^{\sim}_{\mathcal{F}}$ from the equivalence groupoid $\mathcal{G}^{\sim}_{\mathcal{N}}$ computed in Section 2. In the following, we use the following notation for a complex number β

$$\hat{\beta}=\beta \quad \text{if} \quad T_t>0 \quad \text{and} \quad \hat{\beta}=\beta^* \quad \text{if} \quad T_t<0.$$

Theorem III.3. *The equivalence groupoid $\mathcal{G}^{\sim}_{\mathcal{F}}$ of the class $\mathcal{F}$ consists of triple of the form $(F,\tilde{F},\varphi)$, where φ is a point transformation in the space of variables, given by*

$$\tilde{t}=T, \quad \tilde{x}_a=|T_t|^{1/2}O^{ab}x_b+\mathcal{X}^a, \quad \tilde{\psi}=\Psi(t,x,\hat{\psi}), \tag{10a}$$

and the transformed function $\tilde F$ is expressed via F as

$$\tilde F=\frac{1}{|T_t|}\left(\Psi_{\hat\psi}\hat F-i\varepsilon'\Psi_t+i\left(\frac{T_{tt}}{2|T_t|}x_a+\frac{\varepsilon'}{|T_t|^{1/2}}\mathcal{X}^b_tO^{ba}\right)(\Psi_a+\Psi_{\hat\psi}\hat\psi_a)\right)\tag{10b}$$
$$-\frac{1}{|T_t|}\left(\Psi_{aa}+2\Psi_{a\hat\psi}\hat\psi_a+\Psi_{\hat\psi\hat\psi}\hat\psi_a\hat\psi_a\right),$$

where T and X are arbitrary smooth real-valued functions of t, $T_t\neq 0$, Ψ is an arbitrary smooth complex-valued function of t, x and $\hat\psi$, $\Psi_{\hat\psi}\neq 0$, O is an n-dimensional orthogonal matrix. Here $\varepsilon'=\operatorname{sgn}T_t$.

Proof. Consider point transformations (7a)–(7c) in Theorem III.1 where $G=1$, i.e., $H=|T_t|$. Then following [27, Theorem 2] we obtain the transformations (10a). The transformation of $\tilde F$ is obtained by using (10a) into transformation (7c). □

Corollary III.3. *The class $\mathcal{F}$ is normalized. Its equivalence group $G^\sim_{\mathcal{F}}$ is generated by transformations of the form* (10a)–(10b)*. More specifically, $G^\sim_{\mathcal{F}}$ consists of continuous transformations of the above form, where $T_t>0$ and two discrete transformations, the space reflection I_a for a fixed a ($\tilde t=t$, $\tilde x_a=-x_a$, $\tilde x_b=x_b$, $b\neq a$, $\tilde\psi=\psi$, $\tilde F=F$) and the Wigner time reflection I_t ($\tilde t=-t$, $\tilde x=x$, $\tilde\psi=\psi^*$, $\tilde F=F^*$).*

Further, restricting the arbitrary element F by putting $F_{\psi_a}=F_{\psi^*_a}=0$, we obtain the subclass $\mathcal{F}^1$ of equations of the form (3). Then transformations (7c) give us

$$\Psi_{a\hat\psi}=\frac12X^a_bX^b_t\Psi_{\hat\psi},\quad \Psi_{\hat\psi\hat\psi}=0.\tag{11}$$

Integrating these equations we obtain Ψ. We then have the following:

Theorem III.4. *The class $\mathcal{F}^1$ is normalized. Its equivalence group $G^\sim_{\mathcal{F}^1}$ is a subgroup of $G^\sim_{\mathcal{F}}$ and consists of point transformations of the form* (10a)– (10b)*, where*

$$\Psi=\exp\left(\frac{i}{8}\frac{T_{tt}}{|T_t|}\,x_ax_a+\frac{i}{2}\frac{\varepsilon'\mathcal{X}^b_t}{|T_t|^{1/2}}\,O^{ba}x_a+i\Sigma+\Lambda\right)\hat\psi+\Psi^0,$$

and T, $\mathcal{X}$, Λ and Σ are arbitrary smooth real-valued functions of t, $T_t\neq 0$, Ψ^0 is an arbitrary smooth complex-valued function of t and x and O is an n-dimensional constant orthogonal matrix.

We now turn our attention to the class $\mathcal{S}$ of equations of the form (4). This class is a subclass of the class $\mathcal{F}$ with $F=S(t,x,\rho)\psi$. The arbitrary element is now S and it satisfies the additional conditions

$$\psi S_\psi-\psi^*S_{\psi^*}=0,\quad \psi S_\psi+\psi^*S_{\psi^*}\neq 0.\tag{12}$$

With this, we deduce the following result from Theorem III.4:

Theorem III.5. *The class $\mathcal{S}$ is normalized. The equivalence group $G^\sim_{\mathcal{S}}$ of this class is the subgroup of $G^\sim_{\mathcal{F}^1}$, for which $\Psi^0=0$. That is, the group $G^\sim_{\mathcal{S}}$ consists of transformations in the space of variables and the arbitrary element of the form*

$$\tilde t=T,\quad \tilde x_a=|T_t|^{1/2}O^{ab}x_b+\mathcal{X}^a,\tag{13a}$$

$$\tilde\psi=\exp\left(\frac{i}{8}\frac{T_{tt}}{|T_t|}\,x_ax_a+\frac{i}{2}\frac{\varepsilon'\mathcal{X}^b_t}{|T_t|^{1/2}}\,O^{ba}x_a+i\Sigma+\Lambda\right)\hat\psi\tag{13b}$$

$$\begin{aligned}\tilde{S} =&\frac{\hat{S}}{|T_t|}+\frac{2T_{ttt}T_t-3{T_{tt}}^2}{16\varepsilon' {T_t}^3}x_ax_a+\left(\frac{\mathcal{X}^b_t}{T_t}\right)_t\frac{\varepsilon' O^{ba}x_a}{2|T_t|^{1/2}}\\ &+\frac{\Sigma_t-i\Lambda_t}{T_t}-\frac{\mathcal{X}^a_t\mathcal{X}^a_t+inT_{tt}}{4{T_t}^2}.\end{aligned}\tag{13c}$$

Here T, $\mathcal{X}^a$, Λ and Σ are arbitrary smooth real-valued functions of t with $T_t\neq 0$, $\varepsilon'=\operatorname{sgn} T_t$ and $O=(O^{ab})$ is an arbitrary constant $n\times n$ orthogonal matrix.

Note 1. *Theorem* III.5 *is important in the group classification of any subclass of the class $\mathcal{S}$. The equivalence transformations for any such subclass are deduced from the transformations* (13)*.*

Corollary III.4. *For any equation from the class $\mathcal{S}$, $\rho S_{\rho\rho}/S_\rho$ is constant under any transformation which maps this equation to an equation from the same class, excluding I_t. In particular, if $\rho S_{\rho\rho}/S_\rho$ is a real-valued, then it is an invariant of the admissible transformations in the class $\mathcal{S}$.*

To find the equivalence algebra $\mathfrak{g}^{\sim}_{\mathcal{S}}$ of the class $\mathcal{S}$ we use the knowledge of the equivalence group $G^{\sim}_{\mathcal{S}}$ of the class $\mathcal{S}$ as described in Theorem III.5. We evaluate the set of all infinitesimal generators of one-parameter subgroups of the group $G^{\sim}_{\mathcal{S}}$ by representing the the parameter function Σ as $\Sigma=\frac{1}{4}\mathcal{X}^a\mathcal{X}^a_t+\bar{\Sigma}$, where $\bar{\Sigma}$ is a function of t, to ensure the existence of one-parameter subgroup of $G^{\sim}$. Next we successively assume one of the parameter-functions T, O, $\mathcal{X}^a$, $\bar{\Sigma}$ and Λ to depend on a continuous parameter δ and setting the other parameters to their trivial values. That is t for T, the $n\times n$ identity matrix for O and zeroes for $\mathcal{X}^a$, $\bar{\Sigma}$ and Λ. This procedure leads to the components of the associated infinitesimal generator $Q=\tau\partial_t+\xi^a\partial_a+\eta\partial_\psi+\eta^*\partial_{\psi^*}+\theta\partial_S+\theta^*\partial_{S^*}$, where they are computed as

$$\tau=\left.\frac{\mathrm{d}\tilde{t}}{\mathrm{d}\delta}\right|_{\delta=0},\quad \xi^a=\left.\frac{\mathrm{d}\tilde{x}}{\mathrm{d}\delta}\right|_{\delta=0},\quad \eta=\left.\frac{\mathrm{d}\tilde{\psi}}{\mathrm{d}\delta}\right|_{\delta=0},\quad \theta=\left.\frac{\mathrm{d}\tilde{S}}{\mathrm{d}\delta}\right|_{\delta=0}.$$

Then we find the following:

Corollary III.5. *The equivalence algebra $\mathfrak{g}^{\sim}_{\mathcal{S}}$ of the class $\mathcal{S}$ is the algebra,*

$$\mathfrak{g}^{\sim}_{\mathcal{S}}=\langle \hat{D}(\tau),\,\hat{J}_{ab},\,a<b,\,\hat{G}(\chi),\,\hat{M}(\sigma),\,\hat{I}(\zeta)\rangle$$

where τ, $\chi=(\chi^1,\dots,\chi^n)$, σ and ρ run through the set of smooth real-valued functions of t,

$$\begin{aligned}\hat{D}(\tau)&=\tau\partial_t+\frac{1}{2}\tau_tx_a\partial_a+\frac{i}{8}\tau_{tt}x_ax_a\left(\psi\partial_\psi-\psi^*\partial_{\psi^*}\right)\\ &\quad-\left(\tau_tS-\frac{1}{8}\tau_{ttt}x_ax_a+i\frac{\tau_{tt}}{4}\right)\partial_S-\left(\tau_tS^*-\frac{1}{8}\tau_{ttt}x_ax_a-i\frac{\tau_{tt}}{4}\right)\partial_{S^*},\\ \hat{J}_{ab}&=x_a\partial_b-x_b\partial_a,\quad a\neq b,\\ \hat{G}(\chi)&=\chi^a\partial_a+\frac{i}{2}\chi^a_tx_a\left(\psi\partial_\psi-\psi^*\partial_{\psi^*}\right)+\frac{1}{2}\chi^a_{tt}x_a\left(\partial_S+\partial_{S^*}\right),\\ \hat{M}(\sigma)&=i\sigma(\psi\partial\psi-\psi\partial\psi^*)+\sigma_t(\partial_S+\partial_S^*),\\ \hat{I}(\zeta)&=\rho(\psi\partial\psi+\psi^*\partial\psi^*)-i\rho_t(\partial_S+\partial_S^*).\end{aligned}$$

4 Determining equations for Lie symmetries

Consider an equation $\mathcal{L}_S$ from the class $\mathcal{S}$ for a fixed S and let $\mathfrak{g}_S$ the maximal Lie invariance algebra of this equation. Then a vector field in the space of variables (t, x, ψ, ψ^*) belonging to $\mathfrak{g}_S$ is of the form $Q = \tau\partial_t + \xi^a\partial_a + \eta\partial_\psi + \eta^*\partial_{\psi^*}$, where the components τ, ξ^a, η and η^* are smooth functions of (t, x, ψ, ψ^*). The infinitesimal invariance criterion states that

$$Q_{(2)}\left(i\psi_t + \psi_{aa} + S(t,x,\rho)\psi\right)\Big|_{\mathcal{L}_S} = 0,$$

where $Q_{(2)}$ is the second prolongation of the vector field Q. The computation of this expression gives us

$$i\eta^t + \eta^{aa} + (\tau S_t + \xi^a S_a)\psi + \rho S_\rho \eta = 0, \tag{14}$$

where

$$\begin{aligned}
\eta^t &= \mathrm{D}_t\left(\eta - \tau\psi_t - \xi^a\psi_a\right) + \tau\psi_{tt} + \xi^a\psi_{ta},\\
\eta^{ab} &= \mathrm{D}_a\mathrm{D}_b\left(\eta - \tau\psi_t - \xi^c\psi_c\right) + \tau\psi_{tab} + \xi^c\psi_{abc},
\end{aligned}$$

and we recall that D_t and D_a are the operators of total derivatives with respect to t and x_a, respectively. Using equation (14) and take into account that $\psi_t = i\psi_{aa} + iS\psi$ and $\psi_t^* = -i\psi_{aa}^* - iS\psi^*$, we derive the following linear overdetermined system for the coefficients of the vector field Q:

$$\begin{aligned}
&\tau_\psi = \tau_{\psi^*} = \tau_a = 0, \quad \xi^a_\psi = \xi^a_{\psi^*} = 0,\\
&\tau_t = 2\xi^1_1 = \cdots = 2\xi^n_n, \quad \xi^a_b + \xi^b_a = 0, \ a \neq b,
\end{aligned} \tag{15a}$$

$$\eta_{\psi^*} = \eta_{\psi\psi} = 0, \quad 2\eta_{\psi a} = i\xi^a_t, \quad \psi\eta_\psi = \eta, \tag{15b}$$

$$i\eta_t + \eta_{aa} + (\tau S_t + \xi^a S_a)\psi + \rho S_\rho Re\,\eta_\psi + \tau_t S = 0. \tag{15c}$$

The general solution of the subsystems (15a) and (15b) is

$$\begin{aligned}
\tau &= \tau(t), \quad \xi^a = \frac{1}{2}\tau_t x_a + \kappa^{ab}x_b + \chi^a,\\
\eta &= \left(\frac{i}{8}\tau_{tt}x_a x_a + \frac{i}{2}\chi^a x_a + \zeta + i\sigma\right)\psi,
\end{aligned}$$

where τ, χ^a, ζ and σ are smooth real-valued functions of t, and (κ^{ab}) is a constant skew-symmetric matrix. Putting these expressions into the subsystem (15c), we derive the classifying condition for Lie symmetry vector fields of equations from the class $\mathcal{S}$ and we obtain:

Theorem III.6. *The maximal Lie invariance algebra $\mathfrak{g}_S$ of an equation $\mathcal{L}_S$ from the class $\mathcal{S}$ consists of the vector fields of the form $D(\tau) + \sum_{a<b}\kappa^{ab}J_{ab} + G(\chi) + \sigma M + \zeta I$, where*

$$\begin{aligned}
&D(\tau) = \tau\partial_t + \frac{1}{2}\tau_t x_a\partial_a + \frac{1}{8}\tau_{tt}x_a x_a M, \quad J_{ab} = x_a\partial_b - x_b\partial_a, \quad a \neq b,\\
&G(\chi) = \chi^a\partial_a + \frac{1}{2}\chi^a_t x_a M, \quad M = i\psi\partial_\psi - i\psi^*\partial_{\psi^*}, \quad I = \psi\partial_\psi + \psi^*\partial_{\psi^*},
\end{aligned}$$

the parameters τ, χ^a, ζ and σ are arbitrary real-valued smooth functions of t and the matrix (κ^{ab}) is an arbitrary constant skew-symmetric matrix that satisfy the classifying condition

$$\begin{aligned}&\tau S_t+\left(\frac{1}{2}\tau_t x_a+\kappa^{ab}x_b+\chi^a\right)S_a+\zeta\rho S_\rho+\tau_t S\\&=\frac{1}{8}\,\tau_{ttt}x_ax_a+\frac{1}{2}\chi^a_{tt}x_a+\sigma_t-i\zeta_t-i\frac{n}{4}\,\tau_{tt}.\end{aligned}\tag{16}$$

This theorem means that a vector field Q is said to be *a Lie invariance algebra* of an equation $\mathcal{L}_S$ from $\mathcal{S}$ if and only if $(\tau,\chi,\zeta)\neq(0,0,0)$ or $\sigma_t=0$. Varying the arbitrary element S and splitting the classifying condition (16) with respect to the various derivatives of S, we obtain the kernel invariance algebra $\mathfrak{g}^\cap$ of the class $\mathcal{S}$.

Proposition III.1. *The kernel invariance algebra of the class $\mathcal{S}$ is $\mathfrak{g}^\cap=\langle M\rangle$ and its corresponding group $G^\cap_{\mathcal{S}}$ is formed by the transformations $\tilde t=t$, $\tilde x=x$, $\tilde\psi=\psi e^{i\Psi}$, $\tilde S=S$, where Ψ is an arbitrary constant.*

We define $\mathfrak{g}_{\langle\,\rangle}$ as the linear span of all vector fields in Theorem III.6 of all equations when the arbitrary element S varies. That is

$$\mathfrak{g}_{\langle\,\rangle}:=\left\{Q=D(\tau)+\sum_{a<b}\kappa^{ab}J_{ab}+G(\chi)+\sigma M+\zeta I\right\}=\sum_S\mathfrak{g}_S.$$

The nonzero commutation relations between vector fields of $\mathfrak{g}_{\langle\,\rangle}$ are:

$$\begin{aligned}&[D(\tau^1),D(\tau^2)]=D(\tau^1\tau^2_t-\tau^2\tau^1_t),\quad [D(\tau),G(\chi)]=G\left(\tau\chi_t-\frac{1}{2}\tau_t\chi\right),\\&[D(\tau),\sigma M]=\tau\sigma_tM,\quad [D(\tau),\zeta I]=\tau\zeta_tI,\\&[J_{ab},J_{bc}]=J_{ac},\quad a\neq b\neq c\neq a,\\&[J_{ab},G(\chi)]=G(\hat\chi)\quad\text{with}\quad \hat\chi^a=\chi^b,\ \hat\chi^b=-\chi^a,\ \chi^c=0,\ a\neq b\neq c\neq a,\\&[G(\chi),G(\tilde\chi)]=\frac{1}{2}\left(\chi^a\tilde\chi^a_t-\tilde\chi^a\chi^a_t\right)M.\end{aligned}$$

From these commutation relations it is clearly seen that the linear span $\mathfrak{g}_{\langle\,\rangle}$ is a Lie algebra and the subspaces

$$\begin{aligned}&\langle\sigma M\rangle,\quad\langle\zeta I\rangle,\quad\langle G(\chi),\sigma M\rangle,\quad\langle G(\chi),\sigma M,\zeta I\rangle,\quad\langle J_{ab},G(\chi),\sigma M\rangle,\\&\langle J_{ab},G(\chi),\sigma M,\zeta I\rangle,\quad\langle D(\tau),G(\chi),\sigma M,\zeta I\rangle\end{aligned}$$

are ideals of $\mathfrak{g}_{\langle\,\rangle}$. Furthermore, the subspaces $\langle D(\tau)\rangle$, $\langle J_{ab}\rangle$ and $\langle D(\tau),J_{ab}\rangle$ are subalgebras of $\mathfrak{g}_{\langle\,\rangle}$.

Definition III.1. A subalgebra $\mathfrak{s}$ of $\mathfrak{g}_S$ is said to be *appropriate* if there exists an arbitrary smooth function S such that $\mathfrak{s}=\mathfrak{g}_S$.

The push-forward actions of elementary transformations to the vector fields spanning $\mathfrak{g}_{\langle\,\rangle}$ are given by

$$\begin{aligned}&\mathcal{D}_*(T)D(\tau)=D(\tilde\tau),\quad\mathcal{D}_*(T)G(\chi)=G(\tilde\chi),\\&\mathcal{D}_*(T)(\sigma M)=\tilde\sigma\tilde M,\quad\mathcal{D}_*(T)(\zeta I)=\tilde\zeta\tilde I,\end{aligned}$$

$$\begin{aligned}
\mathcal{G}_*(\mathcal{X})D(\tau) &= \tilde{D}(\tau) + \tilde{G}\left(\tau\mathcal{X}_t - \frac{1}{2}\tau_t\mathcal{X}\right)\\
&\quad + \left(\frac{1}{8}\tau_{tt}\mathcal{X}^a\mathcal{X}^a - \frac{1}{4}\tau_t\mathcal{X}^a\mathcal{X}^a_t - \frac{1}{2}\tau\mathcal{X}^a\mathcal{X}^a_{tt}\right)\tilde{M},\\
\mathcal{G}_*(\mathcal{X})G(\chi) &= \tilde{G}(\chi) + \frac{1}{2}(\chi^a\mathcal{X}^a_t - \chi^a_t\mathcal{X}^a)\tilde{M},\\
\mathcal{G}_*(\mathcal{X})J_{ab} &= \tilde{J}_{ab} + G(\hat{\mathcal{X}}) - \frac{1}{2}(\mathcal{X}^a\mathcal{X}^b_t - \mathcal{X}^b\mathcal{X}^a_t)\tilde{M},\\
\mathcal{M}_*(\Sigma)D(\tau) &= \tilde{D}(\tau) + \tau\Sigma_t\tilde{M}, \quad \mathcal{I}_*(\Lambda)D(\tau) = \tilde{D}(\tau) + \tau\Lambda_t\tilde{I}.
\end{aligned}$$

Here tildes over the vector fields mean that these vector fields are expressed in the new variables, where $\tilde{\tau}(\tilde{t}) = (T_t\tau)(T^{-1}(\tilde{t}))$, $\tilde{\chi}(\tilde{t}) = \chi T_t^{1/2}|_{t=T^{-1}(\tilde{t})}$, $\hat{\mathcal{X}}^a = \mathcal{X}^b$, $\hat{\mathcal{X}}^b = -\mathcal{X}^a$, $\hat{\mathcal{X}}^c = 0$, $c \neq a, b$, $\tilde{\sigma} = \sigma(T^{-1}(\tilde{t}))$, $\tilde{\rho} = \rho(T^{-1}(\tilde{t}))$ and in each push-forward by $\mathcal{D}_*(T)$ we should substitute the expression for t given by inverting the relation $\tilde{t} = T(t)$; $t = \tilde{t}$ for the other push-forwards.

For any arbitrary element S in the class of equations (4) the following results hold.

Lemma III.1. $\mathfrak{g}_S \cap \langle \sigma M, \zeta I\rangle = \langle M\rangle$.

Proof. We first prove that the left-hand side is included in the algebra on the right-hand side. Putting $\tau = 0$, $\chi^a = 0$ and $\kappa^{ab} = 0$ in the classifying condition we obtain

$$\sigma_t = 0, \quad \zeta\rho S_\rho = -i\zeta_t.$$

Splitting the second equation with S_ρ gives $\zeta = 0$, so that $\mathfrak{g}_S \cap \langle \sigma M, \zeta I\rangle \subset \langle M\rangle$. The implication in the other direction is clear since the kernel invariance algebra is always contained in the maximal Lie invariance algebra. □

Lemma III.2. $\mathfrak{g}_S \cap \langle \sigma M\rangle = \langle M\rangle$.

Proof. This follows from the previous Lemma. □

Lemma III.3. $\dim \mathfrak{g}_S \leqslant \dfrac{n(n+3)}{2} + 4$ *for any* S *with* $S_\rho \neq 0$.

Proof. The proof is similar to the one given in Lemma 12 in [27]. Thus, fixing the arbitrary element S and putting $\zeta = 0$, the classifying condition (16) yields the following system of linearly ordinary differential equations:

$$\begin{aligned}
\tau_{ttt} &= \gamma^{00}\tau_t + \gamma^{01}\tau + \gamma^{0,a+1}\chi^a + \theta^{0ab}\kappa^{ab},\\
\chi^c_{tt} &= \gamma^{c0}\tau_t + \gamma^{c1}\tau + \gamma^{c,a+1}\chi^a + \theta^{cab}\kappa^{ab},\\
\sigma_t &= \gamma^{n+1,0}\tau_t + \gamma^{n+1,1}\tau + \gamma^{n+1,a+1}\chi^a + \theta^{n+1,ab}\kappa^{ab},\\
\zeta_t &= -\frac{n}{4}\tau_{tt} + \zeta\rho S_\rho + \gamma^{n+2,0}\tau_t + \gamma^{n+2,1}\tau + \gamma^{n+2,a+1}\chi^a + \theta^{n+2,ab}\kappa^{ab},
\end{aligned}$$

where the coefficients γ^{pq} and θ^{pab}, $p = 0, \dots, n+2$, $q = 0, \dots, n+1$, $a < b$, are functions of t. From this it is clear that the upper bound of $\dim \mathfrak{g}_S$ can not exceed the sum of the number of pairs (a, b) of rotations with $a < b$ and the number of arbitrary constants involved in the above system, i.e., $n(n+3)/2 + 5$. But Lemma III.1 shows that this number is reduced by 1 for any S with $S_\rho \neq 0$. This proves the Lemma. □

Lemma III.4. $\dim \mathfrak{g}_S \cap \langle G(\chi), \sigma M\rangle \leqslant 2n + 1$.

Proof. Just as in Lemma III.3, we omit the first equation of the system from the proof and set $\tau = 0$, $\kappa^{ab} = 0$ and then using Lemma III.1 we obtain the stated result. □

5 Schrödinger equations with potentials and modular nonlinearity

We consider the class $\mathcal{V}$ of multidimensional nonlinear Schrödinger equations with potentials and modular nonlinearities of the form (5), where $S = f(\rho) + V(t, x)$ with $f_\rho \neq 0$, $\rho = |\psi|$. This class is singled out from the class $\mathcal{S}$ by the condition $S_{\rho t} = S_{\rho a} = 0$ and $S_\rho \neq 0$ or equivalently by

$$\psi S_{\psi t} + \psi^* S_{\psi^* t} = \psi S_{\psi a} + \psi^* S_{\psi^* a} = 0, \quad \psi S_\psi + \psi^* S_{\psi^*} \neq 0. \tag{17}$$

The point transformations for the class $\mathcal{V}$ are obtained from the point transformations (13). These point transformations are of the form (6) so that they preserve the systems formed by the equations (4), (12) and (17).

Theorem III.7. *The class $\mathcal{V}$ is not normalized. The equivalence group $G^{\sim}_{\mathcal{V}}$ of the class $\mathcal{V}$ consists of point transformations* (13)*, where $T_{tt} = 0$ and $\Lambda_t = 0$. The subclass $\mathcal{V}'$ of $\mathcal{V}$ defined by the condition $\rho f_{\rho\rho}/f_\rho$ ($\equiv \rho S_{\rho\rho}/S_\rho$) is not a real constant, has the same equivalence group. This class is normalized. They are two further cases, corresponding to equations* (5)*:*

$$\mathcal{P}_0 := \{i\psi_t + \psi_{aa} + f(\rho)\psi + V\psi = 0,\ \rho f_{\rho\rho}/f_\rho = -1,\ i.e.,\ f = \delta \ln \rho\},$$
$$\mathcal{P}_\lambda = \{i\psi_t + \psi_{aa} + f(\rho)\psi + V\psi = 0,\ \rho f_{\rho\rho}/f_\rho = \lambda - 1,\ i.e.,\ f = \delta \rho^\lambda,\ \lambda \neq 0\},$$

where δ is nonzero complex constant. These two cases are normalized.

It is clear that the subclasses $\mathcal{V}'$ and $\mathcal{P}_\lambda$, $\lambda \in \mathbb{R}$ have no common equations in the class $\mathcal{V}$, which in turn shows that there are no point transformations connecting equations from these subclasses.

The class $\mathcal{V}$ can be represented as the union of the disjoint normalized subclasses $\mathcal{V}'$, $\mathcal{P}_0$ and $\mathcal{P}_\lambda$. This follows from the fact that the set of admissible transformations of the class $\mathcal{V}$ is the union of the sets of admissible transformations for the corresponding subclasses generated by their equivalence groups. Fore more details we refer to [38].

In the following we denote by $G^{\sim}_{\mathcal{V}'}$, $G^{\sim}_{\mathcal{P}_0}$ and $G^{\sim}_{\mathcal{P}_\lambda}$ the equivalence groups for the classes $\mathcal{V}'$, $\mathcal{P}_0$ and $\mathcal{P}_\lambda$, respectively. In the course of the group classification, we investigate each subclass separately and as mentioned above the classification result for the whole class $\mathcal{V}$ will be the union of the classification results corresponding to each subclass.

Remark III.1. Theorem III.5 together with the results of Sections 6.1–6.3 shows that for each fixed $\lambda \neq 0$, any transformation from $G^{\sim}_{\mathcal{S}}$ is presented as a composition of transformations from the equivalence groups $G^{\sim}_{\mathcal{V}'}$, $G^{\sim}_{\mathcal{P}_0}$ and $G^{\sim}_{\mathcal{P}_\lambda}$.

5.1 General case

Here, we discuss the point transformations and Lie symmetry properties for group classification of the class $\mathcal{V}'$ consisting of equations

$$i\psi_t + \psi_{aa} + f(\rho)\psi + V(t, x)\psi = 0, \quad \rho f_{\rho\rho}/f_\rho \neq \lambda,\ \forall \lambda \in \mathbb{R}. \tag{18}$$

Theorem III.7 then gives us that, in this case, $\Lambda_t = 0$ and $T_{tt} = 0$. Substituting this into the transformations (13) leads to the following result:

Corollary III.6. *If the potential V in equation* (18) *is linear in x, then equation* (18) *is equivalent under a transformation of the form* (13) *to an equation of the form*

$$i\psi_t+\psi_{aa}+f(\rho)\psi=0,\quad \rho f_{\rho\rho}/f_\rho\neq \textit{constant}.$$

Remark III.2. The action of the group $G^\sim_{\mathcal{V}}$ on f is by multiplication by nonzero real constants and by complex conjugation, so we restrict our investigation to the class of equations with a fixed nonlinearity $f(\rho)$ under the assumption that f is determined up to a real multiplicative constant or complex conjugation and the only arbitrary element is then V. We denote by $\mathcal{V}^f$ the class of equations in $\mathcal{V}$ satisfying this restriction.

Theorem III.8. *The class $\mathcal{V}^f$ is normalized. Its equivalence group $G^\sim_{\mathcal{V}^f}$ is given by transformations* (13) *with $T_t=1$ ($T_t=\pm1$ if f is a real-valued function) and $\Lambda=0$.*

Lemma III.5. *The maximal Lie invariance algebra $\mathfrak{g}_V$ of an equation $\mathcal{L}_V$ from $\mathcal{V}^f$ with $\rho f_{\rho\rho}/f_\rho\neq\lambda\in\mathbb{R}$, consists of vector fields of the form $D(c_0)+\sum_{a<b}\kappa^{ab}J_{ab}+G(\chi)+\sigma M$, where c_0 is a nonzero real constant, the parameter functions χ^a and σ are arbitrary real-valued smooth functions of t and the matrix (κ^{ab}) is an arbitrary constant skew-symmetric matrix that satisfy the condition*

$$c_0V_t+(\kappa^{ab}x_b+\chi^a)V_a=\frac{1}{2}\chi^a_{tt}x_a+\sigma_t. \tag{19}$$

The kernel invariance Lie algebra of equations from the class $\mathcal{V}^f$ is $\mathfrak{g}^\cap=\langle M\rangle$.

Proof. We put $S=f(\rho)+V(t,x)$ into the classifying condition (16) and then, noting that $\rho f_{\rho\rho}/f_\rho\neq\lambda\in\mathbb{R}$, we obtain $\tau_t=0$ and $\zeta=0$. The kernel is obtained by varying the arbitrary element V. □

Consider the linear span $\mathfrak{g}_{\langle\,\rangle}$,

$$\mathfrak{g}_{\langle\,\rangle}:=\left\{Q=D(1)+\sum_{a<b}\kappa^{ab}J_{ab}+G(\chi)+\sigma M\right\}=\sum_V\mathfrak{g}_V.$$

The following assertions hold for any potential V in the algebra $\mathfrak{g}_V$:

$$\dim\mathfrak{g}_V\leqslant\frac{n(n+3)}{2}+2,\quad \dim\mathfrak{g}_V\cap\langle G(\chi),\sigma M\rangle\leqslant 2n+1,$$
$$\mathfrak{g}_V\cap\langle\sigma M\rangle=\langle M\rangle.$$

5.2 Logarithmic modular nonlinearity

We study the point transformations and Lie symmetry properties for the class $\mathcal{P}_0$ whose equations of the general form

$$i\psi_t+\psi_{aa}+\delta\psi\ln\rho+V(t,x)\psi=0, \tag{20}$$

where δ is an arbitrary nonzero complex number and V is an arbitrary complex-valued function of t and x. The class $\mathcal{P}_0$ is singled out from the class $\mathcal{S}$ by constraints $S_{\rho t}=S_{\rho a}=0$ and $(\rho S_\rho)_\rho=0$, i.e., $\psi S_{\psi t}+\psi^*S_{\psi^*t}=\psi S_{\psi a}+\psi^*S_{\psi^*a}=0$ and $(\psi\partial_\psi+\psi^*\partial_{\psi^*})^2S=0$.

We can find the point transformations connecting two equations from the class $\mathcal{P}_0$ by direct calculation. However, we have already found point transformations for the class $\mathcal{S}$ in this way, so we can simply obtain them from Theorem III.5. Putting $S=\delta\ln\rho+V(t,x)$ into (13c) we see that we have the following transformations:

Theorem III.9. *The class $\mathcal{P}_0$ is normalized. Its equivalence group $G^{\sim}_{\mathcal{P}_0}$ is given by transformations* (13) *with $T_{tt}=0$. Further, the transformations of δ and V are*

$$\tilde{\delta}=\frac{\hat{\delta}}{|T_t|},\quad \tilde{V}=\frac{\hat{V}}{|T_t|}+\frac{\varepsilon\mathcal{X}^b_{tt}}{2|T_t|^{3/2}}O^{ba}x_a-\hat{\delta}\frac{\Lambda}{|T_t|}-\frac{1}{4}\frac{\mathcal{X}^a_t\mathcal{X}^a_t}{T_t^2}+\frac{\Sigma_t-i\Lambda_t}{T_t}, \tag{21}$$

where the parameter functions $\mathcal{X}^a$, Λ and Σ are arbitrary smooth real-valued functions of t, $\varepsilon=\pm1$ and O is an arbitrary $n\times n$ matrix satisfying $OO^T=I$.

Corollary III.7. *The potential V in* (21) *can be gauged to zero by means of point transformations when $V_{ab}=0$.*

Remark III.3. Arguing as in Remark III.2, we can restrict the arbitrary element $\delta=\delta_1+i\delta_2$ by putting $|\delta|=1$, $\delta_2\geqslant 0$ and assuming V to be the only arbitrary element in the equation. The class $\mathcal{P}^0$ with this restriction is denoted by $\mathcal{P}^\delta_0$ and is the class which we consider below. The equivalence group $G^{\sim}_{\mathcal{P}^\delta_0}$ of this class consists of the point transformations of the form (13), where $T_t=1$ if $\delta_2=\ \mathrm{Im}\ \delta>0$ and $T_t=\pm1$ if $\delta_2=0$.

Putting $S=\delta\ln\rho+V(t,x)$ into the classifying condition (16) and then splitting with respect to ρ yields $\tau_t=0$. We then have:

Lemma III.6. *Any vector field Q from the maximal Lie invariance algebra $\mathfrak{g}_V$ of an equation $\mathcal{L}_V$ from the subclass $\mathcal{P}^\delta_0$ is of the form $D(c_0)+\sum_{a<b}\kappa^{ab}J_{ab}+G(\chi)+\sigma M+\zeta I$, where c_0 is a nonzero real constant, the parameter functions χ^a, σ and ζ are arbitrary real-valued smooth functions of t and the matrix (κ^{ab}) is an arbitrary constant skew-symmetric matrix. The coefficients of these vector fields satisfy the classifying condition*

$$c_0V_t+(\kappa^{ab}x_b+\chi^a)V_a=\frac{1}{2}\chi^a_{tt}x_a+\sigma_t-i\zeta_t-\delta\zeta. \tag{22}$$

As in the previous section the linear span $\mathfrak{g}_{\langle\,\rangle}$ of vector fields from $\mathfrak{g}_V$ for any V is given by

$$\mathfrak{g}_{\langle\,\rangle}:=\left\{Q=c_0+\sum_{a<b}\kappa^{ab}J_{ab}+G(\chi)+\sigma M+\zeta I\right\}=\sum_V\mathfrak{g}_V.$$

Hence for any V

$$\dim\mathfrak{g}_V\leqslant n(n+3)/2+3\quad\text{and}\quad\dim\mathfrak{g}_V\cap\langle G(\chi),\sigma M,\zeta I\rangle\leqslant 2n+2.$$

Lemma III.7. $\mathfrak{g}_V\cap\langle\sigma M,\zeta I'\rangle=\langle M,I'\rangle=\mathfrak{g}^\cap_{P^\delta_0}$.

5.3 Power nonlinearity

The class $\mathcal{P}_\lambda$ of nonlinear Schrödinger equations with potentials and power nonlinearity consists of the equations of the form,

$$i\psi_t+\psi_{aa}+\delta\rho^\lambda\psi+V(t,x)\psi=0, \tag{23}$$

where δ and λ are arbitrary nonzero complex and real constants, respectively, and V is an arbitrary complex-valued potential depending on t and x. This class is derived from the class (4) by imposing the conditions $S_{\rho t}=S_{\rho a}=0$, $(\rho S_\rho)_\rho=\lambda S_\rho$, or equivalently by

$$\begin{gathered}\psi S_{\psi t}+\psi^*S_{\psi^*t}=\psi S_{\psi a}+\psi^*S_{\psi^*a}=0,\\(\psi\partial_\psi+\psi^*\partial_{\psi^*})^2S=\lambda(\psi\partial_\psi+\psi^*\partial_{\psi^*})S.\end{gathered} \tag{24}$$

It follows from the definition of the class $\mathcal{P}_\lambda$ of equations (23) with restrictions (24) that the equivalence transformations will depend on the parameter λ, which means that this class is normalized in the extended sense. Further, λ is an invariant of all admissible transformations in the class (4) so that equations with different values of λ do not transform into one another. To facilitate the group classification of equations from the class $\mathcal{P}_\lambda$, we assume that λ is fixed. Here and below $\mathcal{P}_\lambda$ stands for the class of equations (23) with a fixed value of λ. There are two ways to find the equivalence group $G^{\sim}_{\mathcal{P}_\lambda}$ for the class of equations (23). The first is to find all invertible point transformations in the space $(t, x, \psi, \psi^*, S, S^*)$, under which the system constituted by equations (17) and (24) is invariant. The second is to consider the class $\mathcal{P}_\lambda$ with the involved arbitrary elements δ and V, where

$$i\psi_t + \psi_{aa} + \delta\rho^\lambda\psi + V\psi = 0, \quad V_\psi = V_{\psi^*} = 0, \quad \delta_t = \delta_a = \delta_\psi = \delta_{\psi^*} = 0.$$

and then single out the equivalence group $G^{\sim}_{\mathcal{P}_\lambda}$ from the point transformations (13a)–(13c) of Theorem III.5.

Theorem III.10. *The class $\mathcal{P}_\lambda$ with $\lambda \neq 0$ is normalized. The equivalence group $G^{\sim}_{\mathcal{P}_\lambda}$ consists of point transformations* (13)*, where $e^\Lambda = \mu|T_t|^{-1/\lambda}$, $\mu > 0$. The transformed values of δ and V are*

$$\begin{aligned}\tilde{\delta} &= \frac{\hat{\delta}}{\mu^\lambda},\\ \tilde{V} &= \frac{\hat{V}}{|T_t|} + \frac{2T_{ttt}T_t - 3{T_{tt}}^2}{16\varepsilon' {T_t}^3}x_a x_a + \frac{\varepsilon'}{2|T_t|^{1/2}}\left(\frac{\mathcal{X}^b_t}{T_t}\right)_t O^{ba}x_a\\ &\quad + \frac{\Sigma_t}{T_t} - \frac{\mathcal{X}^a_t\mathcal{X}^a_t}{4{T_t}^2} + i\lambda'\frac{T_{tt}}{T_t^2},\end{aligned}$$

where $\lambda' = \frac{1}{\lambda} - \frac{n}{4}$, T, χ and Σ are real-valued functions of t. As well as this family of continuous transformations, the group $G^{\sim}_{\mathcal{P}_\lambda}$ also contains two discrete transformations: the space reflection I_a for a fixed a ($\tilde{t} = t$, $\tilde{x}_a = -x_a$, $\tilde{x}_b = x_b, b \neq a$, $\tilde{\psi} = \psi$, $\tilde{\delta} = \delta$, $\tilde{V} = V$) and the Wigner time reflection I_t ($\tilde{t} = -t$, $\tilde{x} = x$, $\tilde{\psi} = \psi^*$, $\tilde{\delta} = \delta^*$, $\tilde{V} = V^*$).

Corollary III.8. *A $(1 + n)$-dimensional nonlinear Schrödinger equation with a power nonlinearity of the form* (23) *is equivalent to a nonlinear Schrödinger equation independent of x with respect to a point transformation if*

$$V = h(t)x_a x_a + h^a(t)x_a + \tilde{h}^0(t) + ih^0(t), \tag{25}$$

where h, h^a, h^0 and $\tilde{h}^0$ are real-valued functions of t. More specifically, for $\lambda = 4/n$, any real-valued potential quadratic in x can be transformed to zero. For $\lambda \neq 4/n$, the potential V in (25) *can be gauged to zero if and only if $16(\lambda')^2 h = 2\epsilon\lambda' h^0_t + (h^0)^2$.*

Proof. It is clear that the expression for V in (25) is obtained by putting the transformed potential $\tilde{V} = 0$ in Theorem III.10. For $\lambda \neq 4/n$ we note that (25) and $\tilde{V} = 0$ give us $h^0(t) = -\epsilon\lambda'\frac{T_{tt}}{T_t}$. Differentiating both sides of this expression with respect to t gives $h^0_t = -\epsilon\lambda'\frac{T_{ttt}T_t - T_{tt}^2}{T_t^2}$. We also have $h(t) = -\frac{2T_{ttt}T_t - 3T_{tt}^2}{16T_t^2}$. Solving for $T_{ttt}T_t$ in both $h(t)$ and h^0_t and then equating the two expressions yields the last equality in Corollary III.8. □

Remark III.4. Theorem III.10 implies that any point transformation connecting two equations in the class $\mathcal{P}_\lambda$ acts on δ by multiplication with nonzero real constants and by complex conjugation. We denote by $\mathcal{P}_\lambda^\delta$ the subclass of $\mathcal{P}_\lambda$ with the restriction $|\delta| = 1$ and $\delta_2 \geqslant 0$, as in Remark III.3. The equivalence group $G^\sim_{\mathcal{P}_\lambda^\delta}$ of the class $\mathcal{P}_\lambda^\delta$ consists of the transformations (13a)–(13c), where $T_t > 0$ if $\delta_2 > 0$ and $\mu = 1$. Here and below $\delta_1 = \mathrm{Re}\ \delta$, $\delta_2 = \mathrm{Im}\,\delta$ and we now write $\mathcal{P}_\lambda$ for $\mathcal{P}_\lambda^\delta$ for simplicity of notation.

Theorem III.11. *If two equations from the class $\mathcal{P}_\lambda$ with the parameter values (δ, V) and $(\tilde\delta, \tilde V)$ are transformed each other by point transformations then $\tilde\delta = \delta$. Moreover, since $\delta \neq 0$ any transformation of such type belongs to $G^\sim_{\mathcal{P}_\lambda}$.*

Next consider $S = \delta\rho^\lambda + V(t, x)$. In this case the classifying condition (16) becomes

$$\tau V_t + \left(\frac{1}{2}\tau_t x_a + \kappa^{ab} x_b + \chi^a\right) V_a + \delta(\lambda\zeta + \tau_t)\rho^\lambda + \tau_t V$$
$$= \frac{1}{8}\,\tau_{ttt} x_a x_a + \frac{1}{2}\chi^a_{tt} x_a \sigma_t - i\zeta_t - i\frac{n}{4}\,\tau_{tt}$$

from which we deduce that $\lambda\zeta + \tau_t = 0$. The vector field Q given in Theorem III.6 can be written as $Q = D(\tau) + \sum_{a<b}\kappa^{ab} J_{ab} + G(\chi) + \sigma M - \lambda^{-1}\tau_t I$. Writing $D^\lambda(\tau) = D(\tau) - \lambda^{-1}\tau_t I$ and putting all the above information together we obtain the following result:

Lemma III.8. *Any vector field Q from the maximal Lie invariance algebra $\mathfrak{g}_V$ of an equation $\mathcal{L}_V$ from the class $\mathcal{P}_\lambda$ is of the form $D^\lambda(\tau) + \sum_{a<b}\kappa^{ab} J_{ab} + G(\chi) + \sigma M$. Further, the coefficients of Q satisfy the classifying condition*

$$\tau V_t + \left(\frac{1}{2}\tau_t x_a + \kappa^{ab} x_b + \chi^a\right) V_a + \tau_t V = \frac{1}{8}\,\tau_{ttt} x_a x_a + \frac{1}{2}\chi^a_{tt} x_a + \sigma_t + i\lambda'\tau_{tt}, \quad (26)$$

where $\lambda' = \frac{1}{\lambda} - \frac{n}{4}$.

Lemma III.9. *The kernel invariance Lie algebra of equations from the class $\mathcal{P}_\lambda$ is $\mathfrak{g}^\cap = \langle M\rangle$.*

Consider the linear span

$$\mathfrak{g}_{\langle\,\rangle} := \left\{ Q = D^\lambda(\tau) + \sum_{a<b}\kappa^{ab} J_{ab} + G(\chi) + \sigma M \right\} = \sum_V \mathfrak{g}_V.$$

Then for any V we have

$$\dim \mathfrak{g}_V \leqslant n(n+3)/2 + 4, \quad \mathfrak{g}_V \cap \langle\sigma M\rangle = \langle M\rangle, \quad \dim \mathfrak{g}_V \cap \langle G(\chi), \sigma M\rangle \leqslant 2n + 1,$$

where π^0 is the projection on the space of variable t.

Lemma III.10. *For all V, $\pi^0_*\mathfrak{g}_V$ is a Lie algebra and* $\dim \pi^0_*\mathfrak{g}_V \leqslant 3$. *Moreover,*

$$\pi^0_*\mathfrak{g}_V \in \{0, \langle\partial_t\rangle, \langle\partial_t, t\partial_t\rangle, \langle\partial_t, t\partial_t, t^2\partial_t\rangle\} \bmod \pi^0 G^\sim_{\mathcal{P}_\lambda}, \quad \textit{where} \quad \pi^0_*\mathfrak{g}_{\langle\,\rangle} = \langle\tau\partial_t\rangle.$$

6 Group classification in dimension (1+2)

In this section we present the group classification of the class $\mathcal{V}$. We recall that this class is not normalized but that it is partitioned into the three subclasses $\mathcal{V}'$, $\mathcal{P}_0$ and $\mathcal{P}_\lambda$ consisting of equations whose general forms are (18), (20) and (23), respectively. We first assume the following: the nonlinear term $f(\rho)$ is fixed in $\mathcal{V}'$, we have $|\delta| = 1$ and $\delta_2 \geqslant 0$ in $\mathcal{P}_0$ and $\mathcal{P}_\lambda$ and we then treat the classes $\mathcal{V}^f$, $\mathcal{P}_0^\delta$ and $\mathcal{P}_\lambda^\delta$ with these restrictions and the only arbitrary elements are then the V. Thus, in Section 5 and in the rest of the paper, the Lie symmetry extension is denoted by $\mathfrak{g}_V$. We solve completely the group classification of these classes for $n = 2$. The indices a, b then satisfy $a, b \in \{1, 2\}$ and we sum over repeated indices.

We also use the notations

$$|x| = \sqrt{x_1^2 + x_2^2}, \quad \phi = \arctan x_2/x_1,$$
$$\omega_1 = x_1 \cos\kappa t + x_2 \sin\kappa t, \quad \omega_2 = -x_1 \sin\kappa t + x_2 \cos\kappa t,$$

where in Subsection 6.1 the constant κ is set to be one. For each appropriate subalgebra $\mathfrak{s}$ of $\mathfrak{g}_{\langle\,\rangle}$, we also introduce five integers that are invariant under equivalence transformations: in Section 6.1 and Section 6.3 we define the invariant integers as

$$\begin{aligned}
&k_0 := \dim \mathfrak{s} \cap \langle \sigma M\rangle = \dim \mathfrak{g}^{\cap} = 1,\\
&k_1 := \dim \mathfrak{s} \cap \langle\, G(\chi), \sigma M\rangle - k_0,\\
&k_2 := \dim \mathfrak{s} \cap \langle J,\, G(\chi),\, \sigma M\rangle - k_1 - k_0,\\
&k_3 := \dim \mathfrak{s} - k_2 - k_1 - k_0 = \dim \pi^0_* \mathfrak{s},\\
&r_1 := \operatorname{rank}\{\chi \mid \exists\, \sigma\colon G(\chi) + \sigma M \in \mathfrak{s}\},
\end{aligned}$$

and in Section 6.2 we define the invariant integers as

$$\begin{aligned}
&k_0 := \dim \mathfrak{s} \cap \langle \sigma M, \zeta I\rangle = \dim \mathfrak{g}^{\cap} = 2,\\
&k_1 := \dim \mathfrak{s} \cap \langle G(\chi),\, \sigma M,\, \zeta I\rangle - k_0,\\
&k_2 := \dim \mathfrak{s} \cap \langle J,\, G(\chi), \sigma M, \zeta I\rangle - k_1 - k_0,\\
&k_3 := \dim \mathfrak{s} - k_2 - k_1 - k_0 = \dim \pi^0_* \mathfrak{s},\\
&r_1 := \operatorname{rank}\{\chi \mid \exists\, \sigma, \zeta\colon G(\chi) + \sigma M + \zeta I \in \mathfrak{s}\},
\end{aligned}$$

where π^0 is the projection onto the space of the variable t. It is obvious that in each case

$$\dim \mathfrak{s} = k_0 + k_1 + k_2 + k_3.$$

6.1 General case of nonlinearity

For any V in the class $\mathcal{V}^f$ the following conditions hold for $n = 2$:

$$\dim \mathfrak{g}_V \leqslant 7, \quad k_2 \in \{0, 1\}, \quad k_3 \in \{0, 1\}, \quad r_1 \in \{0, 1, 2\}.$$

Therefore any appropriate subalgebra of $\mathfrak{g}_{\langle\,\rangle}$ is spanned by

- the basis vector field M of the kernel $\mathfrak{g}^{\cap}$,
- k_1 vector fields $G(\chi^{p1}, \chi^{p2}) + \sigma^p M$ with linearly independent tuples (χ^{p1}, χ^{p2}), $p = 1, \dots, p = k_1$,
- k_2 vector fields of the form $J + G(\chi^{01}, \chi^{02}) + \sigma^0 M$,
- k_3 vector fields of the form $D(1) + \kappa^q J + G(\chi^{q1}, \chi^{q2}) + \sigma^q M + \rho^q I$ with $q = k_1 + k_3$.

Theorem III.12. *A complete list of inequivalent Lie symmetry extensions and their corresponding potentials in the class $\mathcal{V}^f$ is given in the following list, where U is an arbitrary complex-valued smooth function of its arguments (or an arbitrary complex constant). The other functions and constants take real values.*

0. $V=V(t,x)$: $\quad\mathfrak g_V=\mathfrak g^\cap=\langle M\rangle$.
1. $V=U(x_1,x_2)$: $\quad\mathfrak g_V=\langle M,\,D(1)\rangle$.
2. $V=U(\omega_1,\omega_2)$: $\quad\mathfrak g_V=\langle M,\,D(1)+J\rangle$.
3. $V=U(t,|x|)+\sigma_t(t)\phi$: $\quad\mathfrak g_V=\langle M,\;J+\sigma(t)M\rangle$.
4. $V=U(|x|)+\varepsilon\phi$, $\varepsilon\in\{0,1\}$: $\quad\mathfrak g_V=\langle M,\,J+\varepsilon tM,\,D(1)\rangle$.
5. $V=U(t,x_2)+\frac14\gamma(t)x_1^2$: $\quad\mathfrak g_V=\langle M,\,G(\chi^{11},0),\,G(\chi^{21},0)\rangle$, where χ^{11} and χ^{21} are linearly independent solutions of the equation $\chi_{tt}=\gamma\chi$, γ is an arbitrary real-valued function of t.
6. $V=U(x_2)+\frac14\gamma x_1^2$, $\gamma\in\{-1,0,1\}$: $\quad\mathfrak g_V=\langle M,\,G(\chi^{11},0),\,G(\chi^{21},0),\,D(1)\rangle$, where χ^{11} and χ^{21} are linearly independent solutions of the equation $\chi_{tt}=\gamma\chi$.
7. $V=U(t,\theta)+\dfrac{\chi^1_{tt}}{4\chi^1}x_1^2+\dfrac{\chi^2_{tt}}{4\chi^2}x_2^2$, $\chi^1\chi^2_t-\chi^2\chi^1_t\neq0$, $\theta=\chi^1x_2-\chi^2x_1$:
   $\mathfrak g_V=\langle M,\,G(\chi^1,\chi^2)\rangle$.
8. $V=U(\omega_2)+\frac14(\beta^2-1)\omega_1^2+\beta\omega_1\omega_2$:
   $\mathfrak g_V=\langle M,\,G(e^{\beta t}\cos t,e^{\beta t}\sin t),\,D(1)+J\rangle$.
9. $V=\frac14h^{ab}(t)x_ax_b+ih^{00}(t)$, $h^{12}=h^{21}$: $\quad\mathfrak g_V=\langle M,\,G(\chi^{p1},\chi^{p2}),\,p=1,\dots,4\rangle$, where $\{(\chi^{p1}(t),\chi^{p2}(t))\}$ is a fundamental set of solutions of the system $\chi^a_{tt}=h^{ab}\chi^b$.
10. $V=\frac14\alpha\omega_1^2+\frac14\beta\omega_2^2+i\nu$, $\alpha\neq\beta$, $\kappa\in\{0,1\}$:
    $\mathfrak g_V=\langle M,\,G(\theta^{p1}\cos\kappa t-\theta^{p2}\sin\kappa t,\;\theta^{p1}\sin\kappa t+\theta^{p2}\cos\kappa t),\,D(1)+\kappa J,\,p=1,\dots,4,\rangle$, where $(\theta^{p1}(t),\theta^{p2}(t))$ are linearly independent solutions of the system $\theta^1_{tt}+2\kappa\theta^2_t=(\kappa^2+\alpha)\theta^1$, $\theta^2_{tt}-2\kappa\theta^1_t=(\kappa^2+\beta)\theta^2$.
11. $V=\frac14h(t)x_ax_a+ih^0(t)$:
    $\mathfrak g_V=\langle M,\,G(\chi^{p1},\chi^{p2}),\,J\rangle$, where $p=1,\dots,4$, $\{(\chi^{p1}(t),\chi^{p2}(t))\}$ is a fundamental set of solutions of the system, $\chi^a_{tt}=h\chi^a$, h and h^0 are real-valued functions of t.
12. $V=\frac14\alpha x_ax_a+i\alpha_0$:
    $\mathfrak g_V=\langle M,\,G(\chi^{11},0),\,G(\chi^{21},0),\,G(0,\chi^{32}),\,G(0,\chi^{42}),\,J,\,D(1)\rangle$,
    where $\{\chi^{11}(t),\chi^{21}(t)\}$ and $\{\chi^{32}(t),\chi^{42}(t)\}$ are fundamental sets of solutions of the equation $\chi_{tt}=\alpha\chi$.

Remark III.5. Here we discuss a few cases for the maximality of essential Lie invariance algebras listed in Theorem III.12 with respect to their corresponding potentials. In some cases we have presented simple necessary and sufficient conditions that provide such inequivalence. In other cases these conditions are not so obvious. Thus in Case 3 the maximality condition for Lie symmetry extension is $U_t\neq0$ or $\sigma_t\neq0$, which excludes the values of V that are $G^\sim$-equivalent to those in Case 4. In Case 5 the condition of maximal Lie symmetry extension is $U_t\neq0$ or $\gamma\notin\{-1,0,1\}$ in order to exclude potentials $G^\sim$-equivalent to those in Case 6. Similarly, potentials in Cases 5–6 and Case 8 are $G^\sim$-inequivalent to those in Cases 9–12 if and only if ($U_{x_2x_2x_2}\neq0$ or $\operatorname{Im}U_{x_2}\neq0$) and ($U_{\omega_2\omega_2\omega_2}\neq0$ or $\operatorname{Im}U_{\omega_2}\neq0$), respectively. To avoid the equivalences between potentials of Case 11 and those of Case 12 we require that $h_t(t)\neq0$ or $h^0_t(t)\neq0$ in Case 11.

Proof. Using k_3, k_2 and r_1 we single out different cases, where the general form of basis vector fields of a Lie symmetry extension is

$$Q^s=D(1)+\kappa^sJ+G(\chi^{s1},\chi^{s2})+\sigma^sM,$$

where the range of s is equal to $\dim\mathfrak g_V-1$, and all the parameter functions are real-valued functions of t.

$\boldsymbol{k_2 = r_1 = 0}$. Any extension of the algebra $\mathfrak{g}_V$ is given by the vector fields of the form $Q^1 = D(1) + \kappa^1 J + G(\chi^{11}, \chi^{12}) + \sigma^1 M$. Up to $G^{\sim}_{\mathcal{V}'}$-equivalence we can set $(\chi^{11}, \chi^{12}) = (0, 0)$, $\sigma^1 = 0$ and reduce Q^1 to $D(1) + \kappa^1 J$ with $\kappa \in \{0, 1\}$. Varying k_3 we obtain three cases, the general Case 0 for $k_3 = 0$ with $\mathfrak{g}_V = \mathfrak{g}^{\cap}$ and Cases 1 and 2 for $k_3 = 1$.

$\boldsymbol{k_2 = 1, r_1 = 0}$. The algebra $\mathfrak{g}_V$ contains, apart from the kernel, a vector field of the form Q^0 with $\kappa^0 = 1$, and additional extension is given by Q^1. The commutation relations of Q^1 and Q^0 together with transformations from $G^{\sim}_{\mathcal{V}'}$ imply that $\chi^{1a} = \chi^{0a} = 0$ and $\kappa^1 = 0$, which gives $Q^0 = J + \sigma^0 M$. If σ^0 is a constant then Q^0 reduces to a single operator J. Otherwise no further reduction is possible. Substituting Q^0 into the classifying condition (19) and integrating the equation in V yields the potential presented in Case 3.

Further, if $\mathfrak{g}_V$ has another extension, it is provided by the vector field $Q^1 = D(1) + \sigma^1 M$. The commutator $[Q^0, Q^1]$ gives $\sigma^0 = t$. Putting Q^1 and Q^0 into the classifying condition (19) gives two independent equations in V whose common solution leads to Case 4.

$\boldsymbol{r_1 = 1}$. If $k_2 = 0$ then the symmetry extension is given by a vector field $Q^1 = G(\chi^{11}, \chi^{12}) + \sigma^1 M$ with $(\chi^{11}, \chi^{12}) \neq (0, 0)$. An additional Lie symmetry extension can be given by another vector field of the same form or a vector field of the form Q^3. Two cases arise. Either the tuple (χ^{11}, χ^{12}) is proportional to a constant tuple or the tuple (χ^{11}, χ^{12}) is not proportional to a constant tuple.

1. Suppose that (χ^{11}, χ^{12}) is proportional to a constant tuple . Then up to $G^{\sim}_{\mathcal{V}'}$-equivalence we can set $\sigma^1 = 0$, $\chi^{12} = 0$ and reduce Q^1 to $G(\chi^{11}, 0)$. Substituting this vector field into the classifying condition (19) yields the potential

$$V = \frac{1}{4}\gamma(t)x_1^2 + U(t, x_2),$$

where γ is a real valued function of t and $\chi^1_{tt} = \gamma\chi^1$. A similar vector field $Q^2 = G(\chi^{21}, 0)$ with $\chi^{21} = h(t)\chi^{11}$ belongs to the algebra $\mathfrak{g}_V$, where χ^{11} and χ^{21} are linearly independent solutions of $\chi^1_{tt} = \gamma\chi^1$, and h is nonzero real-valued function of t. If no further extensions are possible then we have Case 5.

Otherwise, the additional extensions are provided by vector fields $Q^3 = D(1) + \kappa^3 J + G(\chi^{31}, \chi^{32}) + \sigma^3 M$. Up to $G^{\sim}_{\mathcal{V}'}$-equivalence and the commutator $[Q^3, Q^1]$ or $[Q^3, Q^2]$, the vector filed Q^3 reduces to $D(1)$. Substituting $D(1)$ together with $V = \frac{1}{4}\gamma(t)x_1^2 + U(t, x_2)$ into the classifying condition (19), we derive $\gamma = \beta$, where $\beta \in \{-1, 0, 1\}$. This gives Case 6.

2. Suppose next that the tuple (χ^{11}, χ^{12}) is not proportional to a constant tuple, i.e., $\chi^{11}\chi^{12}_{tt} - \chi^{11}_{tt}\chi^{12} \neq 0$. In this case the algebra $\mathfrak{g}_V$ contains only a vector field from the kernel and $Q^1 = G(\chi^{11}, \chi^{12}) + \sigma^1 M$ with linearly independent χ^{11} and χ^{12}. The vector field Q^1 is reduced to $Q^1 = G(\chi^{11}, \chi^{12})$ up to $G^{\sim}_{\mathcal{V}'}$-equivalence. Substituting the components of Q^1 into the classifying condition (19) we derive

$$\chi^{11}V_1 + \chi^{12}V_2 = \frac{1}{2}\chi^{11}_{tt}x_1 + \frac{1}{2}\chi^{12}_{tt}x_2. \tag{27}$$

Letting V_p be a particular solution of the equation (27) such that $V_p = h^{11}(t)x_1^2 + 2h^{12}(t)x_1x_2 + h^{22}(t)x_2^2$, its substitution in equation (27) and splitting with respect to various powers of x yields

$$\chi^{11}h^{11} + \chi^{12}h^{12} = \frac{1}{4}\chi^{11}_{tt}, \quad \chi^{11}h^{12} + \chi^{12}h^{22} = \frac{1}{4}\chi^{12}_{tt}.$$

From $\chi^{11}\chi^{12}_{tt} - \chi^{11}_{tt}\chi^{12} \neq 0$ we can set $h^{12} = 0$, and obtain $h^{11} = \frac{\chi^{11}_{tt}}{4\chi^{11}}$ and $h^{22} = \frac{\chi^{12}_{tt}}{4\chi^{12}}$. If no further extensions are possible then the potentials are given by Case 7. Otherwise the additional symmetry extensions are provided by Q^2, which reduces to $D(1) + \kappa^2 J$, $\kappa^2 \in \{0, 1\}$ up to $G^{\sim}_{\mathcal{V}'}$-equivalence. As the commutator $[Q^1, Q^2]$ belongs to $\mathfrak{g}_V$, we have the equations

$$\chi^{11}_t + \kappa^2\chi^{12} = \beta\chi^{11}, \quad \chi^{12}_t - \kappa^2\chi^{11} = \beta\chi^{12}, \quad \sigma_t = \beta\sigma^1 + \beta_1, \tag{28}$$

where β and β_1 are real constants. The case $\kappa^2 = 0$ necessarily implies the proportionality of the tuple (χ^{11}, χ^{12}), hence $\kappa^2 = 1$. The solutions of equations in (28) jointly with $G^{\sim}_{\mathcal{V}'}$ equivalence depend on the values of β (which can be zero or not) and the general form of potential is presented by Case 8.

Next if $k_2 = 1$ the algebra $\mathfrak{g}_V$ contains the vector fields $Q^0 = J + G(\chi^{01}, \chi^{02}) + \sigma^0 M$ and $Q^1 = G(\chi^{11}, \chi^{12}) + \sigma^1 M$ with $(\chi^{11}, \chi^{12}) \neq (0, 0)$. The commutation relation of Q^0 with Q^1 necessarily implies that $r_1 = 2$, and consequently the case $k_2 = 1, r_1 = 1$ is not possible.

$\boldsymbol{r_1 = 2}$. This means that the algebra $\mathfrak{g}_V$ contains two vector fields of the form $Q^a = G(\chi^{a1}, \chi^{a2}) + \sigma^a M$, where $a = 1, 2$ and $\chi^{11}\chi^{22} - \chi^{12}\chi^{21} \neq 0$. Applying the classifying condition (19) to the components of Q^a gives the equations $V_a = \frac{1}{2}h^{ab}(t)x_b + h^{0a}(t)$, where all the h^{ab}, h^{0a} are real-valued functions of t satisfying the conditions $\chi^a_{tt} = h^{ab}\chi^a$ and $\sigma_t = h^{0a}\chi^a$. This means that the algebra $\mathfrak{g}_V$ contains apart from the kernel, at least four vector fields Q^p, $p = 1, \ldots, 4$. The fact that $V_{12} = V_{21}$ implies that the matrix (h^{ab}) is symmetric and hence the potential V is a quadratic polynomial in x_1 and x_2 with the coefficients being functions of t:

$$V = \frac{1}{4}h^{ab}(t)x_a x_b + h^{0b}(t)x_b + \tilde{h}^{00}(t) + ih^{00}(t).$$

The functions h^{0b} and $\tilde{h}^{00}$ can be set equal to zero up to $G^{\sim}_{\mathcal{V}'}$-equivalence and the potentials reduce to

$$V = \frac{1}{4}h^{ab}(t)x_a x_b + ih^{00}(t). \tag{29}$$

The subclass of the class $\mathcal{V}^f$ with potentials of this form is normalized. Its group classification is partitioned into two normalized subclasses $\mathcal{V}^f_0$ and $\mathcal{V}^f_1$ depending on the values $k_2 = 0$ and $k_2 = 1$, respectively. The subclass $\mathcal{V}^f_1$ (resp. $\mathcal{V}^f_0$) is singled out from the class $\mathcal{V}^f$ by the conditions $h^{12} = h^{21} = 0$, $h^{11} = h^{22}$ (resp. its negation).

We analyze each subclass separately. Consider the subclass $\mathcal{V}^f_0$. We substitute the potentials (29) into the classifying conditions (19) and we obtain the system

$$\chi^a_{tt} = h^{ab}\chi^b, \quad \sigma_t = 0.$$

The equation for the χ^a has a fundamental set of solutions (χ^{p1}, χ^{p2}), $p = 1, \ldots, 4$ and σ is a constant. If no further additional Lie symmetry extensions, the classification for the subclass $\mathcal{V}^f_0$ is given by Case 9. The additional Lie symmetry extension for this case is provided by the vector field $Q^5 = D(1) + \kappa^1 J + G(\chi^{51}, \chi^{52}) + \sigma^5 M$, which is reduced to $D(1) + \kappa^5 J$, $\kappa \in \{0, 1\}$ up to $G^{\sim}_{\mathcal{V}'}$-equivalence. Putting the vector field Q^5 and with the potential (29) into the classifying condition (19) and then splitting with respect to different powers of x_a, yields the following equations

$$h^{11}_t + 2\kappa^5 h^{12} = 0, \quad h^{12}_t + \kappa^5(h^{22} - h^{11}) = 0,$$
$$h^{22}_t - 2\kappa^5 h^{12} = 0, \quad ih^{00}_t = \sigma_t,$$

which, after integrating and rearranging, give us to Case 10.

Next the condition $k_2 = 1$, i.e., the subclass $\mathcal{V}_1^f$, means that apart from the vector fields from the kernel and Q^p, the Lie symmetry extension contains also at least a vector field $Q^0 = J$. In this case we obtain the potentials of the form

$$V = \frac{1}{4}h(t)x_ax_a + ih^0(t), \quad h = \chi_{tt}/\chi, \tag{30}$$

where h and $\tilde{h}^0$ are arbitrary smooth real-valued functions of t. If $k_3 = 0$, then we have no further Lie symmetry extension and we have Case 11. When $k_3 = 1$ the additional Lie symmetry extension is given by vector fields of the form $Q^5 = D(1)+\kappa^1 J+G(\chi^{51},\chi^{52})+\sigma^5 M$, which, up to $G^{\sim}_{\mathcal{V}'}$-equivalence, reduces to $Q^5 = D(1) + \kappa^5 J$ and the form of the potential V is preserved under the equivalence. The commutation relations of Q^5 with Q^p, $p = 0,\dots,4$, show that $\kappa = 0$. Substituting the potential (30) and the components of Q^5 into the classifying conditions (19), we obtain $h = \alpha$ and $h^0 = \alpha_0$, where α and α_0 are real constants. Hence $V = \frac{1}{4}\alpha x_ax_a + i\alpha_0$, which leads to Case 12. □

6.2 Logarithmic modular nonlinearity

We recall that the class of Schrödinger equations with a logarithmic modular nonlinearity with a fixed δ is denoted by $\mathcal{P}_0^\delta$ and consists of equations of the form (20). In the case $n = 2$, for any V, the results from Section 5.2 imply that

$$\dim \mathfrak{g}_V \leqslant 8, \quad \dim \mathfrak{g}_V \cap \langle G(\chi), \sigma M, \zeta I\rangle \leqslant 6,$$
$$k_2 \in \{0,1\}, \quad k_3 \in \{0,1\}, \quad r_1 \in \{0,1,2\}.$$

It then follows that any appropriate subalgebra of $\mathfrak{g}_{\langle\,\rangle}$ is spanned by

- the basis vector fields M and I of the kernel $\mathfrak{g}^\cap$,
- k_1 vector fields $G(\chi^{p1},\chi^{p2})+\sigma^p M+\zeta^p I$ with linearly independent tuples (χ^{p1},χ^{p2}), $p = 1,\dots, k_1$,
- k_2 vector fields $J + G(\chi^{01},\chi^{02}) + \sigma^0 M + \zeta^0 I$,
- k_3 vector fields $D(1) + \kappa^q J + G(\chi^{q1},\chi^{q2}) + \sigma^q M + \zeta^q I$ with $q = k_1 + k_3$.

We will also use the following notation:

$$I' = e^{-\delta_2 t}(\delta_2 I - \delta_1 M) \quad \text{if} \quad \delta_2 \neq 0 \quad \text{and} \quad I' = I + \delta_1 tM \quad \text{if} \quad \delta_2 = 0,$$
$$G'(\chi^a) = G(\chi^a) - \tilde{\chi}^a I - \delta_1 \int \tilde{\chi}^a \mathrm{d}tM \quad \text{with} \quad \tilde{\chi}^a = e^{-\delta_2 t}\int e^{\delta_2 t}\chi^{ab}h^{0b}\mathrm{d}t,$$

where all involved parameters will be explained in the corresponding places.

Theorem III.13. *The complete list of inequivalent cases of V admitting Lie symmetry extensions of the maximal Lie invariance algebra of equation of the form* (20) *is given by the potentials presented below, where the function U is an arbitrary complex-valued smooth of its arguments (or an arbitrary complex constant) and the other functions and constants take real values.*

0. $V = V(t,x)$: $\mathfrak{g}_V = \mathfrak{g}^\cap = \langle M,\, I'\rangle$.
1. $V = U(x_1,x_2)$: $\mathfrak{g}_V = \langle M,\, I',\, D(1)\rangle$.
2. $V = U(\omega_1,\omega_2)$: $\mathfrak{g}_V = \langle M,\, I',\, D(1) + \kappa J,\, \kappa \neq 0\rangle$.
3. $V = U(t,|x|) + (\sigma_t - i\zeta_t - \delta\zeta)\phi$: $\mathfrak{g}_V = \langle M,\, I',\, J + \sigma(t)M + \zeta(t)I\rangle$.

4. $V=U(|x|)+\alpha\phi$: $\mathfrak{g}_V=\langle M,\,I',\,J+\alpha tM,\,D(1)\rangle$.
5. $V=U(t,x_2)+\frac{1}{4}h^{11}(t)x_1^2+(ih^{01}(t)+\tilde h^{01}(t))x_1$:
$\mathfrak{g}_V=\langle M,\,I',\,G'(\chi^{11},0),\,G'(\chi^{21},0)\rangle$, where χ^{11} and χ^{21} are linearly independent solutions of the equation $\chi_{tt}=h^{11}(t)\chi$, h^{11}, h^{01} and $\tilde h^{01}$ are real-valued smooth functions of t.
6. $V=U(x_2)+\frac{\beta_1}{4}x_1^2+(i\beta_2+\beta_3)x_1$:
$\mathfrak{g}_V=\langle M,\,I',\,G'(\chi^{11},0),\,G'(\chi^{21},0),\,D(1)\rangle$, where χ^{11} and χ^{21} are linearly independent solutions of the equation $\chi_{tt}=\beta_1\chi$.
7. $V=U(t,\theta)+\frac{1}{4}h^{11}(t)x_1^2+\frac{1}{4}h^{22}(t)x_2^2+ih^{01}(t)x_1,\ \theta=\chi^{11}x_2-\chi^{12}x_1$:
$\mathfrak{g}_V=\langle M,\,I',\,G'(\chi^{11},\chi^{12})\rangle$, where h^{11}, h^{12} and h^{01} are real-valued smooth functions of t, χ^{11} and χ^{12} are linearly independent, $h^{11}=\chi^{11}_{tt}/\chi^{11}$ and $h^{22}=\chi^{12}_{tt}/\chi^{12}$.
8. $V=U(\omega_2)+\frac{1}{4}(\beta^2-\kappa^2)\omega_1^2+\beta\kappa\omega_1\omega_2-\alpha(i\beta+\delta)\omega_1,\ \beta\alpha\neq0,\ \kappa\neq0$:
$\mathfrak{g}_V=\langle M,\,I',\,G'(e^{\beta t}\cos\kappa t,e^{\beta t}\sin\kappa t),\,D(1)+\kappa J\rangle$.
9. $V=U(\omega_2)-\frac{1}{4}\kappa^2\omega_1^2+(\alpha-i\delta_2)\omega_1,\ \kappa\neq0$:
$\mathfrak{g}_V=\langle M,\,I',\,G'(\cos\kappa t,\sin\kappa t),\,D(1)+\kappa J\rangle$.
10. $V=\frac{1}{4}h^{ab}(t)x_ax_b+ih^{0b}(t)x_b$:
$\mathfrak{g}_V=\langle M,\,I',\,G'(\chi^{p1},\chi^{p2}),\,p=1,\dots,4\rangle$, where $\{(\chi^{p1}(t),\chi^{p2}(t))\}$ is a fundamental set of solutions of the system $\chi^a_{tt}=h^{ab}\chi^b$ and $\zeta^p=-\tilde\chi^p$.
11. $V=\frac{1}{4}\alpha\omega_1^2+\frac{1}{4}\beta\omega_2^2+i\nu_a\omega_a,\ \alpha\neq\beta,\ \forall\kappa$:
$\mathfrak{g}_V=\langle M,\,I',\,G'(\theta^{p1}\cos\kappa t-\theta^{p2}\sin\kappa t,\theta^{p1}\sin\kappa t+\theta^{p2}\cos\kappa t),\,D(1)+\kappa J,\ p=1,\dots,4\rangle$, where $(\theta^{p1}(t),\theta^{p2}(t))$ are linearly independent solutions of the system $\theta^1_{tt}-2\kappa\theta^2_t=(\kappa^2+\alpha)\theta^1$, $\theta^2_{tt}+2\kappa\theta^1_t=(\kappa^2+\beta)\theta^2$, and $\zeta^p=-\tilde\chi^p$.
12. $V=\frac{1}{4}h(t)x_ax_a$:
$\mathfrak{g}_V=\langle M,\,I',\,G(\chi^{p1},\chi^{p2}),\,J\rangle$, where $p=1,\dots,4$ and $\{(\chi^{p1}(t),\chi^{p2}(t))\}$ is a fundamental set of solutions of the system $\chi_{tt}=h\chi$.
13. $V=\frac{1}{4}\beta x_ax_a$:
$\mathfrak{g}_V=\langle M,\,I',\,G(\chi^1,0),\,G(\chi^2,0),\,G(0,\chi^1),\,G(0,\chi^2),\,J,\,D(1)\rangle$,
where $(\chi^1(t),\chi^2(t))$ is fundamental set of solutions of the equation $\chi_{tt}=\beta\chi$.

Remark III.6. The Lie invariance algebras presented in Theorem III.13 are in fact maximal for the corresponding potentials. However, there may be special cases in which a given potential is $G^\sim$-equivalent to potential of another case. Here we list some cases that are easy to single out. In Case 3 the maximality condition for Lie symmetry extensions is $U_t\neq0$ or $\sigma_t\neq0$, which excludes the values of V that are $G^\sim$-equivalent to those in Case 4. The condition for being a maximal Lie symmetry extension in Case 5 is $U_t\neq0$ or $h^{11}_t\neq0$ or $h^{01}_t\neq0$ or $\tilde h^{11}_t\neq0$ to exclude potentials $G^\sim$-equivalent to those in Case 6. In Cases 5–6 and cases 8–9 the potentials are $G^\sim$-inequivalent to those presented in Cases 10–13 if and only if $U_{x_2x_2x_2}\neq0$ or $\operatorname{Im}U_{x_2x_2}\neq0$ and $U_{\omega_2\omega_2\omega_2}\neq0$ or $\operatorname{Im}U_{\omega_2\omega_2}\neq0$. The parameter function $h(t)$ in Case 12 satisfies $h_t(t)\neq0$ to avoid the potentials V that are equivalent to those in Case 13.

Proof. Lie symmetry extensions are obtained by choosing different values for k_2, k_3 and r_1, where the basis vector fields are denoted by

$$Q^s=D(1)+\kappa^sJ+G(\chi^{s1},\chi^{s2})+\sigma^sM+\zeta^sI.$$

Here $0\leqslant s\leqslant\dim\mathfrak{g}_V-2$ and all the parameter functions are real-valued functions of t.

$\boldsymbol{k_2 = r_1 = 0}$. Any symmetry extension is given by the vector field $Q = D(1) + \kappa J + G(\chi^1, \chi^2) + \sigma M + \zeta I$. Up to $G^{\sim}_{\mathcal{P}^{\delta}_0}$-equivalence we can set $\chi^a = \sigma = \zeta = 0$ and reduce Q to $D(1) + \kappa J$. If $k_3 = 0$ Case 0 presents the general potential V. Otherwise we have Case 1 and Case 2.

$\boldsymbol{k_2 = 1, r_1 = 0}$. The algebra $\mathfrak{g}_V$ contains at least one vector field of the form Q^0 with $\kappa^0 = 1$. Up to $G^{\sim}_{\mathcal{P}^{\delta}_0}$-equivalence we can set $\chi^{0a} = 0$ and the vector field Q^0 reduces to $J + \sigma M + \rho I$. If there is no further extension then we set Q^0 into the classifying condition (22) and obtain the potential of Case 3. An additional extension is given by $\langle D(1)\rangle$ with $Q^1 = D(1) + \kappa^1 J + G(\chi^{1a}) + \sigma^1 M + \zeta^1 I$. Using $G^{\sim}_{\mathcal{P}^{\delta}_0}$-equivalence and computing the commutation relation $[Q^1, Q^0]$, and taking linear combinations with the kernel, we can set $\chi^{1a} = 0$, $\kappa^1 = \sigma^1 = \zeta^1 = 0$ and reduce Q^1 to $D(1)$ and Q^0 to $Q^0 = J + \alpha t M + \beta t I$, where α and β are arbitrary real constants. Using the classifying condition (22) for the components of the vector fields Q^0 and Q^1 we obtain two equations for V: $V_t = 0$, $x_1 V_2 - x_2 V_1 = \alpha - i\beta - \delta\beta t$, from which it is evident that $\beta = 0$. The solution is the potential of Case 4.

$\boldsymbol{r_1 = 1}$. Suppose $k_2 = 1$. Then the symmetry algebra $\mathfrak{g}_V$ is spanned by the vector fields M, I', $Q^0 = J + G(\chi^{01}, \chi^{02}) + \sigma^0 M + \zeta^0 I$ and $Q^1 = G(\chi^{11}, \chi^{12}) + \sigma^1 M + \zeta^1 I$, where χ^{11} and χ^{12} are linearly independent. Since $[Q^0, Q^1]$ should belong to $\mathfrak{g}_V$, then

$$[Q^0, Q^1] = G(\chi^{12}, -\chi^{11}) + \frac{1}{2}(\chi^{01}\chi^{11}_t + \chi^{02}\chi^{12}_t - \chi^{01}_t\chi^{11} - \chi^{02}_t\chi^{12})M,$$

which implies $r_1 = 2$, contradicting the fact that $r_1 = 1$. Thus the value $k_2 = 1$ is not possible.

We then look for the Lie symmetry extensions corresponding to $k_2 = 0$. The algebra $\mathfrak{g}_V$ then contains vector fields of the form Q^1 with linearly independent χ^{11} and χ^{12}. The analysis of this case depends whether the tuple (χ^{11}, χ^{12}) is proportional to a constant tuple or not.

1. Let the tuple (χ^{11}, χ^{12}) be proportional to a constant tuple. Then up to $G^{\sim}_{\mathcal{P}^{\delta}_0}$-equivalence we can set $\sigma^1 = 0$, $(\chi^{11}, \chi^{12}) = (\chi^{11}, 0)$ and reduce the vector field Q^1 to $Q^1 = G(\chi^{11}, 0) + \zeta^1 I$. Substituting this vector field into the classifying condition (22) and then integrating the resulting equation for V gives

$$V = U(t, x_2) + \frac{1}{4}h^{11}(t)x_1^2 + ih^{01}(t)x_1 + \tilde{h}^{01}(t)x_1, \tag{31}$$

where $h^{11}(t)$, $h^{01}(t)$ and $\tilde{h}^{01}(t)$ are smooth real-valued functions of t with $h^{11}(t) = \chi^{11}_{tt}/\chi^{11}$, $h^{01}(t) = -(\zeta^1_t + \delta_2\zeta^1)/\chi^{11}$ and $\tilde{h}^{01}(t) = -\delta_1\zeta^1/\chi^{11}$. If $Q^1 = G(\chi^{11}, 0) + \zeta^1 I \in \mathfrak{g}_V$ then a similar vector field $Q^2 = G(\chi^{21}, 0) + \zeta^2 I$ belongs to $\mathfrak{g}_V$ with $\chi^{21} = \gamma(t)\chi^{11}$, where $\gamma(t)$ is a real-valued function of t, $\zeta^2 = e^{-\delta_2 t}\tilde{\chi}^{21}$.

If $k_3 = 0$, the Lie symmetry extension is given by Case 5. If $k_3 \neq 0$ there is an additional extension given by the vector field Q^3, which up to $G^{\sim}_{\mathcal{P}^{\delta}_0}$-equivalence is reduced to $D(1) + \kappa^3 J$. The commutation relations $[Q^3, Q^1]$ and $[Q^2, Q^3]$ give $\kappa^3 = 0$. Putting Q^3 together with the potential in (31) yields $h^{11} = \chi^{11}_{tt}/\chi^{11} = \beta_1$, $h^{01} = \beta_2$ and $\tilde{h}^{01} = \beta_3$ with real constants β_j, $j = 1, \ldots, 3$. This gives Case 6.

2. Next, suppose that the tuple (χ^{11}, χ^{12}) is not proportional to a constant tuple. Then let Q^1 with $\kappa^1 = 0$ be a vector field contained in the algebra $\mathfrak{g}_V$, where χ^{11} and χ^{12}

are linearly independent. Putting the components of this vector field into the classifying condition (22) we obtain

$$\chi^{11}V_1+\chi^{12}V_2=\frac{1}{2}\chi^{11}_{tt}x_1+\frac{1}{2}\chi^{12}_{tt}x_2-i(\zeta^1_t+\delta_2\zeta^1)+\sigma_t-\delta_1\zeta^1. \tag{32}$$

Without loss of generality, let

$$V=\frac{1}{4}h^{11}(t)x_1^2+\frac{1}{2}h^{12}(t)x_1x_2+\frac{1}{4}h^{22}(t)x_2^2+(ih^{01}(t)+\tilde h^{01}(t))x_1+ih^{00}(t)+\tilde h^{00}(t)$$

be a general solution of equation (32), where h^{ab}, h^{01}, $\tilde h^{01}$, h^{00} and $\tilde h^{00}$ are real-valued smooth functions of t. Then, substituting V_1 and V_2 into (32) and collecting the coefficients of the various powers of x gives the following equations

$$\begin{aligned}&\chi^{11}h^{11}+\chi^{12}h^{12}=\chi^{11}_{tt},\quad \chi^{11}h^{12}+\chi^{12}h^{22}=\chi^{12}_{tt},\\ &\chi^{11}h^{01}=-(\zeta^1_t+\delta_2\zeta^1),\quad \chi^{11}\tilde h^{01}=\sigma_t-\delta_1\zeta^1.\end{aligned}$$

The condition $V_{12}=V_{21}$ implies that $h^{12}=h^{21}=0$. Up to $G^{\sim}_{\mathcal{P}^\delta_0}$-equivalence we can set $\tilde h^{01}=h^{00}=\tilde h^{00}=0$. If there are no further extensions we have Case 7, where $\zeta^1=-e^{-\delta_2 t}\int e^{\delta t}\chi^{11}h^{01}\mathrm{d}t$ and $\sigma_t=\delta_1\zeta^1$. Otherwise, an additional vector field is provided by the vector field Q^2, which reduces to $D(1)+\kappa J$ up to $G^{\sim}_{\mathcal{P}^\delta_0}$-equivalence. Then the commutation relation $[Q^1,Q^2]$ gives the equations

$$\begin{aligned}&\chi^{11}_t+\kappa\chi^{12}=\beta\chi^{11},\quad \chi^{12}_t-\kappa\chi^{11}=\beta\chi^{12},\\ &\sigma^1_t=\beta\sigma^1-\alpha_1\delta_1e^{-\delta_2 t}+\alpha_0,\quad \zeta^1_t=\beta\zeta^1+\alpha_1\delta_2e^{-\delta_2 t},\end{aligned}$$

where β, δ_1, δ_2, α_0 and α_1 are real constants. The solution of the first two equations is $(\chi^{11},\chi^{12})=(e^{\beta t}\cos\kappa t,e^{\beta t}\sin\kappa t)$. Then we have two cases: $\beta\neq0$ and $\beta=0$.

For $\beta\neq0$ the solutions for σ^1 and ζ^1 are $\sigma^1=\delta_1\mu e^{-\delta_2 t}+\mu_1e^{\beta t}-\alpha_0\beta^{-1}$ and $\zeta^1=\mu_2e^{\beta t}-\delta_2\mu e^{-\delta_2 t}$ with real constants μ_1, μ_2 and $\mu:=\alpha_1(\beta+\delta_2)^{-1}$. Up to $G^{\sim}_{\mathcal{P}^\delta_0}$-equivalence and linearly combining with elements from the kernel we can preserve the vector Q^2, set $\mu_1=0$ and reduce the vector Q^1 to $G(e^{\beta t}\cos\kappa t,e^{\beta t}\sin\kappa t)+\alpha e^{\beta t}I$, $\alpha:=\mu_2$. Substituting the components of Q^1 and Q^2 in the classifying condition (22) we obtain two independent equations for V whose general solution is given in Case 8.

For $\beta=0$ we find that $\zeta^1=-\alpha_1e^{-\delta_2 t}+\alpha_2$ and $\sigma^1=\alpha_1\delta_1\delta_2^{-1}e^{-\delta_2 t}+\alpha_0t+\alpha_3$, where δ_1, δ_2 α_0, α_1, α_2 and α_3 are real constants. With this we can then combine the resulting expression for Q^1 with constant linear combinations of the elements from the kernel to reduce Q^1 to $Q^1=G(\cos\kappa t,\sin\kappa t)+\alpha_0tM+\alpha_2I$. Q^2 remains unchanged in this process. Substituting the components of Q^1 and Q^2 into the classifying condition (22) we obtain two equations for V:

$$V_t+\kappa(x_1V_2-x_2V_1)=0,\quad \cos\kappa tV_1+\sin\kappa tV_2=-\frac{\kappa^2}{2}\omega_1+(\alpha_0-\delta_1)-i\delta_2,$$

whose solution is given in Case 9.

$\boldsymbol{r_1=2}$. Then the algebra $\mathfrak{g}_V$ contains at least two vector fields Q^1,Q^2 with $\chi^{11}\chi^{22}-\chi^{12}\chi^{21}\neq0$. Putting the coefficients of the parameter functions χ, M and I in the classifying condition (22) yields the following equations for V:

$$\chi^{ab}V_b=\frac{1}{2}\chi^{ab}x_b+\sigma^a_t-i(\zeta^a_t+\delta_2\zeta^a)-\delta_1\zeta^a,$$

which are equivalent to $V_a = \frac{1}{2}h^{ab}(t)x_b + \tilde{h}^{0b}(t) + ih^{0b}(t)$, where all h^{0b}, h^{ab} are real-valued functions of t with

$$\chi^{ab}_{tt} = \chi^{ac}h^{cb}, \quad \sigma^a_t - \delta_1\zeta^a = \chi^{ac}\tilde{h}^{0c}, \quad \zeta^a_t + \delta_2\zeta^a = -\chi^{ac}h^{0c}.$$

Since $V_{12} = V_{21}$ we find that the matrix $H = (h^{ab})$ is symmetric and that V is a quadratic polynomial in (x_1, x_2) with coefficients depending on t:

$$V = \frac{1}{4}h^{ab}(t)x_ax_b + \tilde{h}^{0b}(t)x_b + ih^{0b}(t)x_b + h^{00}(t) + i\tilde{h}^{00}(t),$$

where the functions $\tilde{h}^{0b}$, h^{00}, $\tilde{h}^{00}$ can be set to zero up to $G^{\sim}_{\mathcal{P}^\delta_0}$-equivalence. This reduces the potentials to

$$V = \frac{1}{4}h^{ab}(t)x_ax_b + ih^{0b}(t)x_b. \tag{33}$$

Let us consider the group classification problem of the subclass singled out from the class $\mathcal{P}^\delta_0$ by potentials of the form 33. This subclass is normalized and its group classification depends on subcases $k_2 = 1$ or $k_2 = 0$. Denoting by $\mathcal{P}^\delta_{01}$ and $\mathcal{P}^\delta_{00}$ the subclasses corresponding to these values of k_2, and putting $H = (h^{ab})$ (the matrix formed by the parameter-functions h^{ab} in (33)) we have $H = \Lambda(t)\mathbb{I}$ for some function $\Lambda(t)$ for the class $\mathcal{P}^\delta_{01}$ and $H \neq \Lambda(t)E$ for the class $\mathcal{P}^\delta_{00}$, where E denotes the 2×2 identity matrix.

1. Suppose that $k_2 = 0$. If there is no further Lie symmetry extension we substitute the potentials (33) into the classifying condition (22) and split with respect to various powers of x, which gives the system

$$\chi^a_{tt} = h^{ab}\chi^b, \quad \sigma_t - \delta_1\zeta = 0, \qquad \zeta_t + \delta_2\zeta = -h^{0b}\chi^b. \tag{34}$$

The first system has a fundamental set of solutions (χ^{p1}, χ^{p2}), where the tuples (χ^{p1}, χ^{p2}) are linearly independent and the second system gives $\zeta^p = -e^{-\delta_2 t}\int e^{\delta_2 t}h^{0b}\chi^{pb}$, which leads to Case 10. The additional extensions are provided by vector fields of the form Q^5, where, up to $G^{\sim}_{\mathcal{P}^\delta_0}$-equivalence and linear combinations with the kernel, we can keep the form of V and set the parameter functions χ^{5a}, σ^5 and ζ^5 equal to zero, i.e., $Q^5 = D(1) + \kappa^5 J$. Substituting the components of Q^5 together with the potential (33) in the classifying condition (22) and then splitting with respect to powers of x, we obtain the system (34) and the following equations:

$$\begin{aligned} &h^{11}_t + 2\kappa^5h^{12} = 0, \quad h^{12}_t + \kappa^5(h^{22} - h^{11}) = 0, \quad h^{22}_t - 2\kappa^5h^{12} = 0,\\ &h^{01}_t + \kappa^5h^{02} = 0, \quad h^{02}_t - \kappa^5h^{01} = 0. \end{aligned} \tag{35}$$

If $\kappa = 0$ the solution of equations in (35) implies that the matrix H is a symmetric constant matrix and the functions h^{0b} are real constants. Up to similarity transformations this matrix can be reduced to a constant diagonal matrix with entries α and β where $\alpha \neq \beta$ and $h^{0b} = \nu^b$. If $\kappa \neq 0$ then the solution of differential equations (35) is given by

$$\begin{aligned} &h^{11} = A\cos 2\kappa t + \mu, \quad h^{12} = h^{21} = A\sin 2\kappa t, \quad h^{22} = -A\cos 2\kappa t + \mu,\\ &h^{01} = \nu_1\cos\kappa t - \nu_2\sin\kappa t, \quad h^{02} = \nu_1\sin\kappa t + \nu_2\cos\kappa t, \end{aligned}$$

where A is a nonzero positive real constant, $\kappa := \kappa^5$ a nonzero constant and μ, ν_1 and ν_2 are real constants. Substituting this into the expression for V and rearranging the terms and using the definitions of ω_1 and ω_2 given at the beginning of Section 6, we obtain V as a function of ω_1 and ω_2 as given in Case 11. Note also that the vector $\chi(t) = (\chi^1(t), \chi^2(t))$ satisfies the system $\chi_{tt} = H\chi$. Introducing $\theta^1(t), \theta^2(t)$ by putting $\chi^1 = \theta^1(t)\cos\kappa t - \theta^2(t)\sin\kappa t$ and $\chi^2 = \theta^1(t)\sin\kappa t + \theta^2(t)\cos\kappa t$, then we find, after some calculation, that θ^1, θ^2 satisfy the conditions stated in Case 11.

2. The condition $k_2 = 1$ requires the potentials in (33) to be of the form $V = \frac{1}{4}h(t)x_a x_a$ with an arbitrary real-valued function $h(t)$. If the algebra $\mathfrak{g}_V$ does not contain an operator from $\langle D(1)\rangle$ then we only have vector fields Q^a, where $\kappa^a = 0$ and the tuple $\{(\chi^{p1}(t), \chi^{p2}(t))\}$, $p = 1, \ldots, 4$, is a fundamental set of solutions of the system $\chi_{tt} = h\chi$. If the algebra does contain operator from $\langle D(1)\rangle$, then any additional extension is given by Q^5. Then using the commutation relations with the Q^a and the equivalence group $G^{\sim}_{\mathcal{P}^{\delta}_0}$, the vector field Q^5 can be reduced to $D(1)$. Substituting the components of Q^5 and the expression $V = \frac{1}{4}h(t)x_a x_a$ for the potential into the classifying condition (22) we find that $h(t) = \beta$, where β is a real constant. Hence we obtain Cases 12 and 13. □

6.3 Power nonlinearity

We solve the group classification problem of the class $\mathcal{P}^{\delta}_{\lambda}$ of nonlinear Schrödinger equations with potentials and power nonlinearity with a fixed δ, $\delta \in \mathbb{C}_0$ for $n = 2$. For any potential V in an equation of this class we have

$$\dim \mathfrak{g}_V \leqslant 9, \quad \dim \mathfrak{g}_V \cap \langle G(\chi), \sigma M, \zeta I\rangle \leqslant 5,$$
$$k_2 \in \{0,1\}, \quad k_3 \in \{0,1,2,3\}, \quad r_1 \in \{0,1,2\},$$

and $\lambda' = \frac{1}{\lambda} - \frac{1}{2}$. The algebra $\mathfrak{g}_V$ is spanned by the following vector fields:

- the basis vector field M of the kernel $\mathfrak{g}^{\cap}$,
- k_1 vector fields $G(\chi^{p1}, \chi^{p2}) + \sigma^p M$ with linearly independent tuples (χ^{p1}, χ^{p2}) and $p = 1, \ldots, k_1$,
- k_2 vector fields of the form $J + G(\chi^{01}, \chi^{02}) + \sigma^0 M$,
- k_3 vector fields $D^{\lambda}(\tau^{q-k_1}) + \kappa^q J + G(\chi^{q1}, \chi^{q2}) + \sigma^q M$, $q = k_1+1, \ldots, k_1+k_3$ with linearly independent $\tau^1, \ldots, \tau^{k_3}$.

Theorem III.14. *The complete list of inequivalent Lie symmetry extensions and their corresponding potentials for the class $\mathcal{P}_{\lambda}$ is given below, where U is an arbitrary complex-valued smooth function of its arguments (or an arbitrary complex constant), the other functions and constants take real values.*

0. $V = V(t,x)$: $\mathfrak{g}_V = \mathfrak{g}^{\cap} = \langle M\rangle$.
1. $V = U(x_1, x_2)$: $\mathfrak{g}_V = \langle M,\ D(1)\rangle$.
2. $V = U(\omega_1, \omega_2)$, $\kappa \neq 0$: $\mathfrak{g}_V = \langle M,\ D(1) + \kappa J\rangle$.
3. $V = |x|^{-2}U(\zeta)$, $\zeta = \phi - 2\beta\ln|x|$, $\beta > 0$, $U_\zeta \neq 0$: $\mathfrak{g}_V = \langle M,\ D(1),\ D^{\lambda}(t) + \beta J\rangle$.
4. $V = |x|^{-2}U(\phi)$, $\lambda = 2$, $U_\phi \neq 0$: $\mathfrak{g}_V = \langle M,\ D(1),\ D^{\lambda}(t),\ D^{\lambda}(t^2)\rangle$.
5. $V = U(t,|x|) + \alpha\phi$, $U_t \neq 0$, $\alpha \in \{0,1\}$: $\mathfrak{g}_V = \langle M,\ J + \alpha t M\rangle$.
6. $V = U(|x|) + \alpha\phi$: $\mathfrak{g}_V = \langle M,\ J + \alpha t M,\ D(1)\rangle$.
7. $V = |x|^{-2}U$, $U \neq 0$, $\lambda = 2$: $\mathfrak{g}_V = \langle M,\ J,\ D(1),\ D^{\lambda}(t),\ D^{\lambda}(t^2)\rangle$.
8. $V = U(t, x_2)$: $\mathfrak{g}_V = \langle M,\ G(1,0),\ G(t,0)\rangle$.
9. $V = U(\zeta)$, $\zeta = x_2$: $\mathfrak{g}_V = \langle M,\ G(1,0),\ G(t,0),\ D(1)\rangle$.

10. $V = t^{-1}U(\zeta)$, $\zeta = |t|^{-1/2}x_2$: $\quad \mathfrak{g}_V = \langle M,\, G(1,0),\, G(t,0),\, D^\lambda(t)\rangle$.
11. $V = (t^2+1)^{-1}U(\zeta) - \lambda'(t + (t^2+1)\arctan t)$, $\zeta = (t^2+1)^{-1/2}x_2$:
$\mathfrak{g}_V = \langle M,\, G(1,0),\, G(t,0),\, D^\lambda(t^2+1)\rangle$.
12. $V = x_2^{-2}U$, $U \neq 0$, $\lambda = 2$: $\quad \mathfrak{g}_V = \langle M,\, G(1,0),\, G(t,0),\, D(1),\, D^\lambda(t),\, D^\lambda(t^2)\rangle$.
13. $V = U(t,\omega_2) + \frac{1}{4}(h_{tt} - h)h^{-1}\omega_1^2 + h_t h^{-1}\omega_1$, $h = h(t) \neq 0$:
$\mathfrak{g}_V = \langle M,\, G(h\cos t, h\sin t)\rangle$.
14. $V = U(\omega_2) + \frac{1}{4}(\beta^2 - \kappa^2)\omega_1^2 + \beta\kappa\omega_1\omega_2$, $\beta\kappa \neq 0$:
$\mathfrak{g}_V = \langle M,\, G(e^{\beta t}\cos\kappa t, e^{\beta t}\sin\kappa t),\, D(1) + \kappa J\rangle\rangle$.
15. $V = U(\omega_2) - \frac{1}{4}\kappa^2\omega_1^2 + \alpha\omega_1$, $\kappa \neq 0$:
$\mathfrak{g}_V = \langle M,\, G(\cos\kappa t, \sin\kappa t) + \alpha t M,\, D(1) + \kappa J\rangle$.
16. $V = \frac{1}{4}h^{ab}(t)x_a x_b + ih^{00}(t)$:
$\mathfrak{g}_V = \langle M,\, G(\chi^{p1}, \chi^{p2}),\, p = 1, \ldots, 4\rangle$ where $\{(\chi^{p1}(t),\, \chi^{p2}(t))\}$ is a fundamental set of solutions of the system $\chi^a_{tt} = h^{ab}\chi^b$.
17. $V = \frac{1}{4}\alpha\omega_1^2 + \frac{1}{4}\beta\omega_2^2 + i\nu$, $\alpha \neq \beta$:
$\mathfrak{g}_V = \langle M,\, G(\theta^{p1}\cos\kappa t - \theta^{p2}\sin\kappa t,\, \theta^{p1}\sin\kappa t + \theta^{p2}\cos\kappa t),\, D(1) + \kappa J,\, p = 1, \ldots, 4, \rangle$, $\kappa \neq 0\rangle$, where $(\theta^{p1}(t),\, \theta^{p2}(t))$ are linearly independent solutions of the system $\theta^1_{tt} - 2\kappa\theta^2_t = (\kappa^2 + \alpha)\theta^1$, $\theta^2_{tt} + 2\kappa\theta^1_t = (\kappa^2 + \beta)\theta^2$.
18. $V = ih(t)$: $\quad \mathfrak{g}_V = \langle M,\, G(1,0),\, G(t,0),\, G(0,1),\, G(0,t),\, J)\rangle$.
19. $V = i\beta$, $\beta \neq 0$: $\quad \mathfrak{g}_V = \langle M,\, G(1,0),\, G(t,0),\, G(0,1),\, G(0,t),\, J,\, D(1)\rangle$.
20. $V = i\beta t^{-1}$, $\beta > 0$: $\quad \mathfrak{g}_V = \langle M,\, G(1,0),\, G(t,0),\, G(0,1),\, G(0,t),\, J,\, D^\lambda(t)\rangle$.
21. $V = i(t^2+1)^{-1}\,(2\lambda' t + \beta)$, $\beta > 0$:
$\mathfrak{g}_V = \langle M,\, G(1,0),\, G(t,0),\, G(0,1),\, G(0,t),\, J,\, D^\lambda(t^2+1)\rangle$.
22. $V = 0$:
$\lambda \neq 2$: $\quad \mathfrak{g}_V = \langle M,\, G(1,0),\, G(t,0),\, G(0,1),\, G(0,t),\, J,\, D(1),\, D^\lambda(t)\rangle$.
$\lambda = 2$: $\quad \mathfrak{g}_V = \langle M,\, G(1,0),\, G(t,0),\, G(0,1),\, G(0,t),\, J,\, D(1),\, D^\lambda(t),\, D^\lambda(t^2)\rangle$.

Remark III.7. As in Theorem III.12 and Theorem III.13 we list some conditions for the maximal essential Lie invariance algebras presented in Theorem III.14 in order for the potentials not to be equivalent to other potentials with larger Lie invariance algebras. In some cases we have presented simple necessary and sufficient conditions that provide such inequivalence, but other cases the conditions to be imposed are not so obvious. For example, in Cases 9–11 the condition of being a maximal Lie symmetry extension is $(\zeta^2 U)_\zeta \neq 0$ and $U_{\zeta\zeta\zeta} \neq 0$, and this excludes those V that are $G^\sim$-equivalent to those in Cases 12 and 16–17. Case 8 is already more complicated since in this case we need the condition $(x_2^2 U)_2 \neq 0$ and $U_{222} \neq 0$ to ensure that the potentials are inequivalent to those 16–17 and then we also need conditions to further differentiate them from potentials in 9–11. Similarly, the potentials in Cases 13–15 are $G^\sim$-inequivalent to those in Cases 16–17 if and only if $U_{\omega_2\omega_2\omega_2} \neq 0$ or $\operatorname{Im} U_{\omega_2} \neq 0$. This condition is necessary and sufficient for the maximality of Lie symmetry extensions given in Cases 14 and 15, but for Case 13 we need a further condition to guarantee inequivalence with potentials in Cases 14 and 15. Schrödinger equations for Case 16 are not similar to those for Cases 18–22 if and only if the parameter-functions h's satisfy at least one of the conditions $(h^{12} = h^{21} \neq 0) \vee (h^{11} \neq h^{22})$. The maximality of symmetry extensions and the existence of discrete transformations require that $\beta \neq 0$ in Case 19 and $\beta > 0$ in Cases 20–21, respectively.

Proof. As in the proofs of Theorem III.12 and Theorem III.13 we analyze different cases given by k_2, k_3 and r_1. Here, a basis for a Lie symmetry extension is given by vector fields of the form

$$Q^s = D^\lambda(\tau^s) + \kappa^s J + G(\chi^{s1}, \chi^{s2}) + \sigma^s M,$$

where $0 \leqslant s \leqslant \dim \mathfrak{g}_V - 1$, and all the parameter functions are real-valued functions of t

$\boldsymbol{k_2 = r_1 = 0}$. If $k_3 = 0$ then there is no symmetry extension, $\mathfrak{g}_V = \mathfrak{g}^\cap = \langle M \rangle$ and we have Case 0.

If $k_3 \neq 0$ the algebra $\mathfrak{g}_V$ contains vector fields of the form Q^s, where τ's are linearly independent. It then follows from Lemma III.10 that the span of the vector fields $\{\pi^0_* Q^s\}$ is isomorphic to a subalgebra of the algebra $\mathrm{sl}(2,\mathbb{R})$. This means that the group classification of the class under study is reduced to the classification of subalgebras of the algebra $\mathrm{sl}(2,\mathbb{R})$. Thus following [27, Theorem 18], the algebra $\mathfrak{g}_V$ may include a part from the kernel, additional vector fields $\{D(1)\}$, $\{D(1)+\kappa J\}$, $\{D(1), D^\lambda(t)+\kappa J\}$, $\{D(1), D^\lambda(t), D^\lambda(t^2)\}$. These correspond to Cases 1–4, where $\kappa \neq 0$ in Case 2, $\beta > 0$ and $U_\zeta \neq 0$ (resp. $U_\phi \neq 0$ and $\lambda = 2$) in Case 3 (resp. Case 4).

$\boldsymbol{k_2 = 1, r_1 = 0}$. The algebra $\mathfrak{g}_V$ necessarily contains the vector field Q^0, where χ^{0a} can be set to zero up to $G^\sim_{\mathcal{P}_\lambda}$-equivalence. Substituting the components of Q^0 into the classifying condition (26) leads to an equation for V that gives $V = U(t,|x|) + \alpha\phi$ with $\alpha \in \{0,1\}$ as the solution. If there are no additional symmetry extensions, namely if $k_3 = 0$, then we have Case 5.

If $k_3 \neq 0$ extensions are obtained for different values of k_3.

$k_3 = 1$. This implies that $\mathfrak{g}_V$ includes additional extensions of the form Q^1 with $\tau^1 \neq 0$. Up to $G^\sim_{\mathcal{P}_\lambda}$-equivalence we can set $\tau^1 = 1$, $\chi^{1a} = \sigma^1 = 0$. Then the commutation relation of Q^1 with Q^0 yields $\kappa^1 = 0$ and $Q^0 = J + \alpha t M$, where α is a real constant. Putting the coefficients of Q^1 and Q^2 into the classifying condition (26), we obtain two independent equations in V whose solution is given in Case 6.

$k_3 \geqslant 2$. The algebra $\mathfrak{g}_V$ contains a vector field M from the kernel and the vector fields Q^0 and Q^a where $a = 1,2$, $\kappa^0 = 1$ and τ^a's are linearly independent. Up to $G^\sim_{\mathcal{P}_\lambda}$-equivalence together with commutation relations, Q^0, Q^1 and Q^2 are reduced to $Q^0 = J$, $Q^1 = D(1)$ and $Q^2 = D(t)$. The potential obtained with these vector fields (on using the classifying condition) is $V = |x|^{-2}U$, $U \neq 0$. This potential also admits the symmetry vector field $Q^3 = D^\lambda(t^2)$ for $\lambda = 2$. This gives Case 7.

$\boldsymbol{r_1 = 1}$. Suppose that the algebra $\mathfrak{g}_V$ includes at least the vector field Q^0 and Q^1 with $\kappa^0 = 1$ and $(\chi^{11}, \chi^{12}) \neq (0,0)$. Since the Lie bracket of these vector fields should also belong to the algebra, we find that the $[Q^0, Q^1] \in \mathfrak{g}_V$ gives us the condition $r_1 = 2$. Hence $k_2 = 0$. Cases 8–15 are given by the conditions $k_2 = 0$ and $r_1 = 1$. We then have two possibilities: for the tuple (χ^{11}, χ^{12}) we have either $\chi^{11}_t \chi^{12} = \chi^{11} \chi^{12}_t$ or $\chi^{11}_t \chi^{12} \neq \chi^{11} \chi^{12}_t$.

1. $\chi^{11}_t \chi^{12} = \chi^{11} \chi^{12}_t$. That is, the tuple (χ^{11}, χ^{12}) is proportional to a constant tuple. Then any fixed vector field $Q^1 = G(\chi^{11}, \chi^{12}) + \sigma^1 M$ is $G^\sim_{\mathcal{P}_\lambda}$-equivalent to $G(1,0)$. If $G(1,0) \in \mathfrak{g}_V$ then the vector field $G(t,0)$ also belongs to $\mathfrak{g}_V$. Putting the components of these vector fields into the classifying condition (26) then gives potentials $V = U(t,x_2)$. This is the potential of Case 8. Next, consider the subclass corresponding to the set $\{V = U(t,x_2) \mid U_{222} \neq 0\}$ (this excludes the case $r_1 = 2$ below). This subclass is normalized and its equivalence group consists of point transformations of the form (13a)–(13c) with T fractional linear in t and $\mathcal{X} = c_1 T + c_0$, where c_1 and c_0 are real arbitrary constants. Thus the inequivalent extensions in this subclass are given by the inequivalent subalgebras of the algebra $\langle D(1), D^\lambda(t), D^\lambda(t^2)\rangle$. The maximality condition for Lie symmetries implies that the potentials in Cases 9–11 satisfy $U_{\zeta\zeta\zeta} \neq 0$, $(\zeta^2 U)_\zeta \neq 0$. Substituting the components of these vector fields into the classifying condition (26) gives us equations for V, whose consistency requires $\lambda \in \mathbb{R}$ in Case 11 and $\lambda = 2$ in Case 12.

2. $\chi^{11}_t\chi^{12} \neq \chi^{11}\chi^{12}_t$, (the tuple (χ^{11}, χ^{12}) is not proportional to a constant tuple). In this case, we can, up to $G^\sim_{\mathcal{P}_\lambda}$-equivalence, set $(\chi^{11}, \chi^{12}) = (h\cos t, h\sin t)$, where $h(t)$ is a nonzero function of t. Put $Q^1 = G(h\cos t, h\sin t) + \sigma^1 M$. Any such vector field is $G^\sim_{\mathcal{P}_\lambda}$-equivalent to $Q^1 = G(h\cos t, h\sin t)$. Putting the components of Q^1 into the classifying condition (26) and rearranging one finds the general form of V as given by in Case 13.

Additional symmetry extensions are given by vector fields Q^2 with $\tau^2 \neq 0$. Up to $G^\sim_{\mathcal{P}_\lambda}$-equivalence the parameter functions χ^{1a}, σ^1 can be set to zero and Q^2 reduces to $Q^2 = D(1) + \kappa J$, where $\kappa \neq 0$ since $\kappa = 0$ implies that $\chi^{11}_t\chi^{12} = \chi^{11}\chi^{12}_t$. From the commutator of Q^1 with Q^2 we find that the solutions of the equations in χ and σ depend on the real constant β. The other constants obtained in this way can be assumed to be zero up to $G^\sim_{\mathcal{P}_\lambda}$-equivalence and a linear combination with the kernel. Substituting the components of vector fields of Q^1 and Q^2 obtained in this manner into the classifying condition (26) leads to two independent equations for V whose solutions are presented in Cases 14 and 15, respectively.

$\boldsymbol{r_1 = 2}$. The algebra $\mathfrak{g}_V$ contains at least two vector fields of the form Q^a, where $a = 1, 2$, $\tau = \kappa = 0$ and $\chi^{11}\chi^{22} \neq \chi^{12}\chi^{21}$. Substituting the components of Q^a into the classifying condition (26) yields the system

$$\chi^{ab}V_b = \frac{1}{2}\chi^{ab}_{tt}x_b + \sigma^a_t,$$

which is equivalent to the system $V_b = \frac{1}{2}h^{ab}(t)x_b + h^{0b}(t)$, where the $h(t)$ are smooth real-valued functions of t satisfying $\chi^{ab}_{tt} = \chi^{ac}h^{cb}$ and $\sigma^a_t = \chi^{ac}h^{0c}$. The condition $V_{12} = V_{21}$ implies that the matrix (h^{ab}) is symmetric and thus the integration of the above potential gives V as a quadratic polynomial in x_1 and x_2 with coefficients depending on t. That is

$$V = \frac{1}{4}h^{ab}(t)x_a x_b + h^{0b}(t)x_b + \tilde{h}^{00}(t) + ih^{00}(t),$$

where h^{0b} and $\tilde{h}^{00}$ can be set equal to zero up to $G^\sim_{\mathcal{P}_\lambda}$-equivalence. This means that the reduced form of potentials is

$$V = \frac{1}{4}h^{ab}(t)x_a x_b + ih^{00}. \tag{36}$$

To carry out the group classification of the subclass consisting of equations from the class $\mathcal{P}_\lambda$ with potentials of the form (36), which is normalized, it is convenient to partition this subclass into two subclasses depending on values of k_2. We denote $\mathcal{P}^1_\lambda$ and $\mathcal{P}^0_\lambda$ the subclasses corresponding to the values $k_2 = 1$ and $k_2 = 0$, respectively. The subclasses $\mathcal{P}^1_\lambda$ and $\mathcal{P}^0_\lambda$ are also normalized and $\mathcal{P}^1_\lambda$ is subject to the further condition on the matrix $H = (h^{ab})$ that $H = \Lambda(t)E$ for some function $\Lambda(t)$ whereas $\mathcal{P}^0_\lambda$ is subject to the condition $H \neq \Lambda(t)E$ where E is a 2×2 identity matrix.

For equations from the subclass $\mathcal{P}^0_\lambda$ we have $k_3 \geqslant 1$. The value $k_3 = 0$ means that there is no further extension, which gives Case 16.

We find that up to $G^\sim_{\mathcal{P}_\lambda}$-equivalence is only one-dimensional extensions of the form $Q^5 = D(1) + \kappa J$ with $\kappa := \kappa^5$ are possible. Putting the components of Q^5 together with potentials of the form (36) into the classifying conditions (26) gives the Case 17.

Up to $G^{\sim}_{\mathcal{P}_\lambda}$-equivalence, the subclass $\mathcal{P}^1_\lambda$ consists of equations with potentials of the form $V = ih(t)$, where h is a smooth real-valued function of t. Putting these potentials into the classifying condition (26) and splitting with respect to the powers of x, we obtain the equations

$$\tau_{ttt} = 0, \quad \chi^a_{tt} = 0, \quad \sigma_t = 0, \quad (\tau h)_t = \lambda' \tau_{tt}.$$

From this it follows that the vector fields J, $G(1,0)$, $G(t,0)$, $G(0,1)$, $G(0,t)$ belong to the maximal Lie invariance algebra of any equation with such potential. Additional Lie symmetry extensions are related to subalgebras of the algebra $\langle D(1), D(t), D(t^2)\rangle$. This leads to Cases 18–22. □

7 Conclusion

We have computed the equivalence groupoid and the equivalence group of class $\mathcal{N}$ of generalized Schrödinger equations of the form (1) and used them to obtain the equivalence groups of the classes $\mathcal{F}$, $\mathcal{F}^1$, $\mathcal{S}$ and $\mathcal{V}$ singled out from $\mathcal{N}$ by constrains on the form of the nonlinearities G and F. We have also established the normalization properties of these classes. We have investigated the Lie symmetry properties of the class $\mathcal{S}$ of multidimensional nonlinear Schrödinger equations of the form (4) and solved completely the group classification of the class $\mathcal{V}$ of (1+2)-dimensional nonlinear Schrödinger equations with potentials and modular nonlinearities of the form (5), using the algebraic method.

The point transformations relating nonlinear Schrödinger equations from the class $\mathcal{F}$ singled out in the class $\mathcal{N}$ by the condition $G = 1$ to each other are described by Theorem III.3 and the class $\mathcal{F}^1$ singled out in this class by condition $F_{\psi_a} = F_{\psi^*_a} = 0$ was shown to be normalized. Its equivalence group $G^{\sim}_{\mathcal{F}^1}$ is a subgroup of the group $G^{\sim}_{\mathcal{F}}$. The class $\mathcal{S}$ is described in terms of its Lie symmetry properties. The group $G^{\sim}_{\mathcal{S}}$ is obtained from the group $G^{\sim}_{\mathcal{N}}$ restricted with the cases given by the specific forms of G and F and is described by Theorem III.5. The set of infinitesimal generators of one-parameter subgroup $G^{\sim}_{\mathcal{S}}$ forms an infinite-dimensional equivalence algebra $\mathfrak{g}^{\sim}_{\mathcal{S}}$ of the class $\mathcal{S}$.

Using the Lie infinitesimal criterion, we have obtained the determining equations for the symmetries of equations from the class $\mathcal{S}$. The analysis of these determining equations yields the kernel algebra $\mathfrak{g}^{\cap}$ and the maximal Lie invariance algebra $\mathfrak{g}_S$ of a given equation $\mathcal{L}_S$ from the class $\mathcal{S}$, and the results of this analysis are given in Theorem III.6 and Proposition III.1, respectively. For the class $\mathcal{S}$ we give the nonzero commutations relations of vector fields spanning the linear span $\mathfrak{g}_{\langle\,\rangle}$ of the vector fields coming from the algebra $\mathfrak{g}_V$ and we show that the upper bound dimension of this algebra is $n(n+3)/2+4$ for any S satisfying $S_\rho \neq 0$.

We find that, unlike the class $\mathcal{S}$, the class $\mathcal{V}$ of multidimensional nonlinear Schrödinger equations with potentials and modular nonlinearities is not normalized. To deal with this, we partition this class into three disjoint normalized subclasses V', $\mathcal{P}_0$ and $\mathcal{P}_\lambda$ consisting, respectively, of equations of the form (18), (20) and (23). In order to obtain a complete group classification of these subclasses we have imposed some restrictions on the nonlinear terms: by fixing $f(\rho)$ in $\mathcal{V}'$, setting $|\delta| = 1$ and $\delta_2 \geqslant 0$ in $\mathcal{P}_0$ and $\mathcal{P}_\lambda$, and then the only arbitrary element is considered to be V. These restrictions, together with the normalization properties for the given classes of differential equations, allow us to classify, up to equivalence, all subalgebras of the algebra $\mathfrak{g}_{\langle\,\rangle}$.

For equations of the class $\mathcal{V}'$, the classifying conditions, the kernel algebra and the symmetry algebra $\mathfrak{g}_V$ are given in Lemma III.5. It is also shown that the dimension of any symmetry extension algebra for equations of the class $\mathcal{V}'$ is at most 7. The group classification in this class is performed using three integers $k_3 \in \{0,1\}$, $k_2 \in \{0,1\}$ and $r_1 \in \{0,1,2\}$ that are invariant with respect to equivalence transformations. This gives a complete classification (up to its equivalence) of equations in this class which is given in Theorem III.12.

Similarly, we find that the symmetry extension algebra $\mathfrak{g}_V$ for the class $\mathcal{P}_0$ satisfies the condition $\mathfrak{g}_V \leqslant 8$. The inequivalent Lie symmetry extensions together with their corresponding families of potentials are listed in Theorem III.13.

In the case of the class $\mathcal{P}_\lambda$ we find that $\dim \mathfrak{g}_V \leqslant 9$ for the symmetry algebra $\mathfrak{g}_V$ and we classify completely all subalgebras of $\mathfrak{g}_{\langle\,\rangle}$. In the group classification we have used Theorem III.10 and introduced three integers $k_2 \in \{0,1\}$, $k_3 \in \{0,1,2,3\}$ and $r_1 \in \{0,1,2\}$ invariant under the equivalence group in order to differentiate between the different cases that arise. The results of this classification are summarized in Theorem III.14.

The complete list of inequivalent Lie symmetry extensions together with their corresponding families of potentials in the class $\mathcal{V}$ is given by the union of the corresponding lists that we obtained for the subclasses $\mathcal{V}'$, $\mathcal{P}_0$ and $\mathcal{P}_\lambda$.

Our results generalize the results for the (1+1)-dimensional nonlinear Schrödinger equations with potentials and modular nonlinearities and (1+2)-dimensional cubic Schrödinger equations given in [38].

We intend to exploit the results in this paper to obtain Lie reductions and invariant solutions for the class of nonlinear Schrödinger equations of the form (5) in future publications.

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Paper IV

Admissible transformations of (1+1)-dimensional Schrödinger equations with variable mass

Admissible transformations of (1+1)-dimensional Schrödinger equations with variable mass

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We compute the equivalence groupoid and the equivalence group of the most general class of (1+1)-dimensional generalized nonlinear Schrödinger equations with variable mass and its subclasses. It turns out that this class is not normalized, as well as its subclass of (1+1)-dimensional linear Schrödinger equations with variable mass. However, we prove that this subclass can be mapped by a family of point transformations to the subclass of (1+1)-dimensional linear Schrödinger equations with constant mass equal to one, a class that is normalized.

1 Introduction

The group classification of classes of differential equations is an involved procedure consisting of several steps. It uses several concepts and tools (see [12], [21], [22]) of which the most prominent is that of the *equivalence groupoid* of the class of differential equations. This is the set of admissible transformations, namely those transformations that map one system of differential equations of a given class, to another such system, and under composition of maps they form a groupoid [12], [13]. That is, any admissible transformation in this set is invertible, it contains the identity transformation and the composition $\psi \circ \phi$ of admissible transformations ϕ and ψ is well defined provided that the image of ϕ is in the domain of ψ.

The notion of admissible transformations began with the work by Kingston and Sophocleous while studying the class of generalized Burgers equation [10], where they called such transformations *form-preserving.* Under the name *allowed transformations*, form preserving transformations appeared in the study of the symmetry analysis of the variable-coefficient Korteweg–de Vries equation [27]. Since then, a number of publications involving applications of admissible transformations have appeared [5], [11], [13], [19], [22], [26].

Admissible transformations have been applied to obtain several results in the group classifications of various types of Schrödinger equations, both linear and nonlinear [12], [14], [14], [22], [23], [25], with constant mass or variable mass [17] that are important in the applications [2], [6], [9], [21].

In order to make our study as self-contained as possible, we explain some basic notions related to classes of differential equations.

Let $\mathcal{L}_\theta$ be a system $L(x, u_{(\rho)}, \theta(x, u_{(\rho)})) = 0$ of l differential equations $L^1 = 0$, …, $L^l = 0$ for m unknown functions $u = (u^1, \dots, u^m)$ of n independent variables $x = (x_1, \dots, x_n)$. Here $u_{(\rho)}$ denotes the set of all the derivatives of the functions u with respect to x of order not greater than ρ (the functions u are considered as being derivatives of "order zero"). $L = (L^1, \dots, L^l)$ is a tuple of l fixed functions that depend on x, $u_{(\rho)}$ and θ, where θ denotes the tuple of arbitrary elements $\theta(x, u_{(\rho)}) = (\theta^1(x, u_{(\rho)}), \dots, \theta^k(x, u_{(\rho)}))$. This tuple runs through the set $\mathcal{S}$ of solutions of an auxiliary system of differential equations $S(x, u_{(\rho)}, \theta_{q'}(x, u_{(\rho)})) = 0$ and differential inequalities, such as $\Sigma(x, u_{(\rho)}, \theta_{(q')}(x, u_{(\rho)})) \neq 0$, in which both x and $u_{(\rho)}$ play the role of independent variables, S and Σ are tuples of smooth functions depending on x, $u_{(\rho)}$ and $\theta_{(q')}$. We call the set $\{\mathcal{L}_\theta \mid \theta \in \mathcal{S}\} := \mathcal{L}|_\mathcal{S}$ a *class (of systems) of differential equations* defined by the parameterized form $\mathcal{L}_\theta$ and the set $\mathcal{S}$ (to which the arbitrary elements θ belong).

For a given class $\mathcal{L}|_\mathcal{S}$, let $\mathcal{L}_\theta$ and $\mathcal{L}_{\tilde\theta}$ be two systems of differential equations from this class. The set of point transformations that connect these systems is denoted by $\mathrm{T}(\theta, \tilde\theta)$. A triple $(\theta, \tilde\theta, \varphi)$ consisting of two arbitrary elements θ and $\tilde\theta \in \mathcal{S}$ with $\mathrm{T}(\theta, \tilde\theta) \neq \varnothing$ and a point transformation $\varphi \in \mathrm{T}(\theta, \tilde\theta)$ is called an *admissible transformation* of the class $\mathcal{L}|_\mathcal{S}$. The *equivalence groupoid* of the class $\mathcal{L}|_\mathcal{S}$ is the set of the admissible transformations of this class equipped with the usual composition of transformations: this set is a groupoid and is denoted by $\mathcal{G}^\sim$. For more details on the admissible transformations see [22]. The *maximal Lie symmetry group* G_θ of the system $\mathcal{L}_\theta$ (for a fixed $\theta \in \mathcal{S}$) is a group of transformations that act on the space of independent and dependent variables that preserve the solution set of the system $\mathcal{L}_\theta$, i.e., G_θ coincides with $\mathrm{T}(\theta, \theta)$.

Another important object which can be defined using the notion of point transformations is the (usual) equivalence group. This group was introduced by Ovsiannikov [18] and it plays a fundamental role in group classification. The *usual equivalence group* $G^\sim$ of the class $\mathcal{L}|_\mathcal{S}$ is a collection of point transformations in the space of $(x, u_{(\rho)}, \theta)$, which is projectable onto the space of $(x, u_{(\rho')})$ for any $0 \leqslant \rho' \leqslant \rho$, is compatible with the contact structure on the space $(x, u_{(\rho)})$, and maps each system from the class $\mathcal{L}|_\mathcal{S}$ to another system from the same class. An element $\mathcal{T}$ from $G^\sim$ is called an *equivalence transformation*. Depending on the structure of the point transformations, the notion of the equivalence group can be generalized in various types. From the definition of the usual equivalence group, it follows that the point transformations in the space of independent and dependent variables do not depend on the arbitrary elements of the class under consideration. If we allow this dependence, then the equivalence group is said to be *generalized* and it is denoted as $G^\sim_{\rm gen}$. This means that an element $\mathcal{T}$ belongs to $G^\sim_{\rm gen}$ if for any $\theta \in \mathcal{S}$, $\mathcal{T}\theta \in \mathcal{S}$ and $\mathcal{T}(.,.,\theta(.,.))|_{(x,u)} \in \mathrm{T}(\theta, \mathcal{T}\theta)$. The *extended equivalence group*, denoted by $\hat G^\sim$, consists of those transformations for which the transformations of the arbitrary elements are expressed via old arbitrary elements of the class non-locally. If both cases arise, i.e., transformational components of the equivalence transformations (both independent, dependent and arbitrary elements) are expressed via arbitrary elements non-locally, then the corresponding equivalence group is said to be *generalized extended* and we denote it by $\hat G^\sim_{\rm gen}$. Moreover, if the class $\mathcal{L}|_\mathcal{S}$ and its equivalence group $G^\sim$ are known and $\mathcal{L}|_{\mathcal{S}'}$ is a subclass singled out from the class $\mathcal{L}|_\mathcal{S}$ by imposing additional constraints on the sets $\mathcal{S}'$ and Σ' of the form $\mathcal{S}'(x, u_{(\rho)}, \theta_{(q')}) = 0$, $\quad \Sigma'(x, u_{(\rho)}, \theta_{(q')}) \neq 0$ with respect to the arbitrary elements $\theta = \theta(x, u_{(\rho)})$, where $\mathcal{S}' \subset \mathcal{S}$ is the set of solution of the united system $\mathcal{S} = 0$, $\Sigma \neq 0$, $\mathcal{S}' = 0$, $\Sigma' \neq 0$, then: the equivalence group $G^{\sim}$ of the class $\mathcal{L}|_{\mathcal{S}'}$, $G^{\sim}(\mathcal{L}|_{\mathcal{S}'})$, is called a *conditional equivalence group* of the whole class $\mathcal{L}|_{\mathcal{S}}$ under conditions $\mathcal{S}' = 0$ and $\Sigma' \neq 0$. The conditional equivalence group is said to be *nontrivial* if and only if it is not a subgroup of $G^{\sim}(\mathcal{L}|_{\mathcal{S}})$. It is said to be *maximal* under the above conditions on $\mathcal{S}'$ and Σ' if for any subclass $\mathcal{L}|_{\mathcal{S}''}$ of the class $\mathcal{L}|_{\mathcal{S}}$ containing the subclass $\mathcal{L}|_{\mathcal{S}'}$, we have $G^{\sim}(\mathcal{L}|_{\mathcal{S}'}) \nsubseteq G^{\sim}(\mathcal{L}|_{\mathcal{S}''})$. For more details, see [22], [24].

These different types of equivalence groups are derived from the knowledge of the equivalence groupoid $\mathcal{G}^{\sim}$ of the class $\mathcal{L}|_{\mathcal{S}}$, which is computed using the direct method (see below). This means that the knowledge of the structure of the admissible transformations influences the choice of the group $G^{\sim}$ to be used.

A class of differential equations $\mathcal{L}|_{\mathcal{S}}$ is said to be *normalized (in the usual sense)* with respect to point transformations if the equivalence groupoid $\mathcal{G}^{\sim}$ for this class is generated by its equivalence group $G^{\sim}$. From the various generalizations of the equivalence group there are corresponding notions of normalization [22], [24]. The class $\mathcal{L}|_{\mathcal{S}}$ is said to be *semi-normalized* if any admissible transformation in this class is generated by the transformations from the group $G^{\sim}$ of the whole class and transformations from the point symmetry groups G_θ of initial or transformed systems. Finally, the class $\mathcal{L}|_{\mathcal{S}}$ is said to be *uniformly semi-normalized* if for each $\theta \in \mathcal{S}$ the point symmetry group G_θ of the system $\mathcal{L}_\theta \in \mathcal{L}|_{\mathcal{S}}$, with fixed θ, contains a subgroup N_θ such that the family $\mathcal{N}_{\mathcal{S}} = \{N_\theta \mid \theta \in \mathcal{S}\}$ of all these subgroups satisfies the following properties:

1. $\mathcal{T}|_{(x,u)} \notin N_\theta$ for any $\theta \in \mathcal{S}$ and any $\mathcal{T} \in G^{\sim}$ with $\mathcal{T} \neq \mathrm{id}$.
2. $N_{\mathcal{T}\theta} = \mathcal{T}|_{(x,u)} N_\theta (\mathcal{T}|_{(x,u)})^{-1}$ for any $\theta \in \mathcal{S}$ and any $\mathcal{T} \in G^{\sim}$.
3. For any $(\theta^1, \theta^2, \varphi) \in \mathcal{G}^{\sim}$ there exist $\varphi^1 \in N_{\theta^1}$, $\varphi^2 \in N_{\theta^2}$ and $\mathcal{T} \in G^{\sim}$ such that $\theta^2 = \mathcal{T}\theta^1$ and $\varphi = \varphi^2(\mathcal{T}|_{(x,u)})\varphi^1$.

Here, $\mathcal{T}|_{(x,u)}$ denotes the restriction of $\mathcal{T}$ to the space with local coordinates (x, u). Once the normalization properties of the class of differential equations under study are clearly understood, the choice of the techniques or method for the group classification of this class is easily obtained. The most useful technique applied for normalized classes is the algebraic method [12], [14], [22].

In the present paper, we begin with the study of the admissible transformations for the class $\mathcal{H}$ of equations of the form

$$i\psi_t = H(t, x, \psi, \psi^*, \psi_x, \psi_x^*, \psi_{xx}, \psi_{xx}^*) \quad \text{with} \quad |H_{\psi_{xx}}| \neq |H_{\psi_{xx}^*}|. \tag{1}$$

Here H is a complex-valued smooth function of its arguments and ψ is an unknown complex function of the real independent variables t and x. The subscripts of functions denote differentiation with respect to the corresponding variables. Equations from the class $\mathcal{H}$ are of the most general form that can be considered as generalizations of (1+1)-dimensional Schrödinger equations. We compute the equivalence groupoid $\mathcal{G}^{\sim}_{\mathcal{H}}$ and the equivalence group $G^{\sim}_{\mathcal{H}}$ of the class $\mathcal{H}$ and then show that this class is normalized. Further, we consider the important subclass $\mathcal{A}$ of (1+1)-dimensional generalized nonlinear Schrödinger equations with variable mass, which can be singled out from the class $\mathcal{H}$ by the constraints $H_{\psi_{xx}\psi_{xx}} = H_{\psi_{xx}^*} = 0$. After reparametrization, equations from the class $\mathcal{A}$ have the form

$$i\psi_t + G(t, x, \psi, \psi^*, \psi_x, \psi_x^*)\psi_{xx} + F(t, x, \psi, \psi^*, \psi_x, \psi_x^*) = 0, \quad G \neq 0, \tag{2}$$

where G and F are smooth complex-valued functions of their arguments. We give a complete characterization of the equivalence groupoids and the equivalence groups of this class and its most useful subclasses $\mathcal{N}$ and $\mathcal{M}$, defined as follows: the subclass $\mathcal{N}$ consists of equations singled out from the class $\mathcal{A}$ by the constraints $G_{\psi_x} = G_{\psi^*_x} = F_{\psi_x} = F_{\psi^*_x} = 0$,

$$i\psi_t + G(t,x,\psi,\psi^*)\psi_{xx} + F(t,x,\psi,\psi^*) = 0, \quad G \neq 0. \tag{3}$$

Then specifying further the forms of the arbitrary elements G and F, we obtain the subclass $\mathcal{M}$ of (1+1)-dimensional linear Schrödinger equations,

$$i\psi_t + G(t,x)\psi_{xx} + V(t,x)\psi = 0, \tag{4}$$

where G is a nonzero real-valued smooth function of (t,x) and $G = 1/m(t,x)$ can be assumed as a variable mass depending on (t,x), and V is an arbitrary smooth complex-valued potential depending on t and x.

The structure of this paper is the following: In Section 2 the equivalence groupoids and the equivalence groups of the classes $\mathcal{H}$ and $\mathcal{A}$ are computed. We show that the class $\mathcal{H}$ is normalized and the class $\mathcal{A}$ is not normalized. Then we partition the class $\mathcal{A}$ into two normalized classes with respect to the constrains $G^* = -G$ and $G^* \neq -G$ and their corresponding equivalence groups are derived from the knowledge of the equivalence group for the entire class $\mathcal{A}$. In Section 3 we investigate the admissible transformations of the class $\mathcal{N}$. After showing that this class is not normalized, we partition it into four disjoint subclasses with respect to the conditions $G^* = -G$ and $G^* \neq -G$ in combination with the conditions $(G_\psi, G_{\psi^*}) \neq (0,0)$ and $(G_\psi, G_{\psi^*}) = (0,0)$ and we then construct their equivalence groups. Section 4 is devoted to the study of the admissible transformations of the class $\mathcal{M}$. Its equivalence group is derived from the knowledge of the equivalence group of the class $\mathcal{N}$. It is shown that this class is not normalized and it can be mapped by a family of point transformations to its subclass of (1+1)-dimensional linear Schrödinger equations with constant mass equal to one. The last section includes a summary and suggestions for new directions of research.

2 Equivalence groupoids of covering classes

We consider the class $\mathcal{H}$, which is a superclass for the classes $\mathcal{A}$, $\mathcal{N}$ and $\mathcal{M}$ (that is, it contains them as special cases). In this section, we find the point transformations connecting two equations from this superclass. Here and in the following the subscripts t and x denote differentiation with respect to t and x, the total derivative operator D_μ is defined as $\mathrm{D}_\mu = \partial_\mu + \psi_\mu\partial_\psi + \psi^*_\mu\partial_{\psi^*} + \dots$, μ, $\nu = 0, 1$, $x_0 := t$, $x_1 := x$, and we sum over repeated indices. We look for point transformations of the general form

$$\begin{aligned} &\varphi\colon\ \tilde t = T(t,x,\psi,\psi^*), \quad \tilde x = X(t,x,\psi,\psi^*), \\ &\tilde\psi = \Psi(t,x,\psi,\psi^*), \quad \tilde\psi^* = \Psi^*(t,x,\psi,\psi^*) \end{aligned} \tag{5}$$

with $dT \wedge dX \wedge d\Psi \wedge d\Psi^* \neq 0$ that map a fixed equation $\mathcal{L}_H$ from the class $\mathcal{H}$ to an equation $\mathcal{L}_{\tilde H}$, $i\tilde\psi_{\tilde t} = \tilde H(\tilde t, \tilde x, \tilde\psi, \tilde\psi^*, \tilde\psi_{\tilde x}, \tilde\psi^*_{\tilde x}, \tilde\psi_{\tilde x\tilde x}, \tilde\psi^*_{\tilde x\tilde x})$ of the same class. We can interpret the equation $\mathcal{L}_H$ with its conjugate as a system of two second-order evolution equations with respect to the two unknown functions ψ and ψ^*. This system is nondegenerate with respect to the second-order derivatives ψ_{xx} and ψ^*_{xx} since H, $|H_{\psi_{xx}}| \neq |H_{\psi^*_{xx}}|$. The same interpretation is given by splitting the equation $\mathcal{L}_H$ into its real and imaginary parts and assuming the real and imaginary parts of ψ as the unknown function. We emphasize that when we constrain a condition to the solution set of an equation $\mathcal{L}_H$ from the class $\mathcal{H}$, we should also take into account the complex conjugation of the equation $\mathcal{L}_H$, $-i\psi^*_t = H^*(t,x,\psi,\psi^*,\psi_x,\psi^*_x,\psi_{xx},\psi^*_{xx})$. The equation $\mathcal{L}_H$ is a second-order evolution equation. Then using [21, Lemma 4], we have the following assertion:

Theorem IV.1. *The equivalence groupoid $\mathcal{G}^\sim_{\mathcal{H}}$ of the class $\mathcal{H}$ consists of triples of the form $(H, \tilde{H}, \varphi)$, where φ is a point transformation of the form* (5) *with $T = T(t)$, $T_t \neq 0$, and the arbitrary elements H and $\tilde{H}$ are related by the formula*

$$\begin{aligned}\tilde{H} = &\left(\Psi_\psi - \frac{\mathrm{D}_x\Psi}{\mathrm{D}_xX}X_\psi\right)\frac{H}{T_t} - \left(\Psi_{\psi^*} - \frac{\mathrm{D}_x\Psi}{\mathrm{D}_xX}X_{\psi^*}\right)\frac{H^*}{T_t}\\ &+ \frac{i}{T_t}\left(\Psi_t - \frac{\mathrm{D}_x\Psi}{\mathrm{D}_xX}X_t\right).\end{aligned} \tag{6}$$

Remark IV.1. Here and in the following, the equivalence group $G^\sim$ for any class of (generalized) Schrödinger equations acts on the joint space of second-order jet coordinates $(t, x, \psi_{(2)}, \psi^*_{(2)})$ and the associated arbitrary elements together with the complex conjugates for the complex-valued arbitrary elements. To define elements of $G^\sim$, it suffices to present expressions for the transformation components associated with the variables (t, x, ψ) and the arbitrary elements. Then the transformation components for complex conjugates and derivatives of ψ and ψ^* are obtained by complex conjugation and prolongation, respectively.

Corollary IV.1. *The class $\mathcal{H}$ is normalized in the usual sense. Its usual equivalence group $G^\sim_{\mathcal{H}}$ consists of the point transformations defined in the joint space of second-order jet coordinates $(t, x, \psi_{(2)}, \psi^*_{(2)})$ and arbitrary elements H and H^*, where the transformation components for t, x, ψ and ψ^* are of the form* (5) *with $T = T(t)$, $T_t \neq 0$ and the H-component is given by* (6).

Next consider the subclass $\mathcal{H}'$ that is singled out from the class $\mathcal{H}$ by the constraints $H_{\psi_{xx}\psi_{xx}} = H_{\psi_{xx}\psi^*_{xx}} = H_{\psi^*_{xx}\psi^*_{xx}} = 0$. In other words, the subclass $\mathcal{H}'$ consists of the equations of the form (1), where the arbitrary element H is affine in the second derivative ψ_{xx} and ψ^*_{xx}, i.e., of the equations of the form $i\psi_t + G\psi_{xx} + \breve{G}\psi^*_{xx} + F = 0$, where G, $\breve{G}$ and F are complex-valued smooth functions depending on t, x, ψ, ψ^*, ψ_x and ψ^*_x with $|G| \neq |\breve{G}|$. We can reparameterize the class $\mathcal{H}'$, assuming G, $\breve{G}$ and F as the new arbitrary elements of the class $\mathcal{H}'$.

Corollary IV.2. *The class $\mathcal{H}'$ is normalized with respect to the equivalence group $G^\sim_{\mathcal{H}}$ of the entire class $\mathcal{H}$.*

Further, if $\breve{G} = 0$, i.e., $H_{\psi_{xx}\psi_{xx}} = H_{\psi^*_{xx}} = 0$, we obtain the subclass $\mathcal{A}$ of equations of the form (2). One can derive the equivalence groupoid $\mathcal{G}^\sim_{\mathcal{A}}$ and the equivalence group $G^\sim_{\mathcal{A}}$ of the class $\mathcal{A}$ from the equivalence groupoid $\mathcal{G}^\sim_{\mathcal{H}}$. However, we compute them directly in the following result:

Theorem IV.2. *The equivalence groupoid $\mathcal{G}^\sim_{\mathcal{A}}$ of the class $\mathcal{A}$ consists of triples of the form $((G, F), (\tilde{G}, \tilde{F}), \varphi)$, where the point transformation φ is of the form* (5) *with $T = T(t)$, $T_t \neq 0$,*

$$\text{and, if } G^* \neq -G,\quad X_\psi = X_{\psi^*} = 0,\quad \text{and either } \Psi_\psi = 0 \text{ or } \Psi_{\psi^*} = 0; \tag{7a}$$

and the connection between the arbitrary elements (G, F) and $(\tilde{G}, \tilde{F})$ is given by

$$\tilde{G} = \frac{(\mathrm{D}_xX)^2}{T_t}\begin{cases} G & \text{if } \Psi_\psi \neq 0,\\ (-G^*) & \text{if } \Psi_{\psi^*} \neq 0,\end{cases} \tag{7b}$$

$$\begin{aligned}\tilde{F} = &\left(\Psi_\psi - \frac{\mathrm{D}_x\Psi}{\mathrm{D}_xX}X_\psi\right)\frac{F}{T_t} - \left(\Psi_{\psi^*} - \frac{\mathrm{D}_x\Psi}{\mathrm{D}_xX}X_{\psi^*}\right)\frac{F^*}{T_t}\\ &- \left(\hat{\Delta}\Psi - \frac{\mathrm{D}_x\Psi}{\mathrm{D}_xX}\hat{\Delta}X\right)\frac{\tilde{G}}{(\mathrm{D}_xX)^2} - \frac{i}{T_t}\left(\Psi_t - \frac{\mathrm{D}_x\Psi}{\mathrm{D}_xX}X_t\right).\end{aligned} \tag{7c}$$

Here $\hat{\Delta} = \partial_x^2 + 2\psi_x\partial_x\partial_\psi + 2\psi_x^*\partial_x\partial_{\psi^*} + \psi_x^2\partial_\psi^2 + 2\psi_x\psi_x^*\partial_\psi\partial_{\psi^*} + (\psi_x^*)^2\partial_{\psi^*}^2$.

Proof. A point transformation φ connecting two equations $\mathcal{L}_{GF}$ and $\mathcal{L}_{\tilde{G}\tilde{F}}$ from the class $\mathcal{A}$ is of the form (5). Then, computing the total derivatives D_μ's of the transformations components for $\tilde{\psi}$ and $\tilde{\psi}^*$, we obtain

$$\mathrm{D}_\mu\Psi = \tilde{\psi}_{\tilde{x}_\nu}\mathrm{D}_\mu X^\nu, \quad \mathrm{D}_\mu\Psi^* = \tilde{\psi}^*_{\tilde{x}_\nu}\mathrm{D}_\mu X^\nu. \tag{8}$$

After rearranging and putting $Z := Z(t,x,\psi,\psi^*,\tilde{\psi}_{\tilde{t}},\tilde{\psi}_{\tilde{x}}) = \Psi - \tilde{\psi}_{\tilde{x}_\nu}X^\nu$, we have

$$Z_\psi\psi_{x_\mu} + Z_{\psi^*}\psi^*_{x_\mu} = -Z_{x_\mu}, \quad Z^*_\psi\psi_{x_\mu} + Z^*_{\psi^*}\psi^*x_\mu = -Z^*_{x_\mu}. \tag{9}$$

Solving (9) as with respect to ψ_{x_μ} and $\psi^*_{x_\mu}$ gives

$$\psi_{x_\mu} = -\frac{Z_{x_\mu}Z^*_{\psi^*} - Z^*_{x_\mu}Z_{\psi^*}}{Y}, \quad \psi^*_{x_\mu} = -\frac{Z^*_{x_\mu}Z_\psi - Z^*_\psi Z_{x_\mu}}{Y}, \tag{10}$$

where

$$\begin{aligned} Y :&= Z_\psi Z^*_{\psi^*} - Z_{\psi^*}Z^*_\psi \\ &= \begin{vmatrix} \Psi_\psi & \Psi_{\psi^*} \\ \Psi^*_\psi & \Psi^*_{\psi^*} \end{vmatrix} - \begin{vmatrix} X^\nu_\psi & X^\nu_{\psi^*} \\ \Psi^*_\psi & \Psi^*_{\psi^*} \end{vmatrix}\tilde{\psi}_{\tilde{x}_\nu} - \begin{vmatrix} \Psi_\psi & \Psi_{\psi^*} \\ X^\mu_\psi & X^\mu_{\psi^*} \end{vmatrix}\tilde{\psi}^*_{\tilde{x}_\mu} \\ &\quad + \begin{vmatrix} X^\nu_\psi & X^\nu_{\psi^*} \\ X^\mu_\psi & X^\mu_{\psi^*} \end{vmatrix}\tilde{\psi}_{\tilde{x}_\nu}\tilde{\psi}^*_{\tilde{x}_\mu} \neq 0, \end{aligned} \tag{11}$$

since $(T_\psi, X_\psi, \Psi_\psi, \Psi^*_\psi)$ and $(T_{\psi^*}, X_{\psi^*}, \Psi_{\psi^*}, \Psi^*_{\psi^*})$ are linearly independent since $\mathrm{d}T\wedge\mathrm{d}X\wedge\mathrm{d}\Psi\wedge\mathrm{d}\Psi^* \neq 0$. We apply the total derivative D_x to the both sides of (10), considering the derivatives $\tilde{\psi}_{\tilde{t}}$ and $\tilde{\psi}_{\tilde{x}}$ to be functions of $\tilde{t} = T(t,x,\psi,\psi^*)$ and $\tilde{x} = X(t,x,\psi,\psi^*)$. As a result we obtain the following expression for ψ_{xx},

$$\psi_{xx} = \frac{1}{Y}\mathrm{D}_x\left(Z^*_x Z_{\psi^*} - Z_x Z^*_{\psi^*}\right) - \frac{\mathrm{D}_x Y}{Y^2}\left(Z^*_x Z_{\psi^*} - Z_x Z^*_{\psi^*}\right). \tag{12}$$

We substitute ψ_t and ψ_{xx} in the initial equation $\mathcal{L}_{GF}$ by their expressions given in (10) and (12). For other expressions without tilde's we do not substitute explicitly, but we consider them functions of tilded variables. The resulting equation is to be an identity on the manifold defined by $\mathcal{L}_{\tilde{G}\tilde{F}}$ in the second-order jet space over the space of (t,x,ψ,ψ^*). As a result, after substituting $\tilde{\psi}_{\tilde{t}}$ and $\tilde{\psi}^*_{\tilde{t}}$ in view of $\mathcal{L}_{\tilde{G}\tilde{F}}$ and its conjugate, we are able to split the resulting equation with respect to $\tilde{\psi}_{\tilde{t}\tilde{t}}$ and $\tilde{\psi}^*_{\tilde{t}\tilde{t}}$. We obtain $(D_xT)^2 = 0$, which implies $T_x = 0$, $T_\psi = T_{\psi^*} = 0$. Hence $T = T(t)$ with $T_t \neq 0$ since $\mathrm{d}T\wedge\mathrm{d}X\wedge\mathrm{d}\Psi\wedge\mathrm{d}\Psi^* \neq 0$.

Using this result for T, we use (10) and (12) to obtain expressions for $\tilde{\psi}_{\tilde{t}}$, $\tilde{\psi}_{\tilde{x}}$ and $\tilde{\psi}_{\tilde{x}\tilde{x}}$:

$$\begin{aligned} \tilde{\psi}_{\tilde{t}} &= \frac{1}{T_t}\left(\mathrm{D}_t\Psi - \frac{\mathrm{D}_x\Psi}{\mathrm{D}_x X}\mathrm{D}_t X\right), \quad \tilde{\psi}_{\tilde{x}} = \frac{\mathrm{D}_x\Psi}{\mathrm{D}_x X}, \\ \tilde{\psi}_{\tilde{x}\tilde{x}} &= \frac{1}{(\mathrm{D}_x X)^2}\left(\mathrm{D}_x^2\Psi - \frac{\mathrm{D}_x\Psi}{\mathrm{D}_x X}\mathrm{D}_x^2 X\right). \end{aligned}$$

Substituting these expressions for $\tilde{\psi}_{\tilde{t}}$ and $\tilde{\psi}_{\tilde{x}\tilde{x}}$ into the target equation $\mathcal{L}_{\tilde{G}\tilde{F}}$, we obtain

$$\frac{i}{T_t}\left(\mathrm{D}_t\Psi - \frac{\mathrm{D}_x\Psi}{\mathrm{D}_x X}\mathrm{D}_t X\right) + \frac{\tilde{G}}{(\mathrm{D}_x X)^2}\left(\mathrm{D}_x^2\Psi - \frac{\mathrm{D}_x\Psi}{\mathrm{D}_x X}\mathrm{D}_x^2 X\right) + \tilde{F} = 0. \tag{13}$$

Arranging equation (13) and substituting $\psi_t = iG\psi_{xx} + iF$ and its conjugate $\psi_t^* = -iG^*\psi_{xx}^* - iF^*$, and then collecting the coefficients of $\tilde\psi_{\tilde x\tilde x}$ and $\tilde\psi^*_{\tilde x\tilde x}$, leads to the system

$$
\begin{aligned}
&\left(\Psi_\psi - \frac{\mathrm{D}_x\Psi}{\mathrm{D}_xX}X_\psi\right)\left(\tilde G - \frac{(\mathrm{D}_xX)^2}{T_t}G\right) = 0,\\
&\left(\Psi_{\psi^*} - \frac{\mathrm{D}_x\Psi}{\mathrm{D}_xX}X_{\psi^*}\right)\left(\tilde G + \frac{(\mathrm{D}_xX)^2}{T_t}G^*\right) = 0,
\end{aligned}
\tag{14}
$$

and the remaining terms give us the equation

$$
\begin{aligned}
&\frac{i}{T_t}\left(\Psi_t + iF\left(\Psi_\psi - \frac{\mathrm{D}_x\Psi}{\mathrm{D}_xX}X_\psi\right) - iF^*\left(\Psi_{\psi^*} - \frac{\mathrm{D}_x\Psi}{\mathrm{D}_xX}X_{\psi^*}\right)\right)\\
&-\frac{i}{T_t}\frac{\mathrm{D}_x\Psi}{\mathrm{D}_xX}X_t + \frac{\tilde G}{(\mathrm{D}_xX)^2}\left(\hat\Delta\Psi - \frac{\mathrm{D}_x\Psi}{\mathrm{D}_xX}\hat\Delta X\right) + \tilde F = 0.
\end{aligned}
\tag{15}
$$

The system (14) implies that at least one of the factors in each of the equations vanishes.

The condition that the first factor of the first equation is equal to zero expands to

$$
\begin{vmatrix} X_x & X_\psi \\ \Psi_x & \Psi_\psi \end{vmatrix} - \begin{vmatrix} X_\psi & X_{\psi^*} \\ \Psi_\psi & \Psi_{\psi^*} \end{vmatrix}\psi_x^* = 0,
$$

and since X, Ψ are independent of ψ_x, ψ_x^*, we find that both the determinants are zero. If we suppose that $(X_\psi, \Psi_\psi)^\mathsf{T} \neq (0,0)^\mathsf{T}$ then each of the other columns, $(X_x, \Psi_x)^\mathsf{T}$ and $(X_{\psi^*}, \Psi_{\psi^*})^\mathsf{T}$ is proportional to $(X_\psi, \Psi_\psi)^\mathsf{T}$, which contradicts the condition $\mathrm{d}T \wedge \mathrm{d}X \wedge \mathrm{d}\Psi \wedge \mathrm{d}\Psi^* \neq 0$. Hence, $(X_\psi, \Psi_\psi)^\mathsf{T} = (0,0)^\mathsf{T}$ and thus $(X_{\psi^*}, \Psi_{\psi^*})^\mathsf{T} \neq (0,0)^\mathsf{T}$.

Similarly, we derive that $(X_{\psi^*}, \Psi_{\psi^*})^\mathsf{T} = (0,0)^\mathsf{T}$ and $(X_\psi, \Psi_\psi)^\mathsf{T} \neq (0,0)^\mathsf{T}$ if the first factor of the second equation vanishes. Note that the condition $X_\psi = 0$ and $X_{\psi^*} = 0$ are equivalent since X is a real-valued function and thus $X_\psi = X_{\psi^*}$.

Therefore, the first factors cannot vanish simultaneously and hence at least one of the second factors is zero, which gives (7b). Both second factors are equal to zero only if $G^* = -G$, and then we have no more restrictions for the admissible transformations. If $G^* \neq -G$, one of the first factors vanishes, which is equivalent to (7a).

Further, rearranging equation (15) gives the transformation component (7c) for the arbitrary element F. □

Each element $\mathcal{T}$ of the equivalence group $G^\sim_\mathcal{A}$ generates a family of admissible transformations parameterized by the arbitrary elements (G, F). Hence the projection of $\mathcal{T}$ to the space of variables satisfies all the restrictions for admissible transformations, as given in Theorem IV.2, i.e., $T_x = 0$, $T_\psi = T_{\psi^*} = 0$, $X_\psi = X_{\psi^*} = 0$ and either $\Psi_\psi = 0$ or $\Psi_{\psi^*} = 0$. The components of $\mathcal{T}$ for G and F are uniquely determined by (7b) and (7c). Conversely, a point transformation in the joint space of the jet variables $(t, x, \psi_{(2)}, \psi^*_{(2)})$ and the arbitrary elements (G, G^*, F, F^*) that is consistent with the contact structure of the jet space, belongs to $\mathcal{G}^\sim_\mathcal{A}$ if it satisfies the above conditions. As a result, from Theorem IV.2 it is clear that the equivalence groupoid $\mathcal{G}^\sim_\mathcal{A}$ is not generated by its equivalence group $G^\sim_\mathcal{A}$.

Corollary IV.3. *The class $\mathcal{A}$ is not normalized. Its equivalence group $G^\sim_\mathcal{A}$ consists of the point transformations in the joint space of the jet variables $(t, x, \psi_{(2)}, \psi^*_{(2)})$ and the arbitrary elements (G, G^*, F, F^*), where the components for t, x and ψ are of the form* (5) *with $T = T(t)$, $X = X(t,x)$, $T_tX_x \neq 0$ and either $\Psi_\psi = 0$ or $\Psi_{\psi^*} = 0$, and the components for G and F are of the form* (7b) *and* (7c).

Since the class $\mathcal{A}$ is not normalized, we partition it into the two disjoint subclasses $\mathcal{A}_1$ and $\mathcal{A}_2$ that are defined as follows: $\mathcal{A}_1$ consists of those equations of $\mathcal{A}$ for which $G^* \neq -G$, and $\mathcal{A}_2$ consists of those equations for which $G^* = -G$. We use the usual notation $G^\sim_{\mathcal{A}_1}$ and $G^\sim_{\mathcal{A}_2}$ for the equivalence groups of these subclasses and $\mathcal{G}^\sim_{\mathcal{A}_1}$ and $\mathcal{G}^\sim_{\mathcal{A}_2}$ for their equivalence groupoids. The transformational part φ of an admissible transformation in the class $\mathcal{A}$ necessarily satisfies the condition $\Psi_\psi\Psi_{\psi^*} = 0$ if and only if the initial arbitrary element G is constrained by $G^* \neq -G$. Prolonging the transformation φ to the arbitrary elements G and F according to (7b) and (7c) and taking into account (7a) for the class $\mathcal{A}_1$, gives an equivalence transformation of the respective class. Therefore, the equivalence groupoids $\mathcal{G}^\sim_{\mathcal{A}_1}$ and $\mathcal{G}^\sim_{\mathcal{A}_2}$ are generated by the corresponding equivalence groups, i.e., the subclasses $\mathcal{A}_1$ and $\mathcal{A}_2$ are normalized in the usual sense.

The condition (7b) implies that $G^* = -G$ if and only if $\tilde{G}^* = -\tilde{G}$, and thus $G^* \neq -G$ if and only if $\tilde{G}^* \neq -\tilde{G}$. This means that the partition is invariant with respect to admissible transformations of the class $\mathcal{A}$. In the other words, there are no point transformations between equations that belong to different partition components. This why the partition of the class $\mathcal{A}$ into the subclasses $\mathcal{A}_1$ and $\mathcal{A}_2$ induces the partitioning of the equivalence groupoid $\mathcal{G}^\sim_{\mathcal{A}}$ into the equivalence groupoids $\mathcal{G}^\sim_{\mathcal{A}_1}$ and $\mathcal{G}^\sim_{\mathcal{A}_2}$.

Corollary IV.4. *The class $\mathcal{A}$ is partitioned into the two disjoint normalized subclasses $\mathcal{A}_1$ and $\mathcal{A}_2$ singled out from the class $\mathcal{A}$ by the constraints $G^* \neq -G$ and $G^* = -G$, respectively. Both the subclasses $\mathcal{A}_1$ and $\mathcal{A}_2$ are normalized in the usual sense. The usual equivalence group $G^\sim_{\mathcal{A}_1}$ coincides with the equivalence group $G^\sim_{\mathcal{A}}$ of the entire class $\mathcal{A}$ and the description of the usual equivalence group $G^\sim_{\mathcal{A}_2}$ is obtained from that of the group $G^\sim_{\mathcal{A}}$ by excluding the condition that either $\Psi_\psi = 0$ or $\Psi_{\psi^*} = 0$. Moreover, there are no point transformations that map equations from the class $\mathcal{A}_1$ to equations from the class $\mathcal{A}_2$ or conversely.*

3 Admissible transformations in narrower classes

Here, we study the admissible transformations of the subclasses of the class $\mathcal{A}$ whose arbitrary elements do not depend on the derivatives of the dependent variables.

Theorem IV.2 implies that there is no specific condition on the X-components of the point transformation φ. To ensure the projectability of the components of the point transformations to the space of variables, we need to constrain the functions F and G additionally: we narrow the class $\mathcal{A}$ by requiring $G_{\psi_x} = G_{\psi^*_x} = F_{\psi_x} = F_{\psi^*_x} = 0$ and obtain the class $\mathcal{N}$ consisting of equations of the form (3).

Theorem IV.3. *The equivalence groupoid $\mathcal{G}^\sim_{\mathcal{N}}$ of the class $\mathcal{N}$ consists of triples of the form $((G,F),(\tilde{G},\tilde{F}),\varphi)$, where φ is a point transformation in the space of variables of the form* (5) *whose components are*

$$\begin{aligned}&\tilde{t} = T(t), \quad \tilde{x} = X(t,x),\\ &\tilde{\psi} = \Psi := \Psi^1(t,x)\psi + \Psi^2(t,x)\psi^* + \Psi^0(t,x)\end{aligned} \tag{16a}$$

$$\textit{and, if} \quad G^* \neq -G, \quad \Psi^1\Psi^2 = 0; \tag{16b}$$

the transformed arbitrary elements $\tilde{G}$ and $\tilde{F}$ are given by

$$\tilde{G} = \frac{X_x^2}{T_t}G \quad \textit{if} \quad \Psi^1 \neq 0 \quad \textit{or} \quad \tilde{G} = -\frac{X_x^2}{T_t}G^* \quad \textit{if} \quad \Psi^2 \neq 0, \tag{16c}$$

$$\tilde{F} = \frac{\Psi^1}{T_t}F - \frac{\Psi^2}{T_t}F^* + \left(\frac{X_{xx}}{X_x}\Psi_x - \Psi_{xx}\right)\frac{\tilde{G}}{X_x^2} - \frac{i}{T_t}\left(\Psi_t - \frac{X_t}{X_x}\Psi_x\right), \tag{16d}$$

where T and X are real-valued functions of t, and t and x with $T_tX_x \neq 0$, respectively, and Ψ^1 and Ψ^2 are smooth complex-valued functions of their arguments satisfying the conditions

$$2G\Psi^1_x - \left(\frac{X_{xx}}{X_x}G + i\frac{X_t}{X_x}\right)\Psi^1 = 0,$$
$$2G^*\Psi^2_x - \left(\frac{X_{xx}}{X_x}G^* - i\frac{X_t}{X_x}\right)\Psi^2 = 0. \tag{17}$$

Here $\Psi^0(t,x)$ is a complex-valued smooth function depending on t and x.

Proof. We apply Theorem IV.2 to a pair of equations from the class $\mathcal{N}$. Since both the arbitrary elements G and F of this class do not depend on derivatives of ψ and ψ^*, we can split the relations (7b), (7c) between (G,F) and $(\tilde G,\tilde F)$ with respect to ψ_x and ψ^*_x. Rearranging the results of this splitting, we obtain more restrictive relations (16c), (16d) between (G,F) and $(\tilde G,\tilde F)$ within the class $\mathcal{N}$ and constraints for parameters of transformational part φ:

$$X_\psi = X_{\psi^*} = 0, \quad \Psi_{\psi\psi} = \Psi_{\psi^*\psi} = \Psi_{\psi^*\psi^*} = 0, \tag{18}$$

$$\left(\frac{X_{xx}}{X_x}\Psi_\psi - 2\Psi_{x\psi}\right)\frac{\tilde G}{X_x^2} + \frac{iX_t}{T_tX_x}\Psi_\psi = 0,$$
$$\left(\frac{X_{xx}}{X_x}\Psi_{\psi^*} - 2\Psi_{x\psi^*}\right)\frac{\tilde G}{X_x^2} + \frac{iX_t}{T_tX_x}\Psi_{\psi^*} = 0. \tag{19}$$

Integrating the equations (18) gives the expression for Ψ from (16a), substituting which into (19) leads to the conditions (17). □

By Theorem IV.3 we obtain the usual equivalence group $G^\sim_{\mathcal{N}}$ of the class $\mathcal{N}$ by splitting the conditions (17) with respect to G and G^*. The motivation for this splitting is the fact that $(G_\psi, G_{\psi^*}) \neq (0,0)$, and this leads to

$$X = X(x), \quad \Psi^1 = A^1(t)|X_x|^{1/2}, \quad \Psi^2 = A^2(t)|X_x|^{1/2}, \tag{20}$$

where X is an arbitrary real-valued smooth function of x with $X_x \neq 0$, A^1 and A^2 are complex-valued smooth functions depending on t with $(A^1, A^2) \neq (0,0)$ and $A^1A^2 = 0$.

Corollary IV.5. *The class $\mathcal{N}$ is not normalized. Its equivalence group $G^\sim_{\mathcal{N}}$ consists of point transformations in the space of $(t,x,\psi,\psi^*,G,G^*,F,F^*)$, where the components for t, x and ψ are of the form* (16a) *constrained by* (20) *with $X_x \neq 0$, $(A^1,A^2) \neq (0,0)$ and $A^1A^2 = 0$, and the components for G and F are of the form* (16c) *and* (16d).

In view of Corollary IV.5, the equivalence group $G^\sim_{\mathcal{N}}$ acts only on the space of variables and arbitrary elements and consists of two families of point transformations with either $A^1 = 0$ or $A^2 = 0$. This corollary also means that the equivalence groupoid $\mathcal{G}^\sim_{\mathcal{N}}$ of the class $\mathcal{N}$ is not generated by the equivalence group $G^\sim_{\mathcal{N}}$, i.e., there are more admissible transformations under certain constraints on G, something that complicates the study of the admissible transformations for the class $\mathcal{N}$. In order to give a complete description of the equivalence groupoid $\mathcal{G}^\sim_{\mathcal{N}}$, we partition the class $\mathcal{N}$ into four disjoint subclasses by placing constraints on G,

$$\mathcal{N}_1 := \{\mathcal{L}_{GF} \in \mathcal{N} \mid G^* \neq -G,\ (G_\psi, G_{\psi^*}) \neq (0,0)\},$$
$$\mathcal{N}_2 := \{\mathcal{L}_{GF} \in \mathcal{N} \mid G^* = -G,\ (G_\psi, G_{\psi^*}) \neq (0,0)\},$$

$$\mathcal{N}_3 := \{\mathcal{L}_{GF} \in \mathcal{N} \mid G^* \neq -G,\ G_\psi = G_{\psi^*} = 0\},$$
$$\mathcal{N}_4 := \{\mathcal{L}_{GF} \in \mathcal{N} \mid G^* = -G,\ G_\psi = G_{\psi^*} = 0\},$$

and we then characterize their equivalence groups (which are considered as conditional equivalence groups for the entire class $\mathcal{N}$).

In order to prove that equations from the different subclasses of the partition are not related by point transformations, it is sufficient to show that the constraints that single out each of these subclasses are preserved by admissible transformations of the class $\mathcal{N}$. In other words $G^* = -G$ if and only if $\tilde{G}^* = -\tilde{G}$. Similarly, $G_\psi = G_{\psi^*} = 0$ if and only if $\tilde{G}_{\tilde{\psi}} = \tilde{G}_{\tilde{\psi}^*} = 0$. Both these results follow from the condition (16c). As a result the partition of the class $\mathcal{N}$ induces the partitioning of the equivalence groupoid $\mathcal{G}^\sim_{\mathcal{N}}$ into the equivalence groupoids $\mathcal{G}^\sim_{\mathcal{N}1}, \ldots, \mathcal{G}^\sim_{\mathcal{N}_4}$ of the subclasses $\mathcal{N}_1, \ldots, \mathcal{N}_4$.

The equivalence groups $G^\sim_{\mathcal{N}_1}, \ldots, G^\sim_{\mathcal{N}_4}$ of the subclasses $\mathcal{N}_1, \ldots, \mathcal{N}_4$ can be constructed using Theorem IV.3. The transformational part of an admissible transformation in the class $\mathcal{N}$ satisfies the condition $\Psi^1\Psi^2 = 0$ if and only if the initial arbitrary element G is constrained by $G^* \neq -G$.

Under the constraint $(G_\psi, G_{\psi^*}) \neq (0,0)$, we can split the equations (17) with respect to G and G^* and integrate as above, and this yields X and Ψ as given in (20). If, however, $(G_\psi, G_{\psi^*}) = (0,0)$, then we can integrate these equations immediately to find the expressions for Ψ^1 and Ψ^2:

$$\begin{aligned}\Psi^1 &= A^1(t)|X_x|^{1/2}\exp\left(\frac{i}{2}\int\frac{X_t}{GX_x}\mathrm{d}x\right),\\ \Psi^2 &= A^2(t)|X_x|^{1/2}\exp\left(\frac{i}{2}\int\frac{X_t}{G^*X_x}\mathrm{d}x\right).\end{aligned} \tag{21}$$

Here X is an arbitrary real-valued smooth function of t and x with $X_x \neq 0$, and A^1 and A^2 are complex-valued smooth functions depending on t with $(A^1, A^2) \neq (0,0)$, and the integral denotes a fixed antiderivative of the corresponding integrand with respect to x.

The nonlocal dependence of Ψ on G and G^* according to (21) shows that the subclasses $\mathcal{N}_3$ and $\mathcal{N}_4$ possess transformations that are not point transformations in the joint space of variables and arbitrary elements. The arbitrary tuple of elements (G, F) and $(\tilde{G}, \tilde{F})$ are related non-locally. From (21), we see that for each equation $\mathcal{L}_{GF}$, for a fixed arbitrary element G, we have an infinite number of solutions Ψ^1 and Ψ^2 associated with it. We can extend G and F by the arbitrary complex-smooth functions Φ^1 and Φ^2 non-locally so that Ψ^1 and Ψ^2 are treated as particular solutions of the equations (17) in Φ^1 and Φ^2, respectively. The equivalence transformations allowing such kinds of transformations form a group called the "extended generalized equivalence group" $G^\sim$. In view of (21) these transformations cannot be point transformations, even if we fix the arbitrary element G: this is due to the appearance of the parameter function X depending on t and x.

On the other hand, if $(G_\psi, G_{\psi^*}) \neq (0,0)$, then the point transformations in the space (t, x, ψ, ψ^*), and their prolongations with respect to t and x as well as the arbitrary elements as defined in (16c) and (16d), produce in this joint space transformations (equivalence transformations) which generate the usual equivalence groups $G^\sim_{\mathcal{N}_1}$ and $G^\sim_{\mathcal{N}_2}$.

Corollary IV.6. *The class $\mathcal{N}$ is partitioned into the four disjoint normalized subclasses $\mathcal{N}_1, \ldots, \mathcal{N}_4$ with respect to constraints on G. The classes $\mathcal{N}_1$ and $\mathcal{N}_2$ are normalized in the usual sense and the classes $\mathcal{N}_3$ and $\mathcal{N}_4$ are normalized in the extended generalized sense. The usual equivalence group $G^\sim_{\mathcal{N}_1}$ coincides with the equivalence group $G^\sim_{\mathcal{N}}$ of the* *entire class $\mathcal{N}$ and the description of the equivalence group $G^{\sim}_{\mathcal{N}_2}$ is obtained from that of the group $G^{\sim}_{\mathcal{N}}$ by excluding the condition $A^1A^2 = 0$. The extended generalized equivalence groups $\hat{G}^{\sim}_{\mathcal{N}_3}$ and $\hat{G}^{\sim}_{\mathcal{N}_4}$ consist of transformations whose components for t, x and ψ are of the form* (16a) *constrained by* (21) *with $X_x \neq 0$, $(A^1, A^2) \neq (0, 0)$ (for $\hat{G}^{\sim}_{\mathcal{N}_3}$ we also have $A^1A^2 = 0$), and the components for G and F are of the form* (16c) *and* (16d). *Further, there are no point transformations between equations of the different subclasses of the above partition.*

The classes $\mathcal{N}_3$ and $\mathcal{N}_4$ are difficult and challenging to investigate due to the structure of their admissible transformations as described in the above corollary. In order to avoid this problem, we can use the technique of mapping a class to its subclass as developed in [21], [24]. Thus, in the class $\mathcal{N}_4$ we can set $G = iH(t, x)$, $T_t = \operatorname{sgn} G$ and $X_x = |G|^{-1/2}$, $\Psi^1 = \Psi^2 = 1$ and $\Psi^0 = 0$, where H is a real-valued smooth function of t and x. These transformations imply that $\tilde{G} = i$. Therefore, instead of treating the class $\mathcal{N}_4$ we can treat its subclass $\mathcal{N}_4|_{G=i}$ obtained by that mapping.

Corollary IV.7. *The subclass $\mathcal{N}_4|_{G=i}$ of the class $\mathcal{N}_4$ that consists of equations of the form $i\psi_t + i\psi_{xx} + F(t, x, \psi, \psi^*) = 0$ is normalized in the usual sense. Its equivalence group $G^{\sim}_{\mathcal{N}_4}|_{G=i}$ consists of transformations* (16) *and conditions* (21) *with $X = \varepsilon|T_t|^{1/2}x + X^0(t)$, where X^0 is an arbitrary real-valued function of t and x, and $\varepsilon = \operatorname{sgn} T_t$.*

4 Linear Schrödinger equations with variable mass

We consider the class $\mathcal{M}$ that consists of equations of the form (4). This class is a subclass of $\mathcal{N}_3$, where G is a real-valued function of t and x. The preliminary results for the groupoid of the class $\mathcal{M}$ follows from Theorem IV.3, where $G = 1/m(t, x)$, $m \neq 0$ and $F = V(t, x)\psi$. Then the equations (17) and the structure of T, X and Ψ are preserved, i.e., $T = T(t)$, $X = X(t, x)$ and $\Psi = \Psi^1(\hat{\psi} + \hat{\Phi})$, where T and X are real-valued functions depending on their arguments, and $\Phi = \Psi^0/\Psi^1$ is a complex-valued function of t and x with $T_tX_x\Psi^1 \neq 0$. The equations (16c) and (16d), after rearrangements, give the transformations for the arbitrary elements G and V. Thus we have:

Theorem IV.4. *The equivalence groupoid $\mathcal{G}^{\sim}_{\mathcal{M}}$ of the class $\mathcal{M}$ is constituted by triples $((G, V), (\tilde{G}, \tilde{V}), \varphi)$, where φ is a point transformation in the space of variables whose components are*

$$T = T(t), \quad X = X(t, x), \quad \Psi = \Psi^1(t, x)(\hat{\psi} + \hat{\Phi}(t, x)), \tag{22a}$$

and the transformed arbitrary elements $\tilde{G}$ and $\tilde{V}$ are given by

$$\tilde{G} = \frac{X_x^2}{|T_t|}G, \quad \tilde{V} = \frac{\hat{V}}{|T_t|} - \frac{\Psi^1_{xx}}{\Psi^1}\frac{G}{|T_t|} - \frac{i}{T_t\Psi^1}\left(\Psi^1_t - \frac{X_t}{X_x}\Psi^1_x\right). \tag{22b}$$

Here $T(t)$ and $X(t, x)$ are real-valued functions of their arguments with $T_t \neq 0$ and $X_x \neq 0$, and the complex-valued function $\Phi = \Phi(t, x)$ is an arbitrary solution of the initial equation depending on t and x,

$$i\Phi_t + G(t, x)\Phi_{xx} + V(t, x)\Phi = 0. \tag{23}$$

Moreover, the complex-valued function Ψ^1 satisfies the condition (17).

The point transformations for the class $\mathcal{M}$ described in Theorem IV.4 are difficult to analyze in view of their structure. But since G is a real valued function, the condition $G \neq -G^*$ becomes $G \neq -G$. To obtain the usual equivalence group for this class we can split the condition (17) with respect to G. As a result we obtain the following result:

Corollary IV.8. *The class $\mathcal{M}$ is not normalized. Its usual equivalence group $G^{\sim}_{\mathcal{M}}$ consists of point transformations* (22)*, where $X = X(x)$, $\Psi^1 = A(t)|X_x|^{1/2}$ with $T_tX_x\Psi^1 \neq 0$ and $\Phi = 0$. Here A is a nonzero complex-valued smooth function of t, and X is a real-valued function of x.*

It is known from [21], [24] that we can use the technique of mapping a class to one of its subclasses by admissible transformations from this class and we apply this technique to map the class $\mathcal{M}$ to a subclass. Using transformations (22), we can set $T_t = \operatorname{sgn} G$, $X_x = |G|^{-1/2}$, $\Psi^1 = 1$ and $\Psi^0 = 0$ and gauge the arbitrary element $G = 1$ in the class $\mathcal{M}$. The subclass singled out from this class by gauging $G = 1$ is denoted by $\mathcal{M}|_{G=1}$, and it is similar to the class of (1+1)-dimensional linear Schrödinger equations,

$$i\psi_t + \psi_{xx} + V(t,x)\psi = 0, \tag{24}$$

where V is a complex-valued potential that depends on t and x studied in the paper [12]. This means that any equation from the class $\mathcal{M}$ can be mapped to equation (24). Therefore, the usual equivalence group $G^{\sim}$ for the class (24) computed in [12, Theorem 6] coincides with the usual equivalence group $G^{\sim}|_{G=1}$ of the subclass $\mathcal{M}|_{G=1}$. Consequently, the complete description of the admissible transformations of this subclass is obtained from that of the class (24) and, as a result of this, the group classification of the class $\mathcal{M}$ can be obtained via the group classification of the class (24). That is, the group classification up to $G^{\sim}_{\mathcal{M}}$-equivalence is reduced to the group classification of the class (24) with respect to its equivalence group $G^{\sim}_{(24)}$, which is given below.

Theorem IV.5. *The equivalence groupoid $\mathcal{G}^{\sim}_{(24)}$ of the subclass* (24) *obtained by gauging $G = 1$ in the class $\mathcal{M}$ is given by triples $(V, \tilde V, \varphi)$, where φ is a point transformation in the space of variables of the form* (5) *whose components are*

$$\tilde t = T, \quad \tilde x = \varepsilon|T_t|^{1/2}x + X^0, \tag{25a}$$

$$\tilde\psi = \exp\left(\frac{i}{8}\frac{T_{tt}}{|T_t|}\,x^2 + \frac{i}{2}\frac{\varepsilon\varepsilon' X^0_t}{|T_t|^{1/2}}\,x + i\Sigma + \Upsilon\right)(\hat\psi + \hat\Phi), \tag{25b}$$

and the transformed potential $\tilde V$ is related to V via the equation

$$\begin{aligned}\tilde V = \frac{\hat V}{|T_t|} + \frac{2T_{ttt}T_t - 3T_{tt}^2}{16\varepsilon' T_t^3}x^2 + \frac{\varepsilon\varepsilon'}{2|T_t|^{1/2}}\left(\frac{X^0_t}{T_t}\right)_t x\\ - \frac{iT_{tt} + (X^0_t)^2}{4T_t^2} + \frac{\Sigma_t - i\Upsilon_t}{T_t},\end{aligned} \tag{25c}$$

$T = T(t)$, $X^0 = X^0(t)$, $\Sigma = \Sigma(t)$ and $\Upsilon = \Upsilon(t)$ are arbitrary smooth real-valued functions of t with $T_t \neq 0$, and $\Phi = \Phi(t,x)$ denotes an arbitrary solution of the initial equation. Here $\varepsilon = \pm 1$ and $\varepsilon' = \operatorname{sgn} T_t$.

Corollary IV.9. *The subclass* (24) *is uniformly semi-normalized with respect to linear superposition of solutions. Its usual equivalence group $G^{\sim}_{(24)}$ consists of point transformations of the form* (25) *with $\Phi = 0$.*

5 Conclusion

We have studied the admissible transformations in the class $\mathcal{A}$ of (1+1)-dimensional generalized nonlinear Schrödinger equations with variable mass and its subclasses. A complete description of the equivalence groupoid $\mathcal{G}^{\sim}_{\mathcal{A}}$ for the class $\mathcal{A}$ is provided by Theorem IV.2, and we found that this class is not normalized.

Knowledge of the groupoid $\mathcal{G}^{\sim}_{\mathcal{A}}$ allows us to derive the equivalence groupoid $\mathcal{G}^{\sim}_{\mathcal{N}}$ and the equivalence group $G^{\sim}_{\mathcal{N}}$ of the class $\mathcal{N}$ contained in a wider class $\mathcal{A}$, and singled out by the constraints $G_{\psi_x} = G_{\psi^*_x} = F_{\psi_x} = F_{\psi^*_x} = 0$. We have shown that this class is not normalized. In order to study the admissible transformations of this class, we partitioned it into four disjoint normalized subclasses $\mathcal{N}_1, \dots, \mathcal{N}_4$ with respect to conditions on G so that the union of their equivalence groups is larger than the usual equivalence group $G^{\sim}_{\mathcal{N}}$ of the class $\mathcal{N}$. We then derived their equivalence groups and established the connections between them and the equivalence group $G^{\sim}_{\mathcal{N}}$ of the entire class $\mathcal{N}$, as given in Corollary IV.6.

The subclass $\mathcal{M}$ from the class $\mathcal{N}$ is obtained by putting $G = G(t,x) = 1/m(t,x)$, $m \neq 0$ and $F(t,x,\psi,\psi^*) = V(t,x)\psi$, where m is a real-valued function of (t,x). A complete description of the admissible transformations for this class is provided by Theorem IV.4 and it is shown that this class is not normalized in the usual sense. The method of mappings between classes of differential equations and their subclasses yields the admissible transformations for the class $\mathcal{M}$. Using a family of point transformations from the class $\mathcal{M}$, we are able to gauge the arbitrary element G to $G = 1$ and hence to obtain the class (24) of (1+1)-dimensional linear Schrödinger equations with constant mass equal to one and complex potentials. This means that the equivalence groupoid, the equivalence group and the normalization properties of the subclass of the class $\mathcal{M}$ with $G = 1$ can be obtained from the corresponding objects for (24). These are given in Theorem IV.5 and Corollary IV.9.

Although we have given an exhaustive analysis of the admissible transformations in the class $\mathcal{A}$ and its subclasses, including the class $\mathcal{M}$, the technique of mapping classes to its subclasses used here and the gauging of the arbitrary element G to $G = 1$ in the class $\mathcal{M}$ is not effective in higher dimensions (when $n > 1$). This is due to the fact that we do not have enough functions in the transformations which would allow us to gauge the arbitrary element G to $G = 1$. More specifically, this means that the class of multidimensional linear Schrödinger equations with variable mass cannot be mapped to the class of multidimensional linear Schrödinger equations with constant mass equal to one. Hence new tools and strategies are needed to study the case of higher dimension.

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Paper V

Equivalence groupoid for (1+2)-dimensional linear Schrödinger equations with complex potentials

Equivalence groupoid for (1+2)-dimensional linear Schrödinger equations with complex potentials

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We describe admissible point transformations in the class of (1+2)-dimensional linear Schrödinger equations with complex potentials. We prove that any point transformation connecting two equations from this class is the composition of a linear superposition transformation of the corresponding initial equation and an equivalence transformation of the class. This shows that the class under study is semi-normalized.

1 Introduction

In the theory of group classification of differential equations, the *equivalence groupoid* of a class of differential equations is the set of *admissible transformations* of this class together with the composition operation of admissible transformations [4]. More specifically, an admissible (point) transformation of the class consists of a pair of similar equations (an initial and a target ones) from the class and a point transformation between these equations. The composition of two admissible transformations is well defined whenever the target equation of the first admissible transformation coincides with the initial equation of the second admissible transformation. For each equation from the class, there exists the admissible identity transformation. Each admissible transformation is invertible. The associativity of the composition of admissible transformations directly follows from those of point transformations.

The study of admissible transformations was initiated by Kingston and Sophocleous (see [9]), who called an unformalized version of such transformations *form-preserving* [10]. Form-preserving transformations under the name *allowed transformations* also arose in the course of symmetry analysis of variable-coefficient Korteweg–de Vries equations [20] and variable-coefficient (1+1)-dimensional cubic Schrödinger equations [7]. Later on, formalizing the framework of admissible transformations was initiated [14] and the range of applicability of admissible transformations was extended to various classes of differential equations being important for applications, including nonlinear Schrödinger equations [15], variable-coefficient reaction–diffusion equations [17], [18], [19], eddy vorticity flux parameterizations of the inviscid barotropic vorticity equation [16] and nonlinear wave equations from the theory of elasticity [4].

In order to rigorously pose the problem under consideration, we need to precisely define the notions of classes of differential equations and their equivalence groupoids. Other notions and definitions related to classes of differential equations can be found, e.g., in [4], [15]. Consider a system of differential equations $\mathcal{L}_\theta$: $L(x,u_{(p)},\theta_{(q)}(x,u_{(p)}))=0$, parameterized by the tuple of arbitrary elements $\theta(x,u_{(p)})=(\theta^1(x,u_{(p)}),\dots,\theta^k(x,u_{(p)}))$, where $x=(x_1,\dots,x_n)$ is the tuple of independent variables and $u_{(p)}$ is the set of the dependent variables $u=(u^1,\dots,u^m)$ together with all derivatives of u with respect to x up to order p. The symbol $\theta_{(q)}$ stands for the set of derivatives of θ of order not greater than q with respect to the variables x and $u_{(p)}$. The tuple of arbitrary elements θ runs through the set $\mathcal S$ of solutions of an auxiliary system of differential equations $S(x,u_{(p)},\theta_{q'}(x,u_{(p)}))=0$ and differential inequalities, like $\Sigma(x,u_{(p)},\theta_{(q')}(x,u_{(p)}))\neq 0$, in which both x and $u_{(p)}$ play the role of independent variables and S and Σ are tuples of smooth functions depending on x, $u_{(p)}$ and $\theta_{(q')}$. Then the set $\{\mathcal L_\theta\mid\theta\in\mathcal S\}:=\mathcal L|_{\mathcal S}$ is called a *class of differential equations* defined by the parameterized form $\mathcal L_\theta$ and the set $\mathcal S$ run by the arbitrary elements θ. Denote by $\mathrm T(\theta,\tilde\theta)$ with $\theta,\tilde\theta\in\mathcal S$ the set of point transformations in the space of the variables (x,u) that map the system $\mathcal L_\theta$ to the system $\mathcal L_{\tilde\theta}$. A triple constituted by two arbitrary elements $\theta,\tilde\theta\in\mathcal S$ with $\mathrm T(\theta,\tilde\theta)\neq\varnothing$ and a point transformation $\varphi\in\mathrm T(\theta,\tilde\theta)$ is called an *admissible transformation* of the class $\mathcal L|_{\mathcal S}$. The set of admissible transformations of the class $\mathcal L|_{\mathcal S}$ equipped by the natural composition of such transformations as an algebraic operation is called the equivalence groupoid of this class.

The notion of equivalence group originated from the work of Ovsiannikov [13] and became a powerful tool of which the group classification of differential equations relies on. Thus, the use of equivalence groups constitutes a basis for group classification. Here and in what follows, we are concerned with usual equivalence groups. This notion can be generalized in several ways [8], [14], [15]. The *(usual) equivalence (pseudo)group* $G^\sim=G^\sim(\mathcal L|_{\mathcal S})$ of the class $\mathcal L|_{\mathcal S}$ consists of point transformations in the space of independent and dependent variables and arbitrary elements each of which is projectable to the space of $(x,u_{(p')})$ for any $0\leqslant p'\leqslant p$, is consistent with the contact structure on the space of $(x,u_{(p)})$, and maps every system from the class $\mathcal L|_{\mathcal S}$ to another system from the same class. Its elements are called *equivalence transformations* [15].

Knowing the equivalence groupoid of a class of differential equations simplifies the process of classifying Lie symmetry extensions within this class. In particular, this gives an easy way of finding the corresponding equivalence group. The presence of a nice relation between the equivalence groupoid and the equivalence group makes the entire procedure of symmetry classification less cumbersome and more harmonious and allows for presenting a final classification list in a compact explicit form. Moreover, properties of the equivalence groupoid influence the choice of methods for group classification.

The class $\mathcal L|_{\mathcal S}$ is *normalized (in the usual sense)* if its equivalence groupoid is generated by its (usual) equivalence group and is said to be *semi-normalized* if its equivalence groupoid is generated by the transformations from the equivalence group simultaneously with transformations from symmetry groups of initial or transformed systems [15].

The algebraic method is the best choice for group classification of normalized or specifically semi-normalized classes of differential equations. A number of classes of differential equations that are important for applications are normalized or specifically semi-normalized or can be partitioned into normalized classes and hence were classified within the framework of the algebraic method. See, e.g., [1], [4], [5], [11], [15], [16].

The equivalence groupoid of the class of linear Schrödinger equations with complex potentials has been computed in [11] in the (1+1)-dimensional case. Then properties of the groupoid (more precisely, semi-normalization of the class) was taken as a base for carrying out the group classification of these equations by the algebraic method similarly to [5], where Lie symmetries of nth order ($n\geqslant 2$) of linear ordinary differential equations were classified using semi-normalization of the class of such equations. The final classification result, which is a complete list of inequivalent potentials corresponding to equations with nontrivial Lie symmetries, may be used in quantum theory, quantum field theory, optics and other branches of physics, cf. [2], [3], [6], [12].

The aim of this paper is to study the equivalence groupoid, the equivalence group and normalization properties of the class $\mathcal{F}$ of (1+2)-dimensional linear Schrödinger equations with complex potentials,

$$i\psi_t + \psi_{aa} + V(t,x)\psi = 0. \tag{1}$$

Here and in what follows t, x_1 and x_2 are the real independent variables, $x = (x_1, x_2)$, ψ is the complex dependent variable and V is an arbitrary smooth complex-valued potential depending on t and x. Subscripts of functions denote differentiation with respect to the corresponding variables. In particular, $\psi_t = \partial\psi/\partial t$ and $\psi_{ab} = \partial^2\psi/\partial x_a \partial x_b$. The indices a, b, c, and d run from 1 to 2, and we use summation convention over repeated indices.

The structure of this paper is organized as follows: In Section (2) we compute the equivalence groupoid of the class $\mathcal{F}$. It appears that the transformational parts of admissible transformations are uniformly parameterized by the corresponding initial values of V. The expression for target values of the arbitrary element V in terms of its initial values and transformation parameters is also found. Then the equivalence group of the class $\mathcal{F}$ and normalization properties of this class are described in Section (3). The last section includes a short summary and suggests a direction for the future work.

2 Equivalence groupoid

We find the equivalence groupoid $\mathcal{G}^{\sim}$ of the class $\mathcal{F}$ by using the direct method. We seek for all invertible point transformations of the form

$$\tilde{t} = T(t,x,\psi,\psi^*), \quad \tilde{x}_a = X^a(t,x,\psi,\psi^*), \quad \tilde{\psi} = \Psi(t,x,\psi,\psi^*), \quad \tilde{\psi}^* = \Psi^*(t,x,\psi,\psi^*) \tag{2}$$

with $J = \partial(T, X^a, \Psi, \Psi^*)/\partial(t,x,\psi,\psi^*) \neq 0$ that map a fixed equation from the class $\mathcal{F}$ to another equation of the same class,

$$i\tilde{\psi}_{\tilde{t}} + \tilde{\psi}_{\tilde{x}_a\tilde{x}_a} + \tilde{V}(\tilde{t},\tilde{x})\tilde{\psi} = 0. \tag{3}$$

Hereafter in case of a complex value β we use the notation

$$\hat{\beta} = \beta \quad \text{if} \quad T_t > 0 \quad \text{and} \quad \hat{\beta} = \beta^* \quad \text{if} \quad T_t < 0.$$

Lemma V.1. *Any point transformation $\mathcal{T}$ connecting two equations from the class $\mathcal{F}$ satisfies the conditions*

$$\begin{gathered} T_x = T_\psi = T_{\psi^*} = 0, \quad X^a_\psi = X^a_{\psi^*} = 0, \\ \Psi_\psi = 0 \quad \textit{if} \quad T_t < 0 \quad \textit{and} \quad \Psi_{\psi^*} = 0 \quad \textit{if} \quad T_t > 0. \end{gathered} \tag{4}$$

Proof. The proof is similar to that given in [15]. □

Theorem V.1. *The equivalence groupoid $\mathcal{G}^{\sim}$ of the class $\mathcal{F}$ consists of the triples of the form $(V, \tilde{V}, \mathcal{T})$, where $\mathcal{T}$ is a point transformation in the space of variables, whose components are*

$$\tilde{t} = T, \quad \tilde{x}_a = |T_t|^{1/2} O^{ab} x_b + \mathcal{X}^a, \tag{5a}$$

$$\tilde{\psi} = \exp\left(\frac{i}{8}\frac{T_{tt}}{|T_t|}\, x_a x_a + \frac{i}{2}\frac{\varepsilon' \mathcal{X}^b_t}{|T_t|^{1/2}}\, O^{ba} x_a + \Lambda + i\Sigma\right)(\hat{\psi} + \hat{\Omega}), \tag{5b}$$

and the transformed potential $\tilde V$ is expressed via V as

$$\tilde V = \frac{\hat V}{|T_t|} + \frac{2T_{ttt}T_t - 3T_{tt}{}^2}{16\varepsilon' T_t{}^3}x_ax_a + \frac{\varepsilon'}{2|T_t|^{1/2}}\left(\frac{\mathcal{X}^a_t}{T_t}\right)_t O^{ba}x_a + \frac{\Sigma_t - i\Lambda_t}{T_t} - \frac{\mathcal{X}^a_t\mathcal{X}^a_t + iT_{tt}}{4T_t{}^2}. \tag{5c}$$

Here T, $\mathcal{X}^a$, Λ and Σ are arbitrary smooth real-valued functions of t with $T_t\neq 0$, $\varepsilon' =$ $\operatorname{sgn} T_t$, *$\Omega=\Omega(t,x)$ is an arbitrary solution of the initial equation and $O=(O^{ab})$ is an arbitrary constant 2×2 orthogonal matrix.*

Proof. Let a point transformation $\mathcal{T}$ connect two equations from the class $\mathcal{F}$. Then Lemma (V.1) implies that $T=T(t)$ with $T_t\neq 0$, $X^a=X^a(t,x)$ and $\Psi_{\hat\psi}\neq 0$. Applying the chain rule for total derivatives with respect to t and x to the equality $\tilde\psi(\tilde t,\tilde x) = \Psi(t,x,\psi)$, we derive

$$D_t\tilde\psi(\tilde t,\tilde x) = \tilde\psi_{\tilde t}T_t + \tilde\psi_{\tilde x_b}X^b_t = D_t\Psi, \quad D_a\tilde\psi(\tilde t,\tilde x) = \tilde\psi_{\tilde x_c}X^c_a = D_a\Psi,$$
$$D_bD_a\tilde\psi(\tilde t,\tilde x) = \tilde\psi_{\tilde x_c\tilde x_d}X^c_bX^d_a + \tilde\psi_{\tilde x_d}X^d_{ab} = D_bD_a\Psi,$$

where D_t and D_a are the total derivative operators with respect to t and x_a, respectively. The above equations are equivalent to

$$\tilde\psi_{\tilde t} = \frac{1}{T_t}\left(D_t\Psi - Y^a_bX^b_tD_a\Psi\right), \quad \tilde\psi_{\tilde x_c} = Y^a_cD_a\Psi, \tag{6}$$
$$\tilde\psi_{\tilde x_c\tilde x_d} = Y^a_cY^b_d(D_bD_a\Psi - Y^d_cX^c_{ab}D_d\Psi), \tag{7}$$

where $Y^a_cX^c_b=\delta^a_b$ and δ^a_b is the Kronecker delta. In fact, the vector-function (Y^1,Y^2) is the inverse of the vector-function (X^1,X^2) with respect to x and $Y^a_c=\partial Y^a/\partial\tilde x_c$. We substitute the values of $\tilde\psi_{\tilde t}$ and $\tilde\psi_{\tilde x_c\tilde x_d}$ defined in (6) and (7) into the equation (3) and take into account the expression for $\hat\psi_t$, $\hat\psi_t = i\varepsilon'(\hat\psi_{aa}+\hat V\hat\psi)$. As a result, we derive the equation

$$\frac{i}{T_t}\left(\Psi_t + \Psi_{\hat\psi}(i\varepsilon'\hat\psi_{aa} + i\varepsilon'\hat V\hat\psi) - Y^a_b(\Psi_a + \Psi_{\hat\psi}\hat\psi_a)X^b_t\right) + Y^a_cY^b_c\left(\Psi_{ab} + 2\Psi_{a\hat\psi}\hat\psi_b\right)\right)$$
$$+Y^a_cY^b_c\left(\Psi_{\hat\psi\hat\psi}\hat\psi_b\hat\psi_a + \Psi_{\hat\psi}\hat\psi_{ab} - Y^d_c(\Psi_d + \Psi_{\hat\psi}\hat\psi_d)X^c_{ab}\right) + \tilde V\Psi = 0.$$

Then splitting this equation with respect to various derivatives of $\hat\psi$ and additionally arranging leads to the system

$$Y^a_cY^b_c = 0,\ a\neq b, \quad Y^a_cY^a_c = \frac{1}{|T_t|}, \quad \Psi_{\hat\psi\hat\psi} = 0, \tag{8}$$
$$\frac{2}{|T_t|}\Psi_{a\hat\psi} - \frac{i}{T_t}Y^a_b\Psi_{\hat\psi}X^b_t - \frac{1}{|T_t|}Y^a_c\Psi_{\hat\psi}X^c_{aa} = 0, \tag{9}$$
$$\frac{i}{T_t}\Psi_t - \frac{1}{|T_t|}\hat V\Psi_{\hat\psi}\hat\psi - \frac{i}{T_t}Y^a_b\Psi_aX^b_t + \frac{1}{|T_t|}\Psi_{aa} - Y^d_c\Psi_dX^c_{aa} + \tilde V\Psi = 0, \tag{10}$$

where $\varepsilon' = \operatorname{sgn} T_t$.

We first show that $X^a_{bc}=0$ for all a,b and c. The first two equations in (8) together with the condition $Y^a_cX^c_b=\delta^a_b$ imply that $X^b_a = |T_t|Y^a_b$. Therefore, $X^c_aX^c_b = |T_t|\delta^a_b$, i.e., $X^c_a = |T_t|^{1/2}\,O^{ca}$, where $O=(O^{ca})$ is a 2×2 orthogonal matrix-function of t and

x. Suppose that O is a special orthogonal matrix, i.e., $\det O = 1$. Then the matrix (X^c_a) can be written as

$$(X^c_a) = |T_t|^{1/2} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{11}$$

for a smooth function $\theta = \theta(t,x)$. The representation (11) implies that $X^1_1 = X^2_2 = |T_t|^{1/2}\cos\theta$ and $X^1_2 = -X^2_1 = -|T_t|^{1/2}\sin\theta$, which means that transformation components X^1 and X^1 satisfy the Cauchy–Riemann system

$$X^1_1 = X^2_2, \quad X^1_2 = -X^2_1.$$

Hence both the components X^c are solution of the Laplace equation, i.e., $X^c_{bb} = 0$. Applying the Laplace operator ∂_{bb} to both the sides of the equation $X^c_a X^c_a = |T_t|$, where there is no summation with respect to a, we derive $X^c_{ab}X^c_{ab} = 0$, i.e., $X^c_{ab} = 0$ for all a, b and c. The same result is obtained when $\det O = -1$. Thus, X^c is a linear function in x, which implies that the matrix O may depend only on t. From the previous consideration, the equation (9) becomes

$$\Psi_{a\hat\psi} = \frac{i}{2T_t} X^b_a X^b_t \Psi_{\hat\psi}. \tag{12}$$

Differentiating this equation for $a = 1$ with respect to x_2 and for $a = 2$ with respect to x_1, subtracting the first result to the second and taking into account the coincidence of mixed derivatives, we find the compatibility condition $X^b_1 X^b_{t2} = X^b_2 X^b_{t1}$. The representation (11) together with the compatibility condition results in $\theta_t = 0$. Therefore, the matrix O is a constant orthogonal matrix.

The general solution of the third equation in (8) is $\Psi = \Psi^1\hat\psi + \Psi^0$, where Ψ^0 and Ψ^1 are smooth complex-valued functions of t and x, and $\Psi^1 = \Psi_{\hat\psi} \neq 0$. The integration of the equation (12) gives the expression of Ψ^1,

$$\Psi^1 = \exp\left(\frac{i}{8}\frac{T_{tt}}{|T_t|}\,x_a x_a + \frac{i}{2}\frac{\varepsilon' \mathcal{X}^b_t}{|T_t|^{1/2}}\,O^{ba} x_a + \Lambda + i\Sigma\right), \tag{13}$$

where Λ and Σ are arbitrary smooth real-valued function of t arising in the course of the integration. Finally, we consider the equation (10), which reduces, under the derived conditions to

$$\frac{i}{T_t}\Psi_t - \frac{1}{|T_t|}\hat V \Psi_{\hat\psi}\hat\psi - \frac{i}{T_t}\frac{X^b_a}{|T_t|}X^b_t\Psi_a + \frac{1}{|T_t|}\Psi_{aa} + \tilde V\Psi = 0.$$

Splitting with respect to $\hat\psi$ in view of the representation for Ψ and rearranging, we obtain

$$\tilde V = \frac{\hat V}{|T_t|} - \frac{i}{T_t\Psi^1}\left(\Psi^1_t - \frac{X^b_a X^b_t}{|T_t|}\Psi^1_a\right) - \frac{1}{|T_t|}\frac{\Psi^1_{aa}}{\Psi^1}, \tag{14}$$

$$i\varepsilon'\Psi^0_t - \frac{i}{T_t}X^b_a X^b_t \Psi^0_a + \Psi^0_{aa} + |T_t|\tilde V\Psi^0 = 0. \tag{15}$$

Let us introduce the function $\Omega = \hat\Psi^0/\hat\Psi^1$, i.e., $\Psi^0 = \Psi^1\hat\Omega$. The equation (15) is equivalent to the initial linear Schrödinger equation in terms of Ω. After the substitution of Ψ^1 by its expression from (13) into (14) and then additionally collecting coefficients of x, we derive the final expression for $\tilde V$. □

3 Equivalence group and normalization properties

There are several ways of computing the equivalence group for a given class of differential equations. Some of them are the direct or the infinitesimal method. At the same time, it is unnecessary to do the computation of the equivalence group $G^{\sim}$ of the class $\mathcal{F}$ using one of the above alternatives since we can derive the equivalence group $G^{\sim}$ of the class $\mathcal{F}$ from the knowledge of its equivalence groupoid $\mathcal{G}^{\sim}$.

Corollary V.1. *The (usual) equivalence group $G^{\sim}$ of the class $\mathcal{F}$ consists of point transformations in the space of independent and dependent variables and arbitrary element that are of the form* (5) *with* $\Omega = 0$.

Proof. Let $\mathcal{T}$ be a point transformation connecting two equations from the class $\mathcal{F}$. Then $\mathcal{T}$ is necessarily of the form (5a)–(5b), and potentials of the equations are related by (5c). Any transformation from the group $G^{\sim}$ generates a family of admissible transformations of the class $\mathcal{F}$ and hence it has the form (5). On the other hand, from the definition of equivalence group, the group $G^{\sim}$ contains only point transformations whose components corresponding to variables do not depend on the arbitrary element V. This condition is satisfied if and only if Ω is a common solution for all equations from the class $\mathcal{F}$. The only common solution is $\Omega = 0$. □

Remark V.1. In fact, the whole equivalence group $G^{\sim}$ is generated by the continuous family of transformations of the form (5), where $\Omega = 0$, $T_t > 0$ and $\det O = 1$, and two discrete transformations: the space reflection I_a for a fixed a ($\tilde{t} = t$, $\tilde{x}_a = -x_a$, $\tilde{x}_b = x_b$, $b \neq a$, $\tilde{\psi} = \psi$, $\tilde{V} = V$) and the Wigner time reflection I_t ($\tilde{t} = -t$, $\tilde{x} = x$, $\tilde{\psi} = \psi^*$, $\tilde{V} = V^*$).

Summing up, we state the following

Corollary V.2. *The class $\mathcal{F}$ is semi-normalized. More precisely, for each admissible transformation $(V, \tilde{V}, \mathcal{T})$ in the class $\mathcal{F}$, its transformational part $\mathcal{T}$ is the composition of a linear superposition transformation $\mathcal{T}_1$ of the initial equation with the potential V and the projection of an equivalence transformation $\mathcal{T}_2$ of the class $\mathcal{F}$ to the space variables with $\mathcal{T}_2 V = \tilde{V}$.*

Proof. Consider two fixed similar equations from the class $\mathcal{F}$ with potentials V and $\tilde{V}$ and let $\mathcal{T}$ be a point transformation connecting these equations. From Theorem (V.1), the transformation $\mathcal{T}$ is of the form (5a)–(5b), and the potentials V and $\tilde{V}$ are related by (5c). We define two point transformations. The first transformation $\mathcal{T}^1$ is the point transformation in the variable space with the components $\tilde{t} = t$, $\tilde{x} = x$, $\tilde{\psi} = \psi + \Omega$ with the same Ω as in $\mathcal{T}$. It does not change the potential V and, therefore, is a point symmetry transformation of linear superposition for the initial equation. The second transformation $\mathcal{T}^2$ is the point transformation in the extended space variables and the potential V that is of the form (5) with the same values of parameters as in $\mathcal{T}$ except $\Omega = 0$. Hence it is an equivalence transformation, which connects the equations with potentials V and $\tilde{V}$, $\tilde{V} = \mathcal{T}_2 V$. As a result, the transformation $\mathcal{T}$ coincides with the composition of $\mathcal{T}^1$ and the projection $\mathcal{T}^2|_{(t,x,\psi)}$ of $\mathcal{T}^2$, $\mathcal{T} = \mathcal{T}^2|_{(t,x,\psi)} \circ \mathcal{T}^1$. □

4 Conclusion

The equivalence groupoid of the class $\mathcal{F}$ of (1+2)-dimensional linear Schrödinger equations with complex potentials, which has an interesting algebraic structure, is exhaustively described in Theorem (V.1) using the direct method. The method is standard for the study of point transformations between differential equations. At the same time, it is not quite algorithmic, especially when considering the entire set of admissible transformations of a class of differential equations. The versions of the direct method for finding point symmetry transformations of a single differential equation or equivalence transformations of a class of differential equations are much easier to be realized. Moreover, the computations become trickier and more cumbersome if we increase the dimension of equations. New features also appear in the form of admissible transformations in comparison with the (1+1)-dimensional case. They are not exhausted by the formal extension of the set of space variables. Coupling of space variables leads, in particular, to involving rotations in the corresponding equivalence groupoid. In total, the above makes the entire consideration much more complicated than in (1+1)-dimensions.

Knowing the equivalence groupoid of the class $\mathcal{F}$, we construct the (usual) equivalence group of $\mathcal{F}$ in an easy way, roughly speaking, by singling out families of admissible transformations that are pointwise parameterized by the arbitrary element, which is the potential V for the class $\mathcal{F}$, and having the same transformational part. We relate the equivalence groupoid, the equivalence group and normalization properties of the class $\mathcal{F}$, and this relation is similar to that for the (1+1)-dimensional counterpart of the class $\mathcal{F}$, which is studied in [11]. We show that, roughly speaking, any point transformation connecting two fixed equations from the class $\mathcal{F}$ is the composition of a linear superposition symmetry transformation of the initial equation and an equivalence transformation of this class. In other words, the class $\mathcal{F}$ is semi-normalized in a quite specific way, which guarantees, in view of our experience with the (1+1)-dimensional counterpart of the class $\mathcal{F}$ [11], the effective use of the algebraic method for the exhaustive group classification of the class $\mathcal{F}$. Therefore, the results of the present paper can be considered as a first step in our future work on this classification.

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