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Predicting Global Atmospheric Carbon Dioxide Concentration Dynamics: A Time-Integrated Mathematical Modelling Approach

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Declaration

By signing this document, I, Jean de Dieu MUNEZERO, attest that the research project's content is entirely original work of mine, and that any prior work I or others have done has been duly acknowledged and cited.

Jean de Dieu MUNEZERO, February 2024.

Dedication

To my family and my supervisors ***Prof.Denis NDANGUZA; Dr.Isambi Sailon Imbalawata***.

Abstract

The rise in the concentration of carbon dioxide (CO_2) in the atmosphere is a serious problem that has an impact on the environment and the world's climate. Effective mitigation measures require accurate future CO_2 level predictions and an understanding of the processes causing this increase. Our thesis introduces an enhanced mathematical model of the average global concentration of atmospheric carbon dioxide, based on human emissions. We accomplish so by measuring the behavior of carbon dioxide released into and withdrawn from the atmosphere using a straightforward ordinary differential equation. We calculate best-fit curves to estimate the amount of anthropogenic CO_2 emissions in the atmosphere. In order to determine a model approximation, we estimate the model parameters using the least squares method. We evaluate the model's solution and talk about how fit the model is. The model shows that the constant rise in anthropogenic emissions is what is responsible for the boundary-less increase in atmospheric carbon dioxide concentration. The concentration of CO_2 in the atmosphere will steadily rise if CO_2 emission is not controlled, with increasingly dire consequences.

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1. Introduction

This century has seen a constant increase in the atmospheric content of carbon dioxide (Solomon et al. (2009)). According to scientific findings, the amount of carbon dioxide in the atmosphere has increased to its greatest point in the previous 800,000 years. According to Marques (2020), the concentration of carbon dioxide in the Earth's atmosphere reached 405 parts per million (ppm) in 2017, making it the warmest year on record according to the National Oceanic and Atmospheric Administration (NOAA). This level of concentration has not been reached in 800,000 years. The Scripps Institution of Oceanography states that before the Industrial Revolution, atmospheric carbon dioxide concentrations did not rise above 300 parts per million.

Over the ten years prior to 2010, The growth rate has approached 2.3 ppm. In the last 60 years, the yearly atmospheric CO_2 increase has occurred at a rate that is approximately 100 times higher than previous natural increases, such as those that occurred at the end of the last ice age (Lindsey (2018)). The ocean's acidity, where the potential hydrogen value (PHvalue) appears to be consistently decreasing, can be used to illustrate this growing quantity of carbon dioxide (Barton et al. (2012)). The increase in atmospheric CO_2 concentration is influenced by both population growth and economic expansion. This is caused primarily by an increase in CO_2 emissions brought on by human activity. The primary sources of carbon dioxide in the atmosphere are both human and natural.Decomposition, ocean discharge, and respiration are examples of natural sources of carbon dioxide; nevertheless, burning fossil fuels and deforestation have increased the concentration of carbon dioxide in the atmosphere to a high level, which causes ongoing global warming. The only way to attain extended stabilization of Concentrations of carbon dioxide is to replace fossil fuels with alternatives that will reduce and eventually eliminate carbon dioxide emissions (Adolphson et al. (1992)).

1.1 Motivation and Background

The National Oceanic and Atmospheric Administration (NOAA) reported in 2018 that the amount of carbon dioxide in the atmosphere of the planet has hit 405.0 parts per million and that current levels of CO2 are higher than they have ever been since 800,000 years ago. It was discovered that human consumption of fossil fuels for energy is the primary source of the increase in atmospheric CO_2. Carbon from the atmosphere was removed by plants through photosynthesis over many millions of years, and fossil fuels like coal and oil are merely putting that carbon back into the atmosphere in a matter of centuries. According to the publication's conclusion, atmospheric carbon dioxide concentrations are expected to surpass 900 parts per million by the end of this century if global energy demand keeps rising and is largely satisfied by fossil fuels (Sims (2004)). One of the main causes of global warming is the rise in CO_2 worldwide. According to research, the most significant gas for regulating Earth's temperature is atmospheric CO_2. The earth gets warmer as atmospheric CO_2 rises, and rising surface temperatures have a greater influence on climate change. The ocean and land plants attempt to lower the amount of CO_2 in the atmosphere through the carbon cycle. Plants and the ocean absorb more than half of the anthropogenic CO_2 in the atmosphere. The majority of the excess carbon dioxide will eventually be absorbed by land and sea, but up to 20% of it might stay in the atmosphere for eons. Every carbon reservoir is impacted by these modifications to the carbon cycle. Land plants develop more because of the planet's warmth caused by the high concentration of CO_2 in the atmosphere. Marine life is at risk due to the ocean's increased acidity caused by the high concentration of CO_2 (Keller et al. (2018)). The anthropogenic CO_2 emission triggers a range of responses and feedback mechanisms in the global carbon cycle. The amount of anthropogenic CO_2 that the ocean absorbs is controlled by the carbonate chemistry and ocean circulation. Oceans absorb some amount of carbon due to the long-term increase in atmospheric CO_2 concentration. Accordingly, as CO_2 concentration rises, the ocean's capacity to absorb anthropogenic CO_2 decreases as a result of the carbonate system's decreased buffer capacity and the pace at which surface water and deep water mix, which both restrict the ocean's ability to absorb CO_2 (Farquhar (2001)). Since CO_2 is a greenhouse gas, there is a significant correlation between increases in atmospheric concentrations and temperature variations. Human activity-induced CO_2 emissions have been the primary source of the ocean's and atmosphere's noted warming since the mid-20th century (Showstack (2013)). The challenge in their research (Adolphson et al. (1992)) was figuring out how much CO_2 might be emitted annually without going above the allowed limits of greenhouse gas concentrations. They attempted to study CO_2 emissions allowed for defined atmospheric concentration trajectories. The only method to deal with this issue is to employ climate models that incorporate biogeochemical cycle representations, specifically the carbon cycle. With ideal sub-models and parameterizations, the model considers the functions of the marine and terrestrial biospheres as well as interactions with the ocean. A simple climate model demonstrates that large reductions in CO_2 emissions in the atmosphere, leading to their eventual disappearing, are the only ways to accomplish the long-term stabilization of CO_2 concentrations. A basic model of the atmospheric CO_2 concentration was presented by Spencer (2019). Since CO_2 is primarily anthropogenic, the goal was not to demonstrate that it originates naturally and is the reason of the recent rise in the concentration of CO_2. Rather, the author attempted to utilize a straightforward model to explain the increase in CO_2 concentration and to determine whether any meaningful conclusions could be drawn from it. According to the model, the rate at which atmospheric CO_2 is taken from the atmosphere is proportionate to the amount of CO_2 added to the atmosphere over the point of equilibrium of atmospheric CO_2 concentration. The final model fits the Mauna Loa CO_2 observations data under the assumptions that the CO_2 emission from 2018 will remain constant and that the natural equilibrium CO_2 concentration will be 295 ppm. By 2020, the atmospheric CO_2 concentration is expected to settle at a value of nearly 500 parts per million, according to the numerical solution. These studies attempted to show the significance of atmospheric CO_2 and its impact on the planet, as well as the cause of the rising CO_2 concentration and how the climate system behaves in response to it. However, they were unable to clearly identify the perfect ideal conditions needed to forecast the quantity of of CO_2 in the Earth's atmosphere. Predicting the amount of atmospheric CO_2 remains a challenge (Adolphson et al. (1992)).

1.2 Problem Statement

The rise in the atmospheric concentration of carbon dioxide CO_2 is a critical issue that affects the global climate and environment. Understanding the factors that contribute to this increase and accurately predicting future CO_2 levels is essential for developing effective mitigation strategies. While various experimental methods have been used to study CO_2 levels, mathematical modeling provides a powerful tool for analyzing the underlying dynamics of CO_2 concentrations in the atmosphere. This research is an extension of my work done before and submitted at AIMS Rwanda as a final project in partial fulfilment of the requirements of a Master of Science in Mathematical Sciences (de Dieu MUNEZERO (2019)). Like many previous research on mathematical modeling of atmospheric $CO2$ concentration has yielded valuable insights into the factors influencing $CO2$ levels and their temporal and spatial variations. Our previous work, focused on developing and analyzing a mathematical model to simulate atmospheric $CO2$ concentration based on observational data. This initial study utilized a single dataset of atmospheric $CO2$ measurements to estimate model parameters and investigate the system's behavior. But there is some gap that needed to be addressed in this extension research. The significant limitation of the previous work was the reliance on a single dataset of atmospheric $CO2$ measurements, which may not fully capture the complexity of $CO2$ dynamics. Additionally, the model did not incorporate information on $CO2$ emissions, which are a key driver of atmospheric $CO2$ concentration. We have also introduce oceanic factor which did not presented in previous model, this has a significant role in processes, such as carbon uptake and release in modulating atmospheric $CO2$ levels and their implications for climate change. For these reasons the previous study failed to predict the future value of atmospheric CO_2 concentration. This extended project holds significant implications for climate science and policy by advancing our understanding of the complex interactions driving atmospheric $CO2$ dynamics. By incorporating multiple datasets, emissions data, and oceanic processes into the model, we can improve our ability to predict future changes in atmospheric $CO2$ concentration and assess the effectiveness of mitigation strategies. Furthermore, the exploration of bifurcation behavior and model-data fitting will provide valuable insights into the stability and resilience of the climate system in response to perturbations. In 2020 John P.O'Connor from his research developed a semi-empirical relationship of carbon dioxide emissions with atmospheric CO_2 concentrations. Data from CO_2 emissions due to fossil fuel consumption and land use change were used for the analysis, and it was based on CO_2 concentration calculation using fossil-fuel combustion/cement production emissions with a single, fixed fossil-fuel combustion airborne fraction (AF). The analysis of the data shows that AF has remained constant. However, it may not be assumed that a single value for AF would properly reflect the reported variations in CO_2 concentrations during the industrial period given the wide range of variables, including emission and sink processes, influencing the climate. The Bern Simple Climate Model (BernSCM) based on the emission of CO_2, the flux to the ocean, the land biosphere carbon stock and the time. In the equation the present atmospheric CO_2 was ignored and the CO_2 remove rate by biosphere is equivalent to the shift in carbon stored on land which is determined by the ratio of assimilated terrestrial carbon decay to net primary production does not depend on the current atmospheric CO_2. Trevisan and Luz (2008) propose a mathematical model, based in a modified version of the Lotka-Volterra prey-predator equations, to predict the increasing in CO_2 atmospheric concentration the absorption of CO_2 is proportional to the relative photosynthesis rate which is a temperature dependence. The accurate prediction and understanding of global atmospheric carbon dioxide CO_2 concentration are crucial for addressing climate change. However, existing mathematical models have some limitations such considering natural remove as constant and often fail to capture the complexities of CO_2 dynamics, leading to uncertainties in predictions and hindering effective climate policy formulation and mitigation strategies.

Therefore, the aim for this study is to develop an improved robust mathematical model that can accurately predict atmospheric CO_2 concentrations. Taking into account various factors such as anthropogenic activities, and natural carbon cycles, with CO_2 remove rate which is a function of time with three constant parameters. And this remove rate will be variating between two parameters.

1.3 General Objectives

My thesis's primary goal is to create a suitable model that accurately represents the dynamics of global atmospheric $CO2$ concentration, considering the removal rate as a function of time, and to investigate the interactions between atmospheric $CO2$ levels, emissions, and oceanic processes.

1.4 Specific Objectives

- To Formulate an Enhanced Ordinary Differential Equation (ODE) Model describing the dynamic of atmospheric C02 concentration .

- To Estimate the model parameters using multiple data sources.
- To investigate bifurcation behavior and critical thresholds.
- To Find numerical solutions of the enhanced model.
- To Fit the enhanced model to global CO_2 concentration observed data.

1.4.1 Outlines of the project. There are four parts in this project. The first chapter provides an overview and the background history of CO_2. We also discuss the issue statement that needs to be resolved, outline the primary goals of this study, and outline the approaches that will be taken. The second part reviews the literature and includes information on the origin and cycle of carbon, the impact of CO_2, and an explanation of models developed by other scientists. We develop a mathematical model, estimate parameters, analyze the model, and identify the result in chapter three. Chapter Four presents our recommendation and conclusion, and offers some suggestions based on the model's output.

1.5 Methodology

In order to accomplish the research goals, we require certain methodological elements: Through the use of the UR and AIMS libraries as well as the internet, we study papers from earlier studies on carbon dioxide and mathematical modeling. This aids in the model's solving and analysis as well as the assumptions we make about it. We use a flowchart diagram to construct a model. To confirm the model's validity or suitability, we use secondary data gathered from the Global Carbon Project to determine values for parameters that enable The data-fitting model. We use the least squares method to understand the estimated parameters. By calculating the equilibrium point, we analyze the model and talk about how the numerical solution behaves. We use Matlab software and Runge-Kutta 4th order to solve the problem numerically. The model's numerical solutions will be displayed graphically so that the predictions may be seen. We use the coefficient of determination approach (R^2) in MATLAB software to fit data and evaluate the model.

2. Literature review

The ever-increasing concentrations CO_2 in the Earth's atmosphere represent a critical challenge with far-reaching implications for our planet's climate and ecosystems. As we grapple with the pressing issue of climate change, understanding the dynamics of global atmospheric CO_2 concentration has become paramount. Mathematical modelling emerges as a powerful tool in this endeavor, offering the ability to simulate and predict the complex interplay of various factors influencing CO_2 levels.

2.1 Overview of Atmospheric Carbon Dioxide Concentration

The chemical molecule CO_2 is made up of one carbon atom and two oxygen atoms. One of the gases in our atmosphere is CO_2. It is among the most vital gases that support life as we know it on the Earth. It is found in the atmosphere naturally as a tracing gas,more than 410 parts per million (ppm) in terms of volume, with a current concentration of 0.04 % Brown (2003). The atmosphere contains CO_2 due to both human activity and natural processes. In order to maintain equilibrium in the atmosphere, the

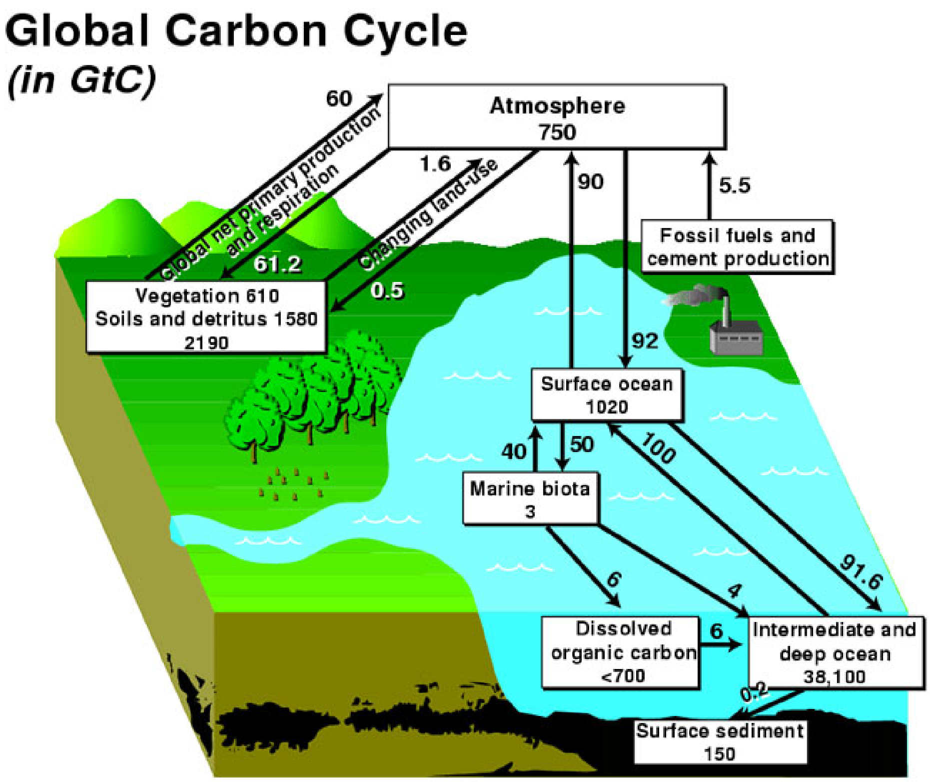


Figure 2.1: Diagram illustrating the storage of carbon and the exchanges amount of carbon in the carbon cycle. Note:GtC= gigatons of carbon(source: EARTH 530, The Critical Zone).

natural cycle permits the release and removal of CO_2.

2.2 Impact of the Atmosphere CO_2 Elevation

The high levels of CO_2 emissions in the atmosphere nowadays have a substantial impact on the environment and change a few natural balances. The two primary effects of such unprecedented conditions—the rise in global temperatures and the resulting drop in ocean pH, or ocean acidification—will be briefly discussed in this thesis.

2.2.1 Global warming. One type of greenhouse gas is carbon dioxide (GHG). Any gas in the atmosphere that, depending on how far away it is from the Sun, both absorbs and releases radiant energy radiation in the thermal infrared is considered a greenhouse gas (GHG). This maintains the atmosphere of the planet is warmer than it would otherwise. The mean surface temperature of Earth is actually 59 degrees Fahrenheit; without CO_2 and other greenhouse gases in the atmosphere, this temperature would be zero Hertzberg et al. (2017). Water vapor, CO_2, nitrous oxide (N2O), methane (CH4) and ozone are the most prevalent greenhouse gas in the environment. The primary greenhouse gas (GHG) released by human activity is CO_2, both in terms of amount released and overall impact on global warming. Because of this, the word CO_2 is occasionally used to simply denote all greenhouse gases. The constant increase in the amount of $CO2$ in the atmosphere is the primary cause of the rise in global temperatures, sometimes known as global warming. The important relationship between temperature and the amount of $CO2$ in the atmosphere that was observed over the cycles of glaciers of the previous few thousands years is one of the most impressive aspects of paleoclimate observations. The temperature rises with an increase in CO_2 concentration. Temperature decreases with decreasing CO_2 concentration Long (1991) & Adolphson et al. (1992).

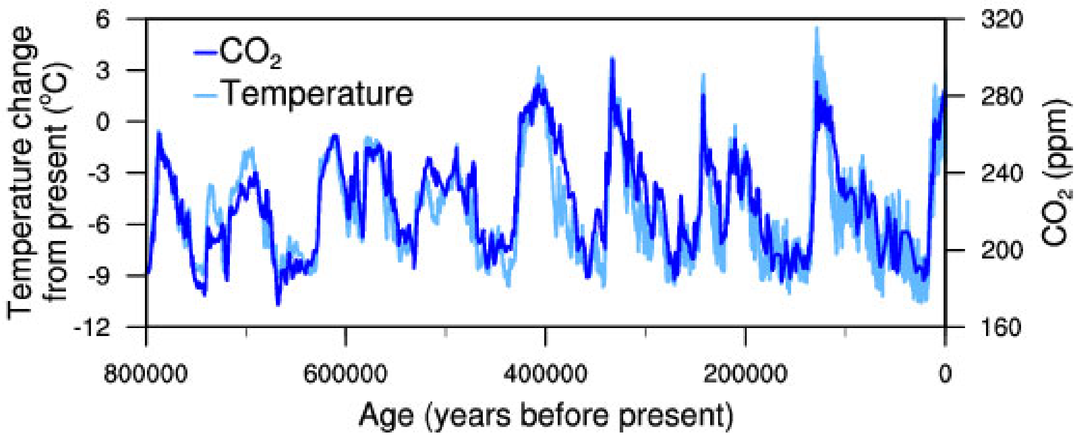


Figure 2.2: Carbon dioxide (dark blue) and Changes in temperature (light blue) are observed from Antarctic's EPICA Dome C ice core ((Wadswarth, 2019)).

Between the mid-1960s and 1980, the world temperature increased by $0.2°C$, resulting in a temperature of $0.4°C$ over the previous century. The earth's average global temperature has been trending upward since the industrial revolution, particularly as CO_2 concentration has been steadily growing. Svante Arrhenius conducted research in 1895 on the potential consequences of increasing carbon dioxide levels about the temperature of Earth. Since then, researchers have improved our knowledge of the greenhouse effect and the potential effects that increased carbon emissions may have on the planet's temperature. The rise in global temperature is the principal effect of increasing atmospheric CO_2 (Hansen et al. (1981) & Adolphson et al. (1992)).

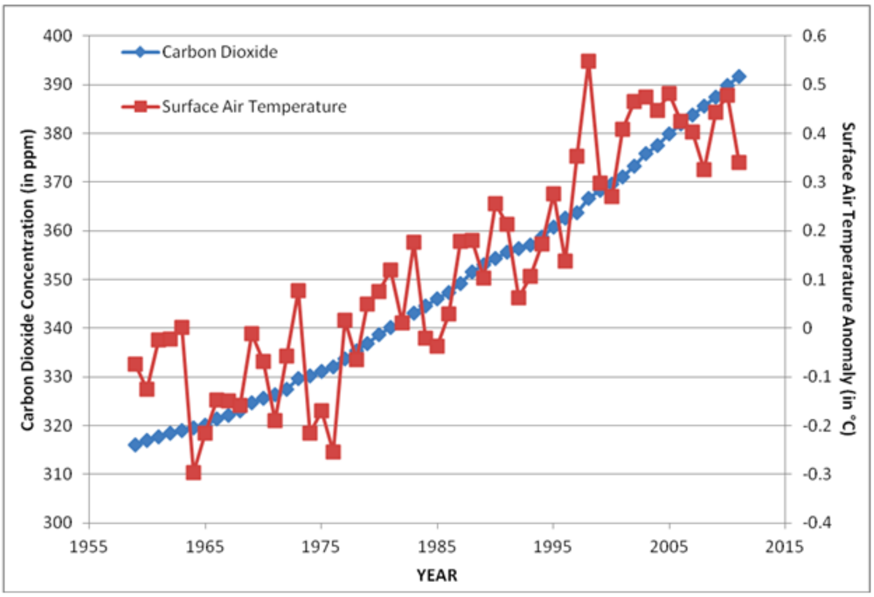


Figure 2.3: Over time, temperature changes (red) and carbon dioxide levels (blue),(source: Alfsen and Skodvin (1998)).

2.3 Overview on Mathematical Modeling

Models explain our perceptions of how the world functions. We attempt to put those views into the mathematical language of mathematical modeling. A mathematical relation that depicts a real-world scenario is called a mathematical model. The process of writing a a mathematical concept applied to a real-world problem is known as mathematical modeling. Next, we resolve the mathematical puzzle and analyze its outcome in relation to the practical issue. The validity of the solution in relation to the actual problem is then examined. Formulation, solution, interpretation, and validation are the key phases in mathematical modeling. Mathematical modeling is crucial because it allows us to forecast

and comprehend occurrences that are naturally experienced. It can also assist us in controlling the environment we live in. A number of goals can be attained with the use of mathematical modeling, including the advancement of scientific knowledge, testing the effects of system modifications, and supporting managers and planners in their tactical and strategic decision-making.

2.4 Prey-Predator Analysis of Atmospheric CO_2 Dynamics

Prey-Predator equations, sometimes referred to as Lotka–Volterra equations, are a set of two nonlinear first-order differential equations that are frequently used to characterize the dynamics of biological systems in which two species interact; the predator and the prey are the two species. This model uses the following pair of equations to describe how the population changes over time:

$$\frac{dx}{dt} = \alpha x - \beta xy, \tag{2.4.1}$$

$$\frac{dy}{dt} = \delta xy - \gamma y, \tag{2.4.2}$$

wherein

Let x be the population of prey.

y stands for the population of predators,

$\frac{dy}{dt}$ and $\frac{dx}{dt}$ denote the two populations' instantaneous growth rates, respectively.

Time is represented by the symbol t, and the positive real parameters α, β, γ, and δ describe how the two species interact.

In this instance, y represents the surface of vegetative biomass, and x represents the CO_2 concentration as prey. In this case α, β, γ, and δ represent, in turn, the pace at which the concentration of CO_2 in the atmosphere is increased, the rate at which vegetation absorbs CO_2, and the ratio of the biomass of plants to its death and to its birth.

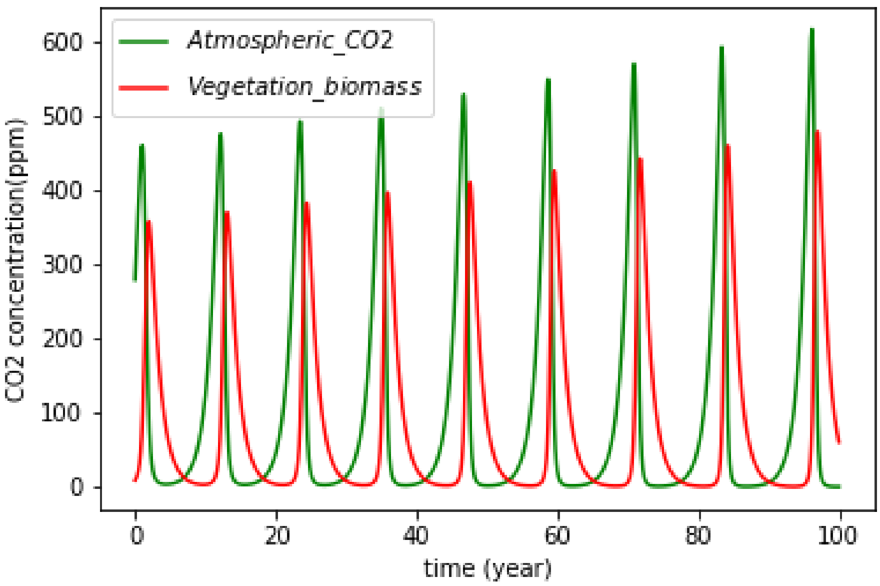


Figure 2.4: Carbon dioxide concentration change (green) and vegetation biomass surface change (red) are shown on the graph.

2.4.1 Solution and Discussion. As the biomass of the vegetation increases in figure 2.4, more CO_2 is absorbed from the atmosphere through photosynthesis. As a result, the concentration of CO_2 in the atmosphere decreased. There will be a currency of CO_2 fertilization when the atmospheric CO_2 decreases and and the amount of plant life is dense, which cause the vegetation biomass surface to drop. We discover that the growth rate of vegetation increases as atmospheric CO_2 increases, and that the increase in vegetation lowers the atmospheric CO_2 concentration. This contributes to illustrating how vegetation plays a major part in imposing the large increase in atmospheric CO_2 concentration.

2.4.2 Study the Stability around critical points. By letting x'= 0 and y' = 0, the critical Point was computed, and We obtain the ones that follow two points:: (0,0) and $(\frac{\gamma}{\delta}, \frac{\alpha}{\beta})$

Jacobian matrix was computed as follow:

$$J = \begin{pmatrix} \alpha - \beta y & -\beta x \\ \delta y & \delta x - \gamma \end{pmatrix}.$$

Around (0,0):

$$J(0,0) = \begin{pmatrix} \alpha & 0 \\ 0 & -\gamma \end{pmatrix}.$$

We obtain $\lambda_1 = \alpha$ and $\lambda_2 = -\gamma$ by determining the eigenvalue (λ) of this matrix.

It demonstrates that if the surface biomass of vegetation and atmospheric CO_2 are both at 0, they will stay that way forever. Therefore, the fixed point at the origin is a saddle point. Nevertheless, we show that it is challenging for the model to calculate both the vegetation biomass surface and the atmospheric CO_2 because the origin's fixed point is saddle point, unstable.

Around $(\frac{\gamma}{\delta}, \frac{\alpha}{\beta})$:

$$J\left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right) = \begin{pmatrix} 0 & \frac{-\beta\gamma}{\delta} \\ \frac{\alpha\delta}{\beta} & 0 \end{pmatrix}.$$

$\lambda_{1,2} = \pm i\sqrt{\alpha\gamma}$ are the eigenvalues.

The center of $\frac{\gamma}{\delta}, \frac{\alpha}{\beta}$ is stable and appealing as both eigenvalues are totally imaginary. This solution shows a fixed point where the temporal, non-zero quantity of both atmospheric CO_2 concentration is sustained. Additionally,the parameters' values α, β, γ, and δ determine the levels or quantities of vegetation and atmospheric CO_2 concentration at which this equilibrium is reached.

Numerous elements that influence the variation in atmospheric CO_2 concentration have been covered. Accelerating the rate of photosynthesis is one of the significant effects of rising CO_2 concentrations. A higher rate of photosynthesis causes plants to absorb more CO_2, grow more quickly, and increase their rate of reproduction, which further increases CO_2 consumption. In this study, we introduce a model that accounts for the interaction between plants and atmospheric carbon to forecast an elevation of the atmospheric concentration of CO_2. In essence, it uses a modified predator-prey model to calculate a numerical biomass parameter and a potential time growth of Ca. This is done by accounting for the rise in photosynthetic rates, which is the outcome of the rise in the atmospheric concentration of (CO_2), Ca. Any living thing that uses photosynthesis is included in this characteristic, regardless of whether it is a vegetable or not.

2.5 Previous Modeling Approaches

2.5.1 Model of predator-prey modified for plant-atmospheric carbon dynamic. According to this model, which was modified to address the issue of plants and CO_2 interaction, the plants were viewed as the predator and the CO_2 as the inorganic prey (not reproducing) (Trevisan and Luz (2008)). Since the CO_2 molecule cannot replicate on its own, the growth of its concentration, indicated by Ca (in parts per million), is solely dependent on the emissions of CO_2 in the atmosphere, indicated Q(t), the absorption (photosynthesis), which is thought to be proportionate to the surface of vegetative biomass, indicated by P(t), as well as with relation to the relative rate of photosynthesis A(Ca, T), where T is the temperature. This is how the two of equations is displayed (where K is a fixed value.):

$$\frac{dCa}{dt} = -A(Ca, T) * P + kQ(t), \tag{2.5.1}$$

$$\frac{dP}{dt} = -e * P + f * P * A(Ca, T). \tag{2.5.2}$$

When the recorded data and the model fit are compared, it is discovered that the results are almost same. The model parameters are calculated to produce results that are more in line with the CO_2 concentration data for the years 1800–2000. After fitting a quick change of the concentration over time between 1990 and 2005, they found that it evolved essentially linearly on a short time scale, taking into account absorption by the ocean of about 30% of anthropogenic emissions, continuous emissions (7 GtC/y), rate of vegetative death of 0.78, and f = 0.015. Once more, they see from the model that the parameters required to match a long historical period may differ from those required to fit short range data. The way the absorptions are taken into account is the primary distinction between this model and the Intergovernmental Panel on Climate Change. when there is less than a unity vegetation mortality rate, The apparatus is capable of still maintain A consistent level of concentration even with continuous released emissions.

2.5.2 The Bern Simple Climate Model. The Bern Simple Climate Model (BernSCM) is a well-known simplified climate model that is often used for educational and research purposes. It's designed to provide a basic understanding of the interactions between greenhouse gas emissions, atmospheric concentrations, and global temperature changes. The model is particularly useful for demonstrating key concepts related to the climate system, carbon cycle, and climate sensitivity Strassmann and Joos (2018). BernSCM offers a basic understanding of carbon dioxide's role in climate change However, like any model, the BernSCM has its limitations. It uses a simple linear relationship between emissions and atmospheric carbon dioxide concentrations. This oversimplification neglects important nonlinearities and feedback mechanisms present in the real carbon cycle, such as the varying uptake rates by different carbon sinks. Assumptions about Equilibrium Concentrations, BernSCM assumes a constant equilibrium concentration of $C02$ and does not account for changes in natural carbon reservoirs or their responses to climate change. Further more BernSCM is sensitive to parameter values. The accuracy of the model's predictions is highly sensitive to the values assigned to its parameters, such as the climate sensitivity parameter λ and the radiative forcing coefficient α. Small changes in these values can lead to significant differences in model outcomes.

2.5.3 Dynamical Analysis of Carbon Concentration Model Due to the Interaction with Biomass Production Based on Predator-Prey. The model was created to characterize the dynamical behavior of the climatic vector of CO_2 and predict its stability. It is based on the interaction between carbon consumption owing to biomass production. To demonstrate the dynamic relationship between the carbon concentration C and the growth rate of the biomass N, they developed a straightforward mathematical model based on a variation of the Lotka-Volterra predator-prey model (Hafizan et al. (2019)). The model was fitted using experimental data on biomass abundance.

$$\frac{dC}{dt} = -vN + aC,$$
$$\frac{dN}{dt} = -eC + fNA(L,T).$$

They modified the predator-prey model for the CO2 interaction issue in their model. They view the biomass density as the predator and the carbon content as the prey. The adduction rate a (mg/days), which is dependent on the carbon concentration (mg/l), is denoted by C. The density of generated phytoplankton N (ppm) is inversely correlated with the vegetative biomass distribution, or v. The in vitro treatment parameter A(t) includes variables like light density L and temperature T (T0 is room temperature):

$$A(L,T) = \frac{bLT}{126 + \dfrac{T_0}{T}}.$$

The model is stable as a consequence of the dynamic analysis and can be used to predict the results of experimental tests. Instead of 25 days, this model needs longer time to validate the data, which is currently being collected. The model might require a larger data set to test its robustness before it can be used as a provision for the atmospheric CO_2 gas emission. While the model offers useful information, it also has several drawbacks, such as a restricted depiction of human activities and other factors that greatly affect variations in CO_2 concentration, such as changes in land use, deforestation, industrial emissions, economic factors, etc.

3. Model Development and Analysis

3.1 Mathematical Formulation of the Model

The high concentration of CO_2 in the atmosphere has become a major issue, and it is difficult to predict how much CO_2 will be in the atmosphere in the future. We take into consideration the simple mathematical model that may be applied to this topic to show how to utilize an ordinary differential equation to control the growth of a given population or product in the presence of some causative. We analyze the change in atmospheric CO_2 concentration as a function of increasing human emissions due to population growth and industrialization, which both seem to increase exponentially, in order to develop the model. Population of the planet accurately reflects atmospheric CO_2. Population growth and human emission CO_2 have been growing exponentially together (Hofmann (2011)). The amount of CO_2 in the atmosphere affects the CO_2 change as well. Since the vegetation absorb the amount of CO_2 according to the present atmospheric CO_2 concentration and ocean uptake the amount of CO_2 according to the emission amount of CO_2 in the atmosphere. We present the model as follow.

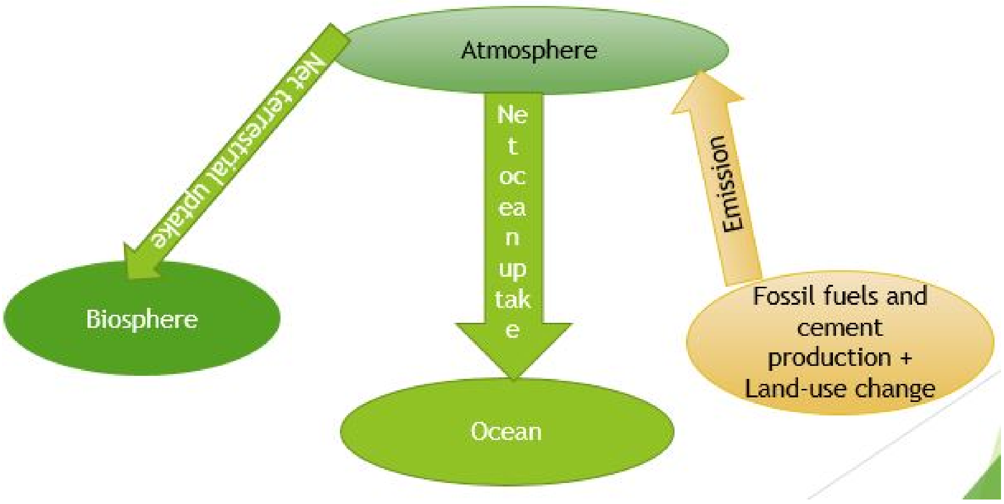


Figure 3.1: Flowchart followed to write the equation of the model

$$
\begin{aligned}
&\frac{dC_a}{dt} = (k-f)e - \beta(t)C_a, \\
&\frac{de}{dt} = ae,
\end{aligned}
\tag{3.1.1}
$$

where

$$\beta(t) = \beta_0 + \frac{\beta_1 - \beta_0}{1 + exp(-qt)}.$$

Ca: represent the atmospheric CO_2 concentration, e: represent the total global emission, k: is representing the continuous emission coefficient e, f: is presenting the sinking rate of CO_2 release into ocean, a: is the rate at which emissions are increasing, $\beta(t)$: is representing the speed at which CO_2 leaves the atmosphere through natural carbon cycle (biomass surface and terrestrial). β_0 is the rate before the CO_2 removal, The rates following CO_2 elimination are denoted by β_1, the decreasing rates by q, and the time by t. This function $\beta(t)$ decreases from β_0 to β_1. According to the work done (ARTELS (2015)) trying to estimate the amount of CO2 absorbed by the ocean. The ocean is able to absorb around 30 percent of all the carbon released as CO_2 from all human activities. In this work the value of coefficient f is taken to be 0.30 (Raghuvanshi et al. (2006)).

3.2 Data

We have got secondary annual data on the concentration of carbon dioxide in atmosphere from 1000 up to 2019 (1019 observations), collected from our world in data (Global Carbon Project). It consists of the average yearly concentration of carbon dioxide in the atmosphere. We have also got a global annual $C0_2$ emission dataset containing the historical emission CO_2 by fossil fuel and cement production + land use change from 1850 up to 2020 (170 observations) (Ritchie and Roser (2020)) . If we try to look into the CO_2 atmospheric concentration historical data,We can observe that there is a very slight annual variation in concentration within the range of 1000 to 1940. Around the 1940s, the atmospheric CO_2 concentration began to drastically fluctuate. In this research, we develop a model to fit data from the 1940s to 2015.

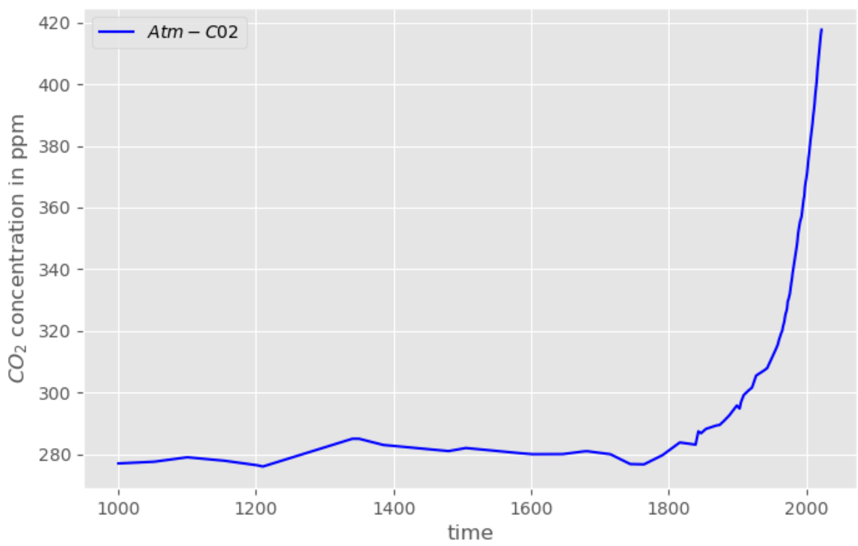


Figure 3.2: Observations of atmospheric CO_2 concentrations data starting from 100 to 2020 (Source:Compiled by Earth Policy Institute, with long term historical data from Worldwatch Institute)

From figure 3.3 & 3.2 the increasing rate of atmospheric CO_2 is very high in the period, from 1940 to 2019 compare to the increasing rate in the past which seems to be almost constant.

In figure 3.4 from 1850 up to 1940 the growth in emissions was relatively slow then from 1940 the became high. From 3.5 the emission increasing continuously. We see that prior to the Industrial Revolution, emissions were very low. Growth in emissions was still relatively slow until the mid-20th century. In 1950 the world emitted 6 billion tonnes of CO_2. By 1990 this had almost quadrupled, reaching more than 22 billion tonnes. Emissions have continued to grow rapidly; we now emit over 34 billion tonnes each year. In addition we can see that from figure 3.4 the CO2 emissions drop sharply at the end trying

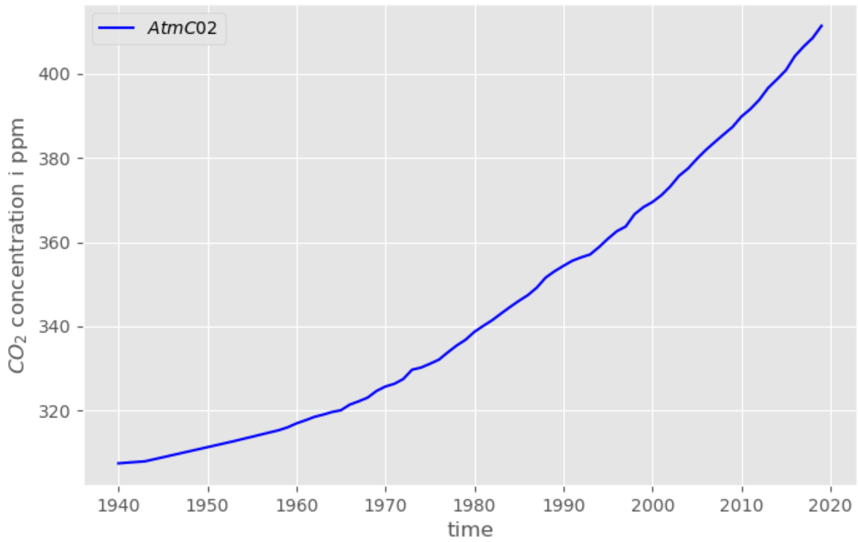


Figure 3.3: Observations of atmospheric CO_2 concentrations data, starting from 1940 to 2019, (Source: Compiled by Earth Policy Institute, with long term historical data from Worldwatch Institute)

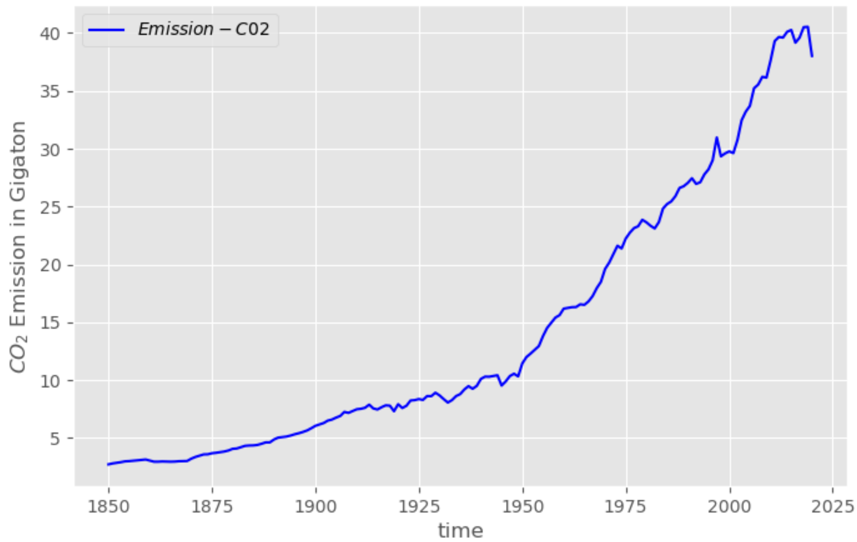


Figure 3.4: Global CO_2 emissions into the environment from fossil fuel consumption + land-use changes starting from 1850 to 2022 (Source: Our word in data, Global carbon project), https://www.globalcarbonproject.org/

to zoom in by plotting the small range approaching to the end we can see that the emission dropped in 2020 this may caused by COVID-19 see figure 3.6.

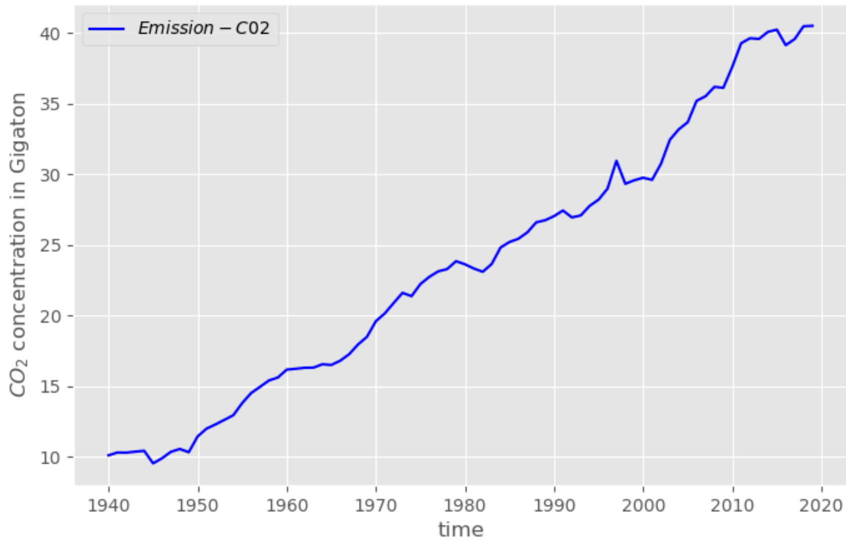


Figure 3.5: Global CO_2 emissions into the environment from fossil fuel consumption + land-use change, starting form 1940 to 2019 (Source: Our word in data, Global carbon project)

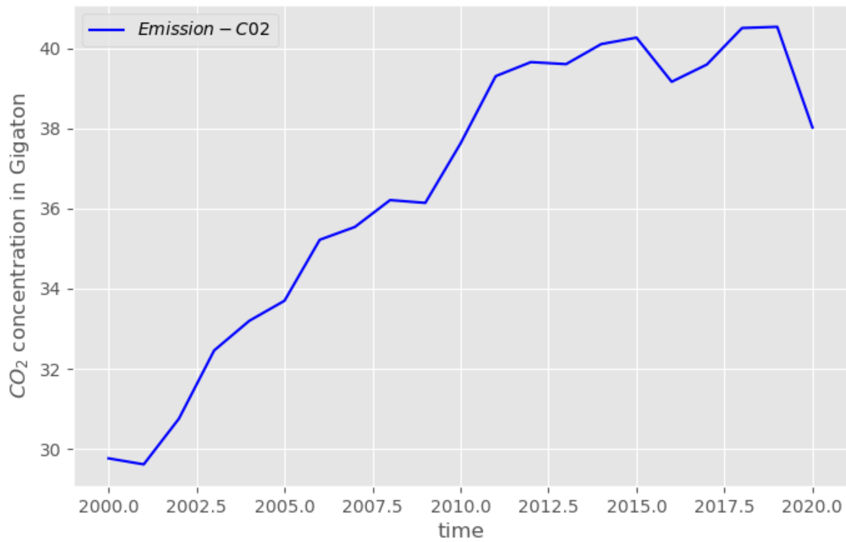


Figure 3.6: Global CO_2 emissions into the environment from fossil fuel consumption + land-use change, starting form 2000 to 2020 (Source: Our word in data, Global carbon project)

3.3 Parameters Estimation

In order for the model to fit the data, we must estimate the parameter value in order to find the expected results. The Least Squares approach was employed in this part to estimate the parameters. Identifying the estimators that reduce the sum square error.The difference between the expected response value and the dependant value, expressed as a sum of squares. Examine the nonlinear model with n observations in matrix form:

$$Y = f(X, \beta) + \epsilon$$

The explanatory variables are represented by a $n \times m$ matrix, denoted by X. Model parameters are represented by the $m \times 1$ vector β. A nx1 vector of errors is ϵ. and A $n \times 1$ vector Y represent response or dependant variable. Therefore, the value that minimizes the objective function is the estimator of β.

$$Q = \sum_{i=1}^{n} \Big(y_i - f(X_{ij,\beta_j}) \Big)^2 \tag{3.3.1}$$

Finding the value of $\hat{\beta}$ requires solving the following equation for $\hat{\beta}$

$$\frac{\partial Q}{\partial \beta}|_{\beta=\hat{\beta}} = 0. \tag{3.3.2}$$

Using the atmospheric CO_2 concentration and CO_2 emission from fossil fuel and cement production plus land use change historical data starting from 1940 up to 2015 the parameters β_0, β_1, a, k and q were estimated. We use python as computation tool, the R square was obtained to be $R^2 = 0.9960$. In this case, an R-squared value of 0.9960 means that 99.60% of the variability in the dependent variable is accounted for by the independent variables included in the model. This implies that the model provides a very good fit to the data, as it explains a high percentage of the observed variation.

$$R^2 = 1 - \frac{RSS}{TSS},$$

where RSS is sum of squares of residuals and TSS is total sum of squares. In the context of estimating parameters in a dynamical system, the value of R-square, denoted as R^2, represents the coefficient of determination. R^2 is a statistical measure that indicates the proportion of the variance in the dependent variable that can be explained by the independent variables in a regression model. A value of $R^2 =$ 0.9960 implies that 99.60% of the variability in the observed data can be explained by the estimated parameters in the model. In other words, the model is able to capture a large portion of the underlying relationships between the independent variables and the dependent variable. Having a high R^2 value suggests that the estimated parameters provide a good fit to the data and that the model is likely to be reliable in predicting the behavior of the dynamical system.Using the atmospheric CO_2 concentration and CO_2 emission from fossil fuel and cement production plus land use change historical data starting from 1940 up to 2019 the parameters β_0, β_1, a, k and q were estimated. We use python as computation tool, the R square was obtained to be $R^2 = 0.9960$. In this case, an R-squared value of 0.9960 means that 99.60% of the variability in the dependent variable is accounted for by the independent variables included in the model. This implies that the model provides a very good fit to the data, as it explains a high percentage of the observed variation. 3.1.

To estimate the parameters, the initial values of e and Ca are considered to be 307.41 and 10.11 respectively, since from the dataset the value of Ca and e in 1940 was 307.41 and 10.11 respectively.

3.4 System stability analysis

In this section, we examine the model's behavior to determine if it is stable or unstable.

Parameter	Initial value	Estimated value
K	0.95	0.3758
β_0	0.9	0.9000
β_1	0.0019	0.00103
q	0.5167	0.5167
a	0.005	0.0176.

Table 3.1: Estimated parameter values

3.4.1 Points of equilibrium. The critical points of the model was computed in the following way, we let;

$$\frac{dCa}{dt} = 0,$$
$$\frac{de}{dt} = 0.$$

Now following point was obtained; (0,0).

Jacobian matrix :

Conducting stability analysis involves analyzing the eigenvalues of the system's Jacobian matrix, which provide insights into the behavior of our model around its equilibrium point. The eigenvalues will tell us whether the system is stable, indicating that it will converge to a steady state, or unstable, suggesting that the system will exhibit oscillations or diverge over time.

$$J(ca, e) = \begin{pmatrix} -\beta(t) & k - f \\ 0 & a \end{pmatrix},$$

Determining (λ),the matrix J(ca,e)'eigenvalues.

$$det \begin{pmatrix} -\beta(t) - \lambda & k \\ 0 & a - \lambda \end{pmatrix} = 0.$$

$$(-\beta(t) - \lambda)(a - \lambda) = 0$$

At this point, we have two eigenvalues, $\lambda_1 = a$, where a is constant value, and λ_2. $\lambda_2 = -\beta(t)$ is a time-dependent function. Prior to discussing λ_2, we must first study the $\beta(t)$.

3.4.2 Studying the rate of removal $\beta(t)$ over time.. We analyze several values of q, both positive and negative, in order to discuss $\beta(t)$. Beginning with the case where q is positive.

The CO_2 rate removal is decreasing over time in figure 3.7. Ocean warming and deforestation are two factors that contribute to the declining rate of atmospheric CO_2 elimination. Deforestation, or the cutting down of trees, reduces the amount of plant life that can absorb carbon dioxide from the atmosphere, thus lowers the quantity of CO_2 taken from the planet atmosphere. And this demonstrates

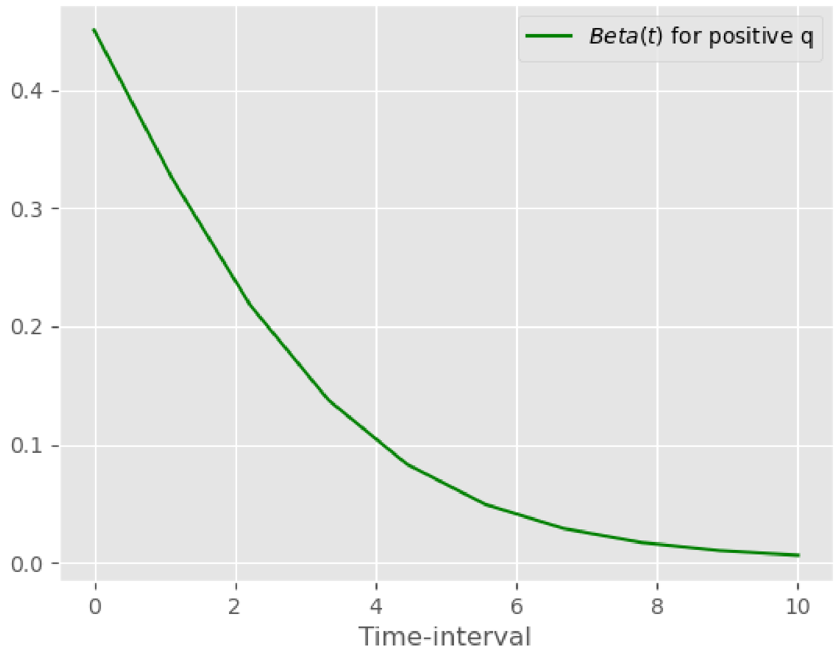


Figure 3.7: Depiction of the CO_2 elimination rate graph for positive q, in the atmosphere

how deforestation adds to the rising atmospheric CO_2 concentration. Climate change is another factor because oceanic absorption of CO_2 is temperature-dependent. A greater amount can be absorbed by colder water, while a lesser amount can be absorbed by warmer water Farquhar (2001). According to certain scientific findings, the rate of atmospheric removal of CO_2 will drop as the seas warm and their capacity to absorb it will diminish. The case where q is negative, we saw different scenario.

The CO_2 rate removal is seen to be increasing over time in figure 3.8. Reforestation is the primary factor that raises the rate of atmospheric removal of CO_2. Reducing deforestation and expanding forests by planting a large number of trees will increase the quantity of CO_2 absorbed by plant life.

When the parameter q is positive, it causes the value of $\beta(t)$ to decrease; conversely, if it's not positive, it causes the $\beta(t)$ value to increase. Parameter q can be influenced by various factors, such as reforestation or deforestation. Equation 3.1.1 uses both graphs in figures 3.7 and 3.7 to illustrate $\beta(t)$.

We can determine the maximum and minimum values of $\beta(t)$ mathematically. The same as we illustrated by taking into account positive and negative q in figures 3.7 and 3.8. The elimination rate for Positive q is declining over time. It will attain its maximum value at time t=0 and, at time $t \to \infty$, it will reach its minimum value.

Now let consider for $t \to 0$,

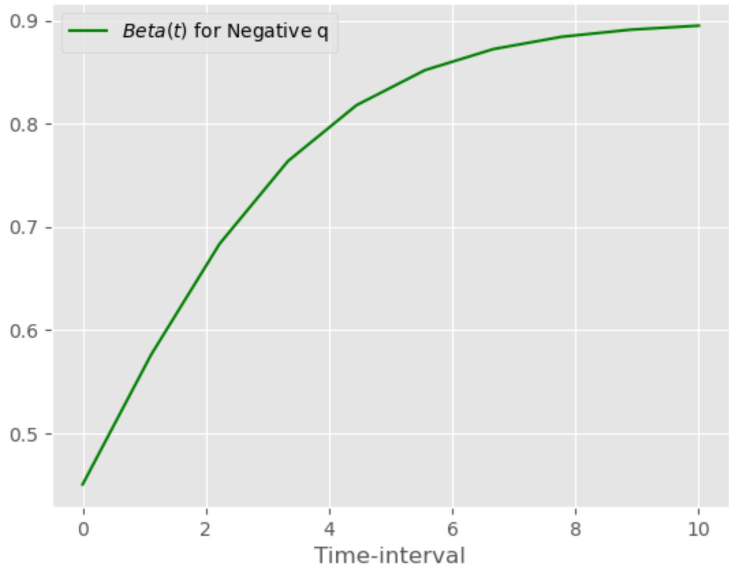


Figure 3.8: Graph of CO_2 removal rate from atmosphere with negative q

$$\begin{aligned}\lim_{t\to 0}\beta(t) &= \lim_{t\to 0}\beta_0 + \frac{\beta_1-\beta_0}{1+\exp(-qt)}\\ &= \frac{\beta_1-\beta_0}{2}\\ &= 0.45095.\end{aligned}$$

At $t\to\infty$ The number was obtained using table 3.1.

$$\begin{aligned}\lim_{t\to\infty}\beta(t) &= \lim_{t\to\infty}\beta_0 + \frac{\beta_1-\beta_0}{1+\exp(-qt)}\\ &= \beta_1.\end{aligned}$$

The values of β_0 and β_1 are known. The rate is gradually rising for negative q. $\beta(t)$ has a low value at time t=0 and a greatest value at $t\to\infty$. Reflecting on $t\to 0$, The $\beta(t)$ value, is the same as when we assume that the Rate q is declining positively.

$$\begin{aligned}\lim_{t\to 0}\beta(t) &= \lim_{t\to 0}\beta_0 + \frac{\beta_1-\beta_0}{1+\exp(-qt)}\\ &= \frac{\beta_1+\beta_0}{2}\\ &= 0.45095.\end{aligned}$$

As above, the number was obtained using table 3.1. At $t \to \infty$ In this instance, On $(-qt)$, the negative sign will be positive. as we believe q to be negative.

$$\begin{aligned}\lim_{t\to\infty} \beta(t) &= \lim_{t\to\infty} \beta_0 + \frac{\beta_1 - \beta_0}{1+\exp(-qt)} \\ &= \beta_0 + \frac{\beta_1 - \beta_0}{\exp(\infty)} \\ &= \beta_0.\end{aligned}$$

It is now evident that $\beta(t)$ varies in value between β_1 and β_0.

The analysis of the eigenvalue will come next in order to comprehend the variability of $\beta(t)$.

3.5 Examine Eigenvalue Behavior When Parameters Are Varied

In this section we will discuss about the behavious of eigenvalue λ_2 for all three parameters q, β_0 and β_1 in the equation 3.1.1. Where we fix all other parameters keep one changing with different constant values. Furthermore, we will perform bifurcation analysis to identify critical points where the system's behavior changes qualitatively

3.5.1 Fix other parameters changing q. From above in subsection 3.4.2 it is clear and obvious that for any value of q, $\beta(t)$ is always positive since λ_2 is negative values of $\beta(t)$ ($\lambda_2 = -\beta(t)$) then λ_2 is always negative for any value of q see figure 3.9

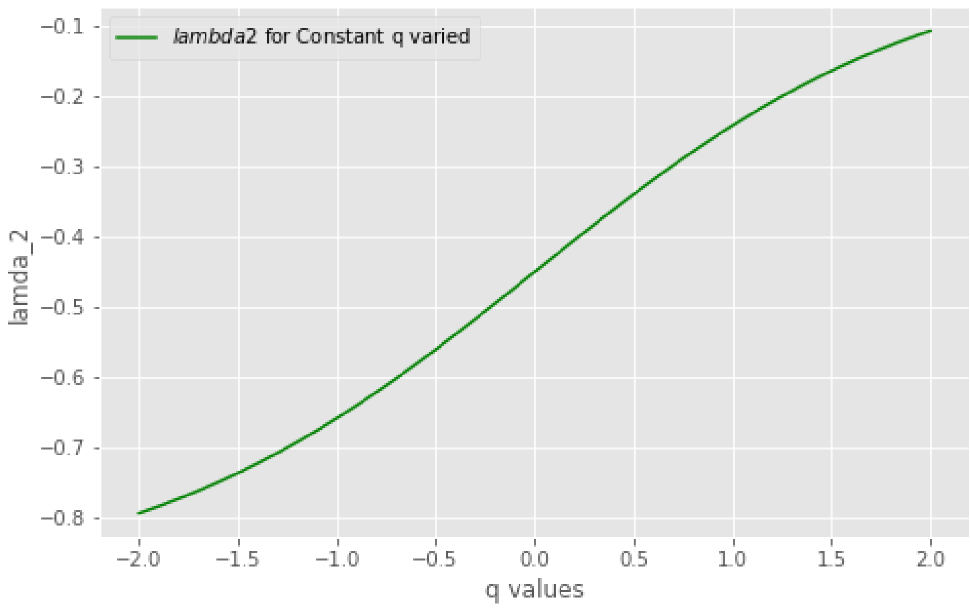


Figure 3.9: Plotting eigenvalue λ_2 under variation of parameter q

Figure 3.9 illustrates how the eigenvalue varies with time, rising throughout time and remaining consistently negative. Since $\beta(t)$ is always positive for all values of the constant parameter q, where the other parameters are fixed and positive, it cannot be positive.

$$\lambda_2 < 0.$$

We also take into account the Parameter a, which might be positive or negative, when analyzing the eigenvalue. λ_1 is positive and λ_2 is negative if an is positive $(a > 0)$.

$$\lambda_2 < 0 < \lambda_1.$$

Figure 3.10 illustrates the unstable saddle point at equilibrium point in this scenario using a phase plane trajectory. This reveals that the system is not stable and that there is no stability. The amount of $CO2$ in the atmosphere will always rise if nothing is done. In order to stabilize atmospheric $CO2$, human activity is required. For instance, increasing the surface area of plants can either raise or reduce how quickly $CO2$ is taken out of the atmosphere.

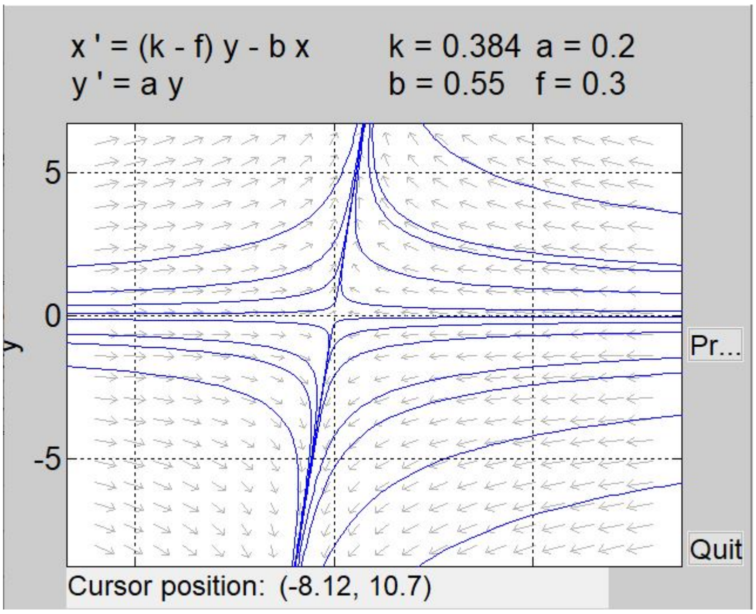


Figure 3.10: Phase portrait When a is greater than zero $(a > 0)$

When $(a < 0)$ is negative, λ_1 and λ_2 are also negative, indicating that both eigenvalues are less than zero.

$$\lambda_1 < 0, \quad \lambda_2 < 0.$$

Since all trajectories in this instance point toward the equilibrium point, the critical point is inappropriate node and the system is thus asymptotically stable (see figure 3.11). When the a is negative, it indicates, the amount of CO_2 emissions are declining Gradually. This indicates that the system is approaching its stable state. Stated differently, the atmospheric concentration of CO_2 can attain equilibrium if the amount of CO_2 emission is reduced.

3.5.2 Fix other parameters changing β_0. In this case we observe the change in qualitative behavious of the system as the constant parameter β_0 varied with different values while other parameter are fixed. It is easy to understand that bifurcation is expected to occur when at least one of the eigenvalues of the dynamical system becomes zero or pure imaginary. In our case we took eigenvalue λ_2. By considering

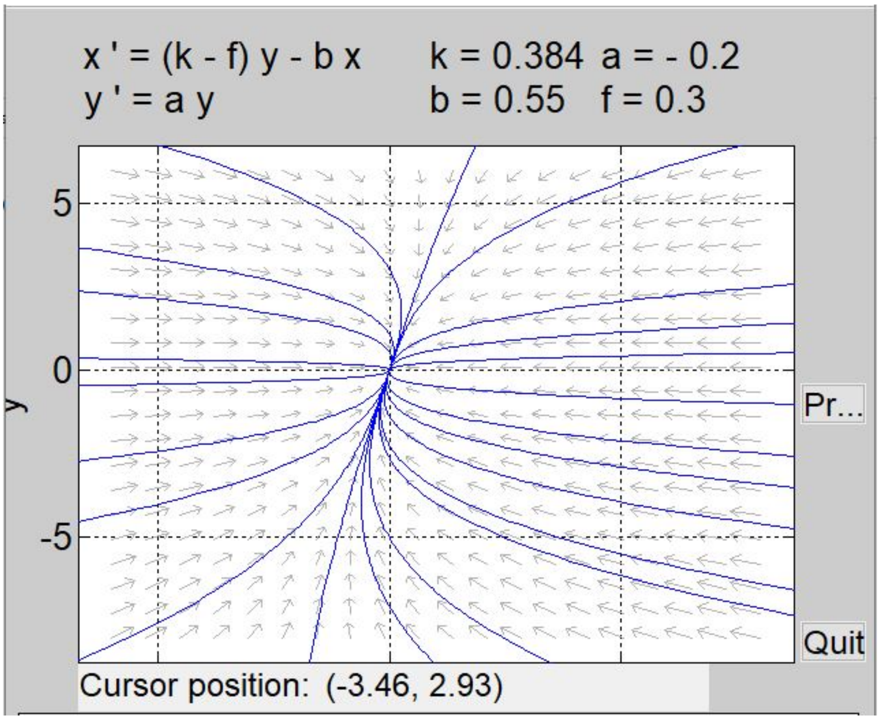


Figure 3.11: Phase portrait for a less than zero $(a < 0)$

the removal rate $\beta(t)$ we can compute for which value of β_0 the eigenvalue λ_2 is less or is greater than zero or the value for which eigenvalue (λ_2) is equal to zero. Consider,

$\lambda_2 < 0$

since $\lambda_2 = -\beta(t)$,

$$-\beta_t < 0,$$

replace $\beta(t)$ by it value from 3.1.1

$$-\beta_0 + \frac{\beta_1 - \beta_0}{1 + exp(-qt)} < 0,$$
$$\beta_0 > -\beta_1 \exp(qt).$$

So we observe that the equilibrium point is stable when $\beta_0 > -\beta_1 \exp(qt)$ and $a < 0$, it a saddle node when $\beta_0 > -\beta_1 \exp(qt)$ and $a > 0$ or when $\beta_0 < -\beta_1 \exp(qt)$ and $a < 0$ and the equilibrium point is unstable node when $\beta_0 < -\beta_1 \exp(qt)$ and $a > 0$ see figure 3.14, the value at $\beta_0 = -\beta_1 \exp(qt)$ is bifurcation value for any value of t. Using the estimated value of parameters a bifurcation occurs at the parametric value $\beta_0 = -0.00071578$ where as β_0 increase the equilibrium point change from any unstable saddle point to an asymptoticly stable improper node when a in negative constant. See figure 3.12.

From figure 3.12 the eigenvalue λ_2 is positive for the values of $\beta_0 < -\beta_1 \exp(qt)$ and changing to be negative for the values of $\beta_0 > -\beta_1 \exp(qt)$. When we fix the constant a to be negative, the system is

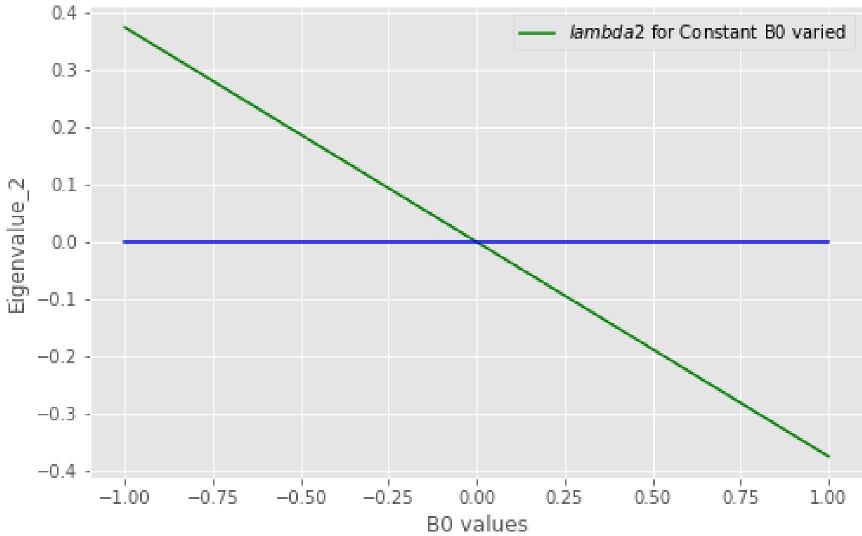


Figure 3.12: Eigenvalue λ_2 under variation of constant β_0

unstable for the values of $\beta_0 < -\beta_1 \exp{(qt)}$ and the system will be stable for the values of $\beta_0 > -\beta_1$ see figure 3.14.

3.5.3 Fix other parameters changing β_1. In this case we observe the change in qualitative behavious of the system as the constant parameter β_1 change with different values while others keep unchanged. Considering the removal rate $\beta(t)$ we can compute for which values of β_1 the eigenvalue λ_2 is zero as we did for parameter β_0.

$$
\begin{aligned}
&\lambda_2 < 0, \\
&-\beta_t < 0, \\
&-\beta_0 + \frac{\beta_1 - \beta_0}{1 + exp(-qt)} < 0, \\
&\beta_1 > -\beta_0 \exp{(-qt)}.
\end{aligned}
$$

Noting the sign of λ_1 and λ_2 we can observe that the equilibrium point at the origin is stable when $\beta_1 > -\beta_0 \exp{(-qt)}$ and $a < 0$, it a saddle node when $\beta_1 > -\beta_0 \exp{(-qt)}$ and $a > 0$ or when $\beta_1 < -\beta_0 \exp{(-qt)}$ and $a < 0$, and the equilibrium point is unstable node when $\beta_1 < -\beta_0 \exp{(-qt)}$ and $a > 0$ see figure 3.14 the value at $\beta_1 = -\beta_0 \exp{(-qt)}$ is bifurcation value for any value of t. A bifurcation occurs at the parametric value $\beta_1 = -0.5368..$ obtained by using the estimated values of parameters. as for β_0 when β_1 increases, the equilibrium point changes from any unstable saddle point to asymptoticly stable improper node. See figure 3.13 In this case the eigenvalue λ_2 is positive for the values of $\beta_1 < -\beta_0 \exp{(qt)}$ and changing to be negative for the values of $\beta_1 > -\beta_0 \exp{(qt)}$ figure 3.13. When we fix the constant a to be negative, the system is unstable for the values of $\beta_1 < -\beta_0 \exp{(qt)}$ and the system will be stable for the values of $\beta_1 > -\beta_0$ see figure 3.14.

3.5.4 Phase portrait. This is a picture that illustrates how our dynamical system behaves over time. It will include graph that show the trajectories that the system can follow through its state space, which

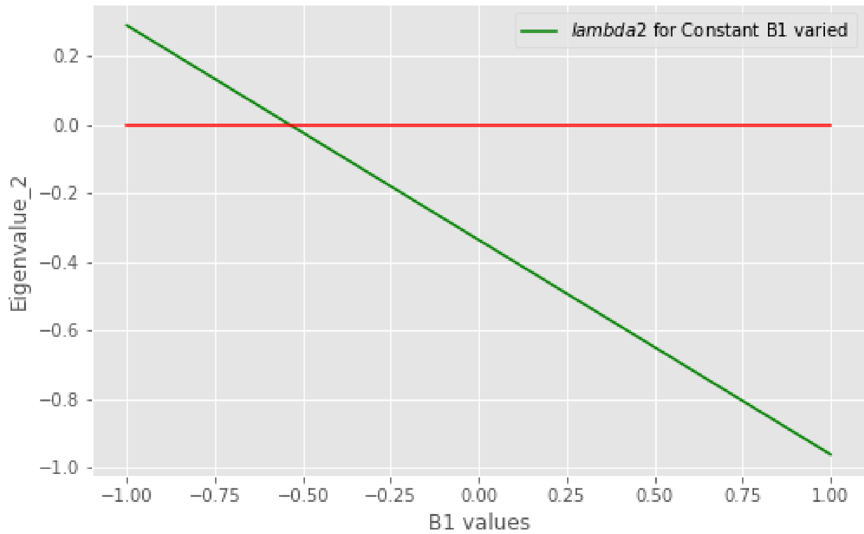


Figure 3.13: Bifurcation behaviours for β_1

is the set of all possible states that the system can occupy. The state space is usually represented by a set of axes, with each axis representing one of the system's variables which in our system are Ca and e. It will help us to analyze and interpret a phase portrait requires careful observation and understanding of the system's behavior. And it will provide valuable insights into the dynamics of the system and help predict its future behavior.

See figure 3.14 the constant b stands for $\beta(t)$ and its values depends on the value of constant β_0 and β_1. Where we have $b < 0$ that when $\beta_0 < -\beta_1 \exp(-qt)$ or$\beta_1 < -\beta_0 \exp(-qt)$ the same $b > 0$ when $\beta_0 > -\beta_1 \exp(-qt)$ or$\beta_1 > -\beta_0 \exp(-qt)$.

3.5.5 Analyse the model with a constant removal rate. In this section, we examine the model's behavior to determine if it is stable or unstable under constant β.

Equilibrium Point

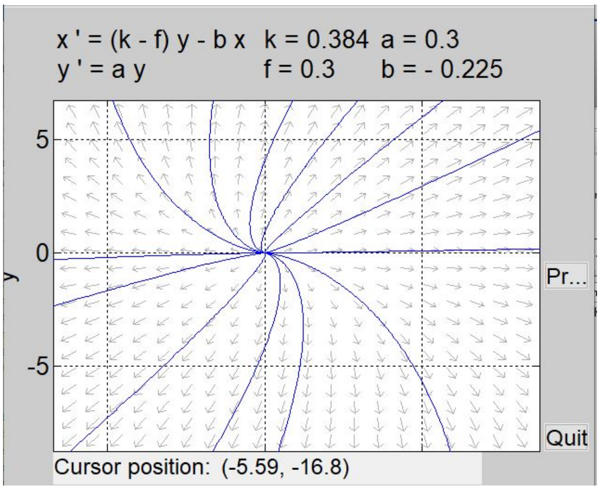
As we computed before, in order to determine the model's critical points, we let;

$$\frac{dCa}{dt} = 0$$
$$\frac{de}{dt} = 0$$

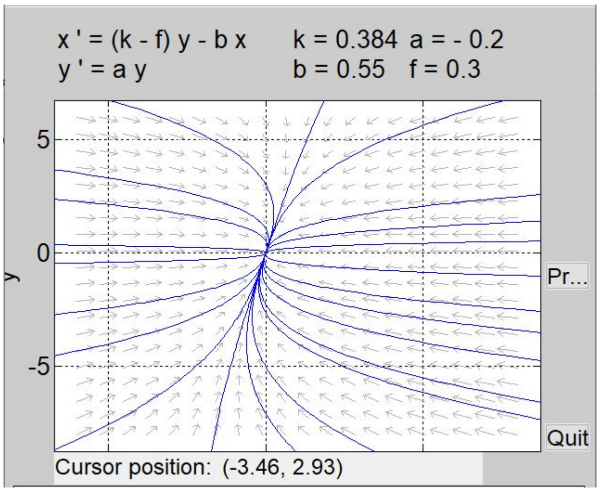
And we obtain (0,0) as equilibrium point.

The Jacobian matrix is :

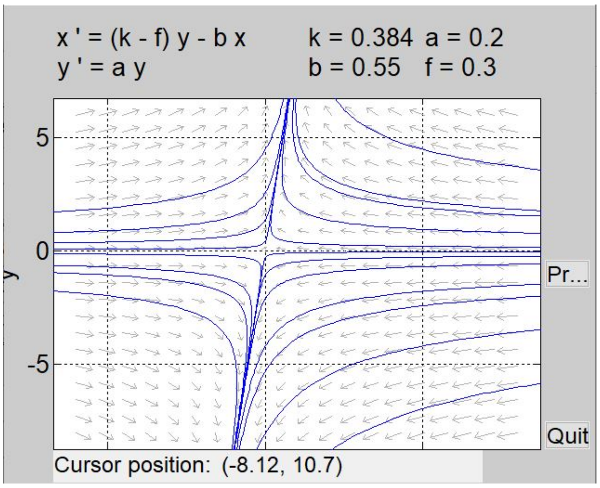
$$J(Ca, e) = \begin{pmatrix} -\beta & k - f \\ 0 & a \end{pmatrix}.$$



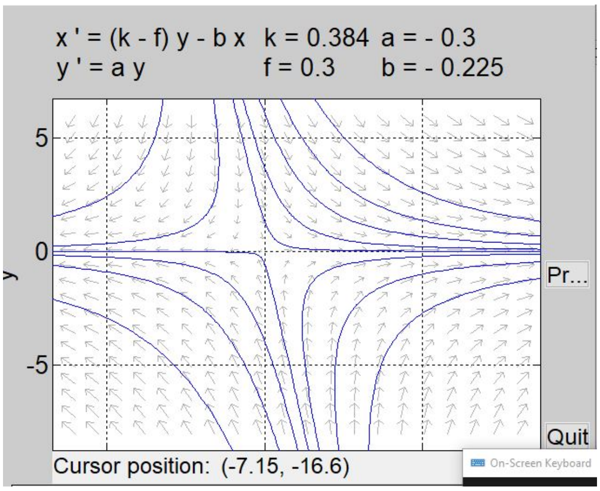
(a) Phase portrait for $a > 0$ and $b < 0$: Unstable node



(b) Phase portrait for $a < 0$ and $\beta_0 > -\beta_1 e^{(-qt)}$ or$\beta_1 > -\beta_0 e^{(-qt)}$: stable



(c) Phase portrait for $a > 0$ and $b > 0$: Unstable saddle point



(d) Phase portrait for $a < 0$ and $b < 0$: Unstable saddle point

Figure 3.14: Phase portrait by changing values of parameters (a, β_0, β_1)

Determine the eigenvalues of matrix J(ca,e) (λ) we have got;

$$det \begin{pmatrix} -\beta - \lambda & k \\ o & a - \lambda \end{pmatrix} = 0.$$

$$\lambda^2 + \beta\lambda - a\lambda - a\beta = 0$$

$$\lambda^2 + (\beta - a)\lambda - a\beta = 0$$

$$\begin{aligned} D &= (\beta - a)^2 + 4a\beta \\ &= (\beta)^2 - 2\beta a + a^2 + 4a\beta \\ &= (\beta + a)^2 \end{aligned}$$

$$\lambda_1, \lambda_2, = \frac{-(\beta - a) \pm (\beta + a)}{2}.$$

$$\begin{aligned} \lambda_1 &= \frac{-\beta + a + \beta + a}{2} \\ &= a \end{aligned}$$

$$\begin{aligned} \lambda_2 &= \frac{-\beta + a - \beta - a}{2} \\ &= -\beta \end{aligned}$$

Two eigenvalues, λ_1 and λ_2, are obtained; $\lambda_1 = a$, and $\lambda_2 = -\beta(t)$ where the values of a and $\beta(t)$ are constant. If a is positive we have one positive eigenvalue and one negative eigenvalue since the removal rate is taken to be positive.In this case the equilibrium point is saddle point and he system is unstable. If a is negative we have two negative eigenvalue .In that case the equilibrium point will be proper node and the system will be stable.

In this case of constant removal rate. We anlyse the removal rate as a fixed constant which does not change. But from the literature the removal rate is changed by certain factors such as vegetation surface, deforestation,....et. Those factors are not included in the model when considering removal rate β to be constant. But for the case of our original model with removal rate as function of time with different other parameter the information of those factor are capture in the model.

3.6 Investigation of the Outcome and Numerical Resolution

We show the model's numerical solutions in this section. The initial quantity of Ca = 307.41 ppm is taken from the dataset, and it corresponds to the CO_2 concentration in 1940. We use e = 10.11 Gigaton as our initial emission value. Python was used as the software and the Runge-Kutta method (4th order) as the techniques to produce the numerical result.

The CO_2 concentration in figure 3.15 is rising without end. We can see from the graph that the concentration of CO_2 in the atmosphere increases with time. The rate of growth for the upcoming year is greater than the rate for the preceding year. There is no limit to the rising amount of CO_2 in the atmosphere.

In figure 3.16 the emission of CO_2 is increasing continuously over the time. This increase is due mainly to human activities. As the population increase the global energy demand increase led by emerging markets and developing economies. Demand for all fossil fuels is on course to grow significantly in nowadays with both coal and gas are rising.

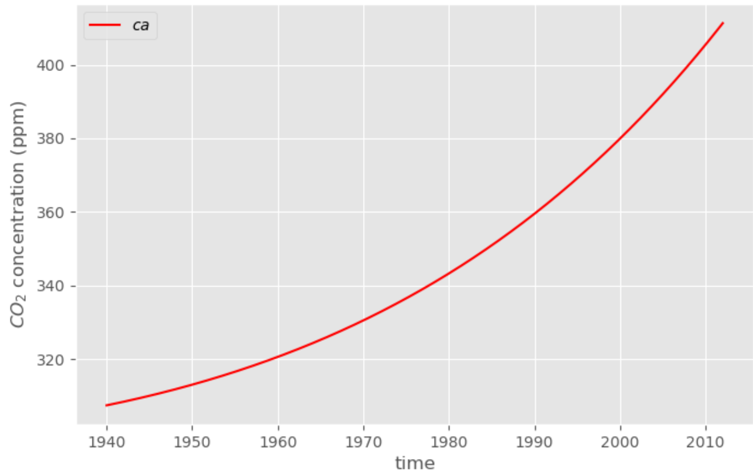


Figure 3.15: Numerical solution of the model, atmospheric CO_2 concentration against time

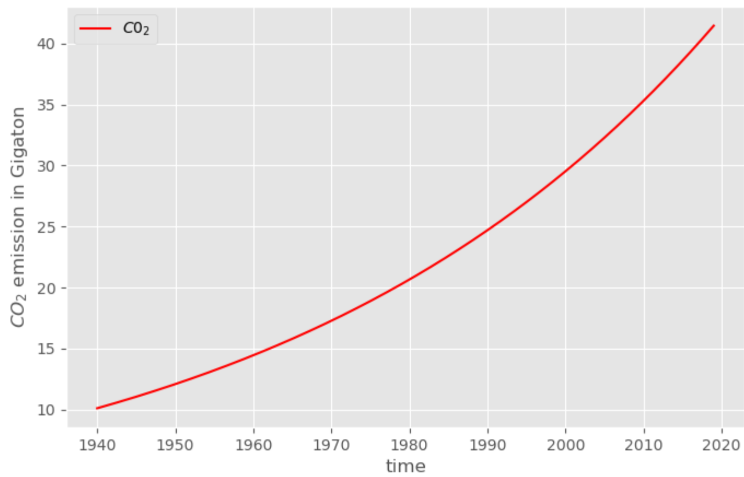


Figure 3.16: Global emission levels of CO_2 from fossil fuels and land-use changes exponential curve from the model.

3.7 Sensitivity Analysis

Understanding the parameter's significance within the model is made easier by the analysis of sensitivity. Within this section, we examine the numerical resolution that was produced through altering the values of some coefficients while maintaining the same values for others. We took into account the various values of factor a in our model equation and plotted the solution where the values of the other parameters are fixed. The numerical solutions for the values of $a = 0.03$ and $a = 0.0$ are plotted in figures 3.17 and 3.18.

For the value of $a = 0.0300$ in figure 3.17 as we accelerate the rate of emission, the emission soon rises.

We plot two graphs in figures 3.17 & 3.18 with two alternative values for the factor a. When compared

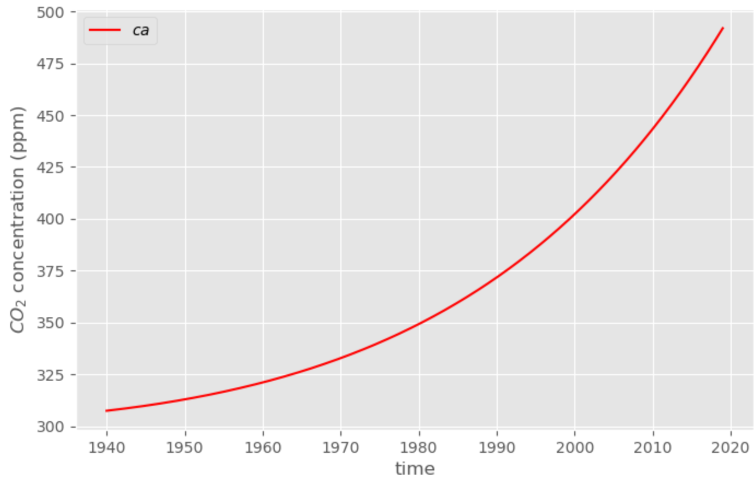


Figure 3.17: Numerical solution for $a = 0.030$

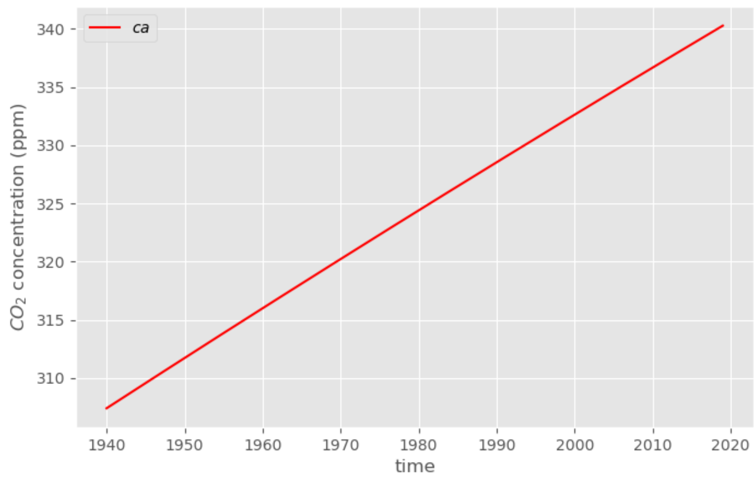


Figure 3.18: Numerical solution for $a = 0$

to the graph in figure 3.18, the graph in figure 3.17 depicts a numerical calculation of (Ca), the atmospheric CO_2 concentration, given $a = 0.030$, It is the value where the rate of CO_2 emission is highest. As a result, The amount of CO_2 in the atmosphere increased dramatically and without limits. It differs from the graph in Figure 3.18 where it is assumed that the atmospheric CO_2 emission rate is zero, or $a = 0.0$. This means that the emission is steady. The atmospheric CO_2 concentration rises with boundaries since the emission is continuous. The concentration of CO_2 in the atmosphere follows a concave downward curve. The rate of rise in the following year will be smaller than in the previous year since the atmospheric CO_2 content is dropping with time, as shown in figure 3.18. The concentration of CO_2 in the atmosphere is examined in the following part by assuming a constant CO_2 emission. To achieve this, we will go over the analytical solution of Ca.

3.8 An analytical solution in the case of a constant emission (e).

This section will address the analytical solution for a constant amount of CO_2 released into the atmosphere, which denotes an annual amount of CO_2 emitted into the atmosphere. The following is the equation of the model is expressed as follows::

$$\frac{dC_a}{dt} = (k - f)e - \beta(t)C_a,$$
$$\frac{de}{dt} = 0,$$

As the emission is constant, the derivative of emission e with respect to time t which the emission rate is zero.

$$\frac{dC_a}{dt} = p - \beta(t)C_a \quad where \quad p = e(k - f). \tag{3.8.1}$$

We can now use the method for solving linear ordinary differential equations to solve the equation (3.8.1) analytically. The equation can be expressed as follows:

$$\frac{dC_a}{dt} + \beta(t)C_a = p. \tag{3.8.2}$$

Let

$$\mu(t) = \exp\left(\int_{1940}^{2015} \beta(t)dt\right).$$

When $u(t)$ is multiplied by both sides of equation (3.8.2), we obtain,

$$\mu(t) \times \frac{dC_a}{dt} + \mu(t) \times \beta(t)C_a = \mu(t) \times p,$$
$$\exp\left(\int_{1940}^{2015} \beta(t)dt\right) \times \frac{dC_a}{dt} + \exp\left(\int_{1940}^{2015} \beta(t)dt\right) \times \beta(t)C_a = \exp\left(\int_{1940}^{2015} \beta(t)dt\right) \times p,$$
$$\left[\exp\left(\int_{1940}^{2015} \beta(t)dt\right) \times ca\right]' = p \times \exp\left(\int_{1940}^{2015} \beta(t)dt\right). \tag{3.8.3}$$

By integrating both side of equation (3.8.2)

$$\int \left[\exp\left(\int_{1940}^{2015} \beta(t)dt\right) \times ca\right]' dt = \int p \times \exp\left(\int_{1940}^{2015} \beta(t)dt\right) dt,$$
$$\exp\left(\int_{1940}^{2015} \beta(t)dt\right) \times ca + c_1 = \frac{p}{\beta(t)} \times \exp\left(\int_{1940}^{2015} \beta(t)dt\right) + c_2,$$
$$\exp\left(\int_{1940}^{2015} \beta(t)dt\right) \times ca = \frac{p}{\beta(t)} \times \exp\left(\int_{1940}^{2015} \beta(t)dt\right) + c. \tag{3.8.4}$$

Where c_1, c_2, c are constant and $c = c_2 - c_1$ In order to obtain the overall equation, we multiply by $e^{-\int \beta(t)dt}$ on both sides of equation (3.8.4) . As a general solution, We arrive to the subsequent equation.:

$$ca(t) = \frac{p}{\beta(t)} + c \times \exp\left(-\int_{1940}^{2015} \beta(t)dt\right) \tag{3.8.5}$$

We must find the exact solution after determining the general one. and to determine the value of c, we apply to the initial condition. Our model was constructed to fit data from 1940, when the CO_2 concentration in the atmosphere was 307. The initial state,

$$Ca(1940) = 307.41$$

We determine the c value in order to arrive at the solution. We just need to enter the initial condition to accomplish this, to get equation in order to solve for c. Now go ahead and get started;

$$307.41 = \frac{p}{\beta(1940)} + c * \exp\left(-\int_{1940}^{2015} \beta(t)dt\right), \tag{3.8.6}$$

$$c = \left(307.41 - \frac{p}{\beta(1940)}\right) * \exp\left(\int_{1940}^{2015} \beta(t)dt\right), \tag{3.8.7}$$

where

$$\int \beta(t) = \int \left(\beta_0 + \frac{\beta_1 - \beta_0}{1 + \exp(-qt)}\right) dt$$
$$= \beta_0 t + \frac{(\beta_1 - \beta_0) ln(\exp(qt)) + 1)}{q}.$$

The exact solution can be obtained by substituting the values of c and $\int \beta(t)$ into the general solution of equation (3.8.5). This is shown in the graph below.

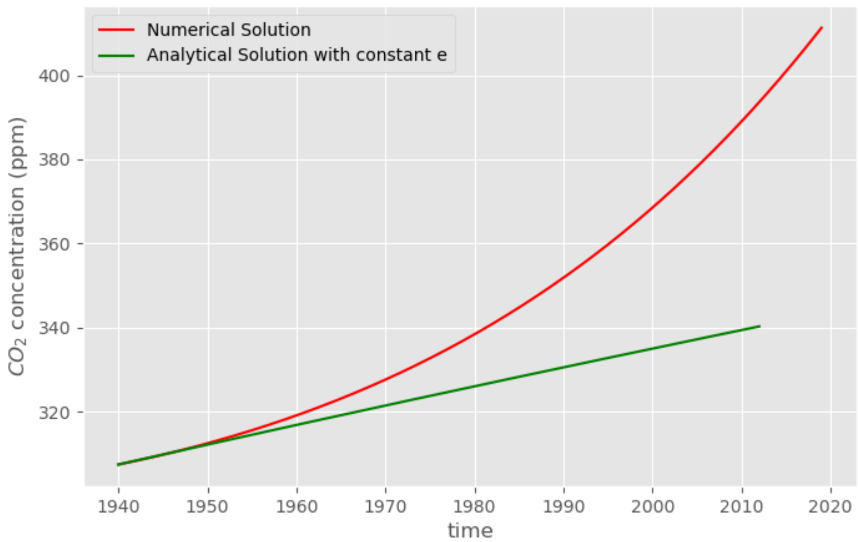


Figure 3.19: Comparison of the numerical solution (red curve) and the analytical solution (green curve) when the emission (e) is constant.

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The graph in figure 3.19 is comparing the numerical solution and the analytical solution with constant emission (e). The analytical solution showing that the atmospheric CO_2 seems to increase linearly at constant emission of CO_2 while the numerical solution showing exponential growth of atmospheric $C0_2$

concentration. And the increasing rate at constant e in analytical solution is smaller than the growth rate in numerical solution. Upon examining the graph, we can see that, in 2010 with constant emission, the atmospheric CO_2 concentration reached 340 ppm, but, in other circumstances, it reached that level around 1981.

3.8.1 The Model's Fitness. We must determine whether the model fits the gathered data in order to determine whether it will be helpful for prediction. In this study, R-squared is computed to assess the model's fitness. A statistical indicator of how closely the data match the expected regression line is called R-squared. R-squared measures the goodness of fit, or the degree to which the data fit the regression model. Is occasionally referred to as the determination coefficient or the multiple determination coefficient. R-squared, is computed by dividing the overall Error sum of squares divided by total sum squares owing to regression error.

$$R - Squared = \frac{SSregression}{SStotal}.$$

The ideal value for the value of R-squared is 1, which is consistently in the range between 1 and 0. R^2 score will be larger the smaller the analysis of error regression proportionate to the total error. The value of R-squared for our model within this research is 0.9985. The model's fitness is high because its R-squared value is quite near to 1. In chapter three we have estimated parameter for our model using

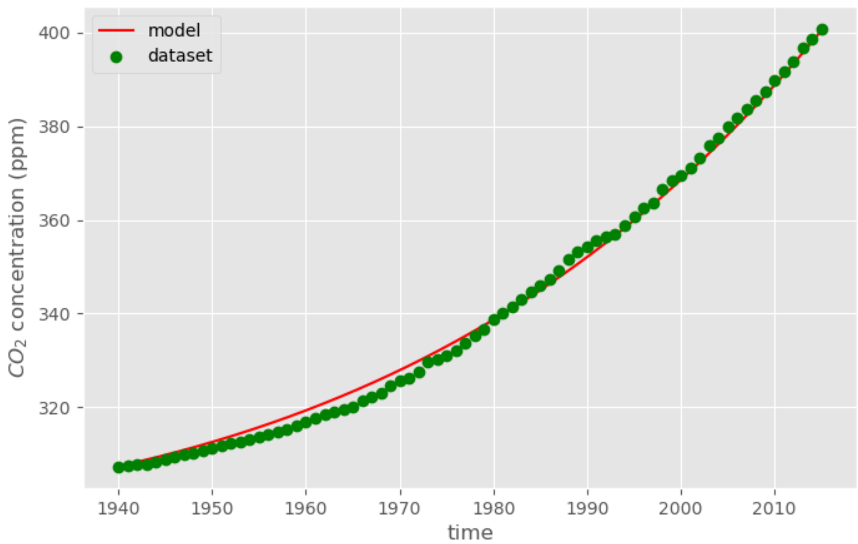

Figure 3.20: Comparison of predicted CO_2 concentrations with observations 1940 –2015. Curve is best fit to data

historical data starting from 1940 to 2015. Figure 3.20 shows the model's curve and historical data on the same graph to see how The model matched the range of data 1940—2015 which we used to estimate the parameters and by looking on the graph we can see that the model fit the data. To see whether the model can fit the rest of the data so that can be used in predicting the future value of atmospheric $C0_2$ concentration, we plot the graph comparing the model's numerical solution to historical observed data within the range from 1940 to 2019. see 3.21.

To put it briefly, we exhibit the recorded data and the model's numerical solution on the same graph in figures 3.21 & 3.20. The model suited the data, as evidenced by the close proximity of the two curves

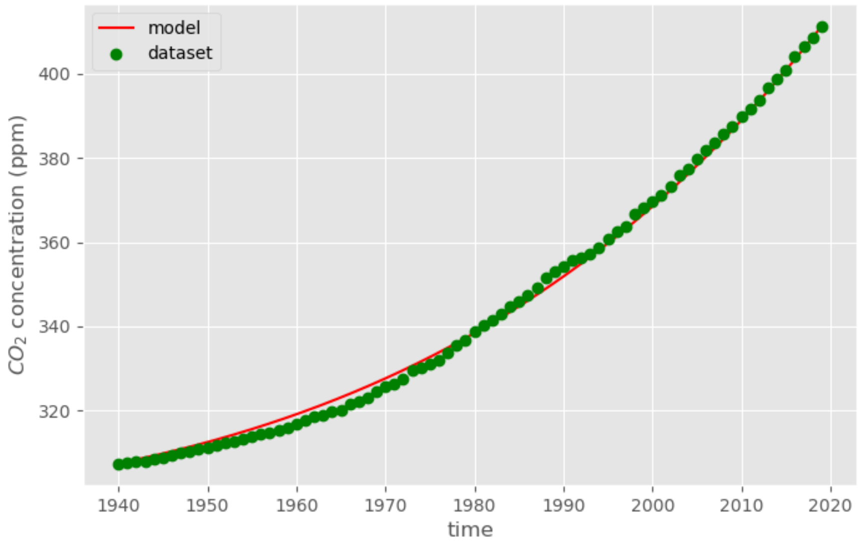

Figure 3.21: Comparison of predicted CO_2 concentrations with observations 1940 –2019. Curve is best fit to data

when compared. Since the model's curve closely resembles the curve of the actual data, we can state that our model can be utilized to make predictions.

3.9 Simulation and Prediction using the model

We can use the developed mathematical model to predict the atmospheric carbon dioxide concentration.

Year	Prediction		Historical Data	
	Ca(ppm)	dCa/dt(ppm/y)	Ca(ppm)	dCa/dt(ppm/y)
2000	365.98	1.94	369.52	2.06
2005	375.67	2.16	379.8	2.01
2010	386.47	2.4	389.85	2.2
2015	398.46	2.66	400.86	2.62
2020	411.78	2.95	413.94	
2025	426.55	3.27		
2030	442.91			

Figure 3.22: Ca predictions, according to the present model and historical data

4. Concluding remarks and Recommendation

In this project we presented an enhanced mathematical model which is an ordinary differential equation based on incorporating key factors which are emission of CO_2, removal processes and Ocean absorption. Compared to the previous model, which fitted data from 1940 to 2010 and while excluded the oceanic effect. The new model fits data on the concentration of CO_2 in the atmosphere observed between 1940 and 2019 years in a different way.Through the parameter estimation process by using least square method, we successfully estimated the values of the model parameters by fitting the model to the historical data of atmospheric CO_2 concentration and emission from fossil fuel and land use change, data from 1940 to 2015 The Ocean absorption factor taken to be 0.3 (Riebeek) and the emission growth exponentially. This allowed us to obtain a better understanding of the underlying processes driving the global atmospheric CO_2 concentration. The numerical solution of the model using Runge-Kutta fourth other method provide an approximation of the CO_2 concentration over time.This numerical solution enables us to observe the behavior and trends of atmospheric CO_2 concentration and simulate various scenarios, such as change in emission, or policy interventions. Fitting the model to CO_2 data, the R-Squared with value 0.96 close to 1, indicates that the model explain a large proportion of the variation in the observed CO_2 concentration data. This suggests that the model is a good fit to the data, capturing the underlying trends, patterns, and dynamics of the global CO_2 concentration to a significant degree. There are certain anomalies in the data that warrant discussion. Figure 3.4 shows that CO_2 emissions sharply decrease at the end. However, Figure 3.3 atmospheric CO_2 seems to be increasing at the same rate. It is interesting to enlarge the data in this figure 3.6. We hope that the COVID-19 lock-downs, which altered the world economy, are to blame for this. The implications and applications of the model are diverse and have broad-reaching effects on climate change mitigation, policy decisions, scientific understanding, and sustainable development.

In conclusion, this extended project represents a significant advancement in the field of mathematical modeling of atmospheric CO_2 concentration. By addressing key limitations of previous research and incorporating new datasets and factors into the model, we aim to provide a more comprehensive and accurate representation of CO_2 dynamics and their implications for climate change. The results of this study may help guide the formulation of climate policies and advance initiatives to lessen the negative effects of human-caused CO_2 emissions on the planet's climate system.

Further research should explore incorporating additional factors, such as regional variations, feedback mechanisms, and the impact of climate change on CO_2 concentration dynamics. This would improve the accuracy and realism of the model and enable more precise predictions of future CO_2 concentration trends. The MCMC method is also advised to be used for model validation in the future since it can account for uncertainty in parameter estimates and provides a thorough framework for estimating parameter distributions and managing numerous difficulties related to model validation.

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