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Applications of Renormalization Group Methods to Cosmological Problems

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Abstract

Despite the remarkable progress in understanding the large-scale structure ofthe universe, the nature of its very early moments and its accelerated expansion are still among the biggest puzzles in modern physics. The inflation theory improved our understanding of the early universe by discussing solutions of some relevant problems. The inflation scenario has to be considered with the standard Friemann-Lemaitre-Robertson-Walker cosmological model. One could imagine a condensate of a scalar field, called the inflaton field, which moves toward the minimum of a scalar potential during the inflation period.

The scenario of inflation proposed by Alan Guth is based on the existence of a false vacuum state before the Big Bang. The inflaton field was trapped in this metastable vacuum state, then via the quantum tunneling the inflaton field freed itself and the inflation period began. This provided a amount of negative energy that forced the cosmic space to expand faster than light immediately before the Big Bang. Some problems in this scenario are related to the proper explanation of the reheating mechanism and the production of radiation.

Several models have been proposed over the last decades to explain the inflation phase of the early universe. Some of these models, such as the quadratic large field inflation, are excluded by the recent Planck observations of the cosmic microwave background. A reasonable candidate model is the slow-roll model. This model replaces the quantum tunneling mechanism of the inflaton field with a slowly rolling field. This requires the inflationary potential to be almost flat in the beginning of inflation. One can use this model to analyze which inflationary potential is viable and comparable with the observations. But this can leave us to ask why inflationary field should start at this particular flat situation. It seems to be present a fine tuning problem, referred to as the "initial conditions problem". One of the proposed solutions to fix this problem is provided by the study of Renor-

malization Group (RG) running of the inflationary potential changing the energy scale (or equivalently with the cosmic time scale) in the era before the inflation. This can provide the smooth exit of the inflaton field from the false vacuum to the true vacuum and get the slow inflation scenario.

Another big puzzle of modern cosmology is the cosmological constant problem. This constant was invented in the beginning by Einstein himself to get the static universe case which was believed to be true at that time. After discovering the accelerated expansion of the universe, the cosmological constant is the candidate to represent the vacuum energy with negative pressure. The problem is the great discrepancy between the theoretically predicted value from quantum theory for this vacuum energy and its measurable value. The disagreement is notoriusly very huge, which implies that there is something missing in the theory. One of the possible solutions to this issue is again related to the RG running of the inflation potential. The inflaton scalar field potential can be usually expanded in terms of the field around its minimum. This produces a field-independent term of the potential. The cosmological constant can be represented by this field-independent constant term. In the context of non-perturbative quantum gravity theory, the RG scaling of this constant term can be then investigated. The Functional Renormalization Group (FRG) method can be applied to provide a connection between the ultraviolet (UV) and the infrared (IR) values of this constant term. The recently observed cosmological constant (or vacuum energy) value can be considered to be equivalent to the IR value of the constant field-independent term of the inflation potential. This not only can provide a viable solution to the cosmological constant problem but also relate the phase of the accelerated expansion of the recent universe with its early time inflation phase. The running scale that can be considered here is the usual momentum (or energy) scale k, which can correspond in this scenario to the cosmic time or the temperature of the universe.

The structure of the thesis is the following. Chapter 1 is divided into two parts. In Section 1.1 the basic concepts of the FRG method are summarized. This method is based on deriving the functional equation of the RG scaling of the effective action. The features of the effective action in the UV and IR limits are also discussed. Since the exact functional equation cannot be solved exactly, approximation techniques have to be used. The so-called local potential approximation is also summarized in the same section. Section 1.2 includes a more detailed discussion of cosmic inflation, the problems with previous cosmological models, and how they can be addressed by the inflation scenario.

Chapter 2 is divided into three parts. In Section 2.1 a summary of previous work on the RG flow of the inflationary potential is introduced. The massive sine-Gordon potential is considered to be the inflationary potential. We show that the RG flow of the scalar potential starts from a concave potential at the UV scale (i.e. at the Planck scale) tending to be more convex at a lower energy scale (i.e. at the so-called grand unified theory scale) which release the vacuum expectation value of the scalar field to initiate the inflation. This provides a possible mechanism to allow the slow-roll inflation scenario takes place. In Section 2.2 the RG evolution of the inflation potential is discussed based on the slow-roll inflation scenario. This could be obtained by fixing the values of the scaling parameters of potential by the measuring values of the slow-roll inflation parameters. In Section 2.3 the same analysis is repeated by considering a higher Fourier harmonic of the MSG model. The RG evolution of the double massive sine-Gordon is studied and it is found that including this higher harmonic does not affect the conclusions given in Section 2.2.

Chapter 3 is divided into three parts. In Section 1.3 a brief discussion is introduced of the FRG evolution of the field-independent term of the inflation potential in the context of asymptotic safe (non-perturbatively renormalizable) quantum gravity theory. Section 3.2 includes a brief summary of previous work on the RG evolution of the field-independent term in the case of the minimal coupling between the scalar field and gravity. Subtraction schemes are discussed to remove the spurious divergences appear in the value of the cosmological constant in the UV scale. In Section 3.3 the non-minimal coupling case is studied by adding the term $\frac{1}{2}\xi R\varphi^2$ to the gravitational part of the scaling effective action. The modified RG evolution equations are obtained and compared with those obtained in Section 3.2.

Contents

Chapter 1

Introduction

The renormalization group (RG) technique plays a crucial role in many areas of physics, and it is used to determine the long-distance properties at low energy scales of many physical systems and to search for viable UV completions of the physical theories that diverge perturbatively at high energy scales. It also provides us with a natural framework to understand theoretical models that exhibit distinct behavior on different energy scales where the degrees of freedom can be correlated over long distances. The origin of renormalization goes back to statistical mechanics and hydrodynamics, where the physics of phase transitions and critical phenomena could be better understood by using this concept. A more systematic description of this concept is based on the property of changing the observational scale of a theory. The RG method has been extensively used for many purposes. Some of its more important goals are: (i) to remove the spurious UV divergences [1]; (ii) to describe the scale dependence of the running physical parameters and to classify these parameters according to their impact on the dynamics of the theory around its critical points. [2]; (iii) to study the critical behavior of local field variables [3]; (iv) to resume the perturbation expansion in quantum field theory [4, 5]; and (v) to get viable solutions to the strongly coupled theories. Furthermore, the

RG method provides a methodical and efficient way of organizing our understanding of complicated dynamical systems. This way is called coarse-graning, which successively eliminates degrees of freedom. The distinctive feature of the RG method is that it retains the influence of the eliminated subsystem by rescaling the couplings on the rest. The key idea of the FRG method is applying (v) in the framework of functional formalism to search for a general algorithm to address non-perturbative (not necessarily critical) theories.

A strategy typical of RG procedures for the resummation of the perturbation expansion is to introduce and develop effective vertices rather than evaluating higher-order corrections. The result is an iterative procedure where the contributions to the effective vertices are computed at a given step in the leading order of the perturbation expansion. The higher-order corrections are reproduced by inserting these contributions in the next-step graph. In the traditional application of the RG procedure, one can follow the evolution of a few coupling constants only. There is an improvement of this method in order to follow the evolution of a large number of couplings. This can be realized by using functional formalism where the RG evolution is constructed for the generator function for all coupling constants. The resultant running equation is called the Functional Renormalization Group (FRG) equation.

The FRG technique is a modern approach to implementing the Wilsonian RG, which allows one to derive a non-perturbative approximation scheme that goes beyond the standard approaches of perturbative RG. The FRG is based on an exact functional flow equation for the running effective action. The word functional refers to the key idea of this method of formulating functional integrals in dimensional space or dimensional spacetime to evaluate the free energy and the correlation functions, respectively. However, the FRG method is a useful tool for analyzing the non-perturbative perspective of physical theories and it is also useful in its perturbative formulation for studying several systems as, for example, in the case of disordered systems [6, 7, 8]. The necessity of a functional approach, in this case, is the existence of an infinite number of marginal operators at the upper or lower critical dimension. So, in some disordered systems, it is not possible to restrict ourselves to a finite number of coupling constants and the form of functions of the field must be considered.

The formulation of the FRG based on an exact flow equation for scale-dependent effective action (or free energy in the context of statistical physics) and the generating functional of one-particle irreducible vertices is successful in constructing non-perturbative approximation schemes. [9, 10, 11, 12, 13]. The functional may be seen as a coarse-grained free energy that includes only fluctuations with energies or momenta larger than a scale k. The field function represents the microscopic degrees of freedom which could be classical such as the magnetization in the magnetic materials or quantum mechanical, either fermionic or bosonic.

1.1 The RG Flow Equation and The Local Potential Approximation

There are several ways to arrive at a functional RG equation. The simplest one is following Wilson-Kadanoff blocking in continuous space-time. It leads to a functional differential equation describing the cut-off dependence of the bare action. There are other schemes that work by suppressing the modes smoothly when the cut-off is changing. The running parameters of the bare action that are followed by these RG schemes can be related to the observables only qualitatively. So, it is beneficial to transform the RG equation for the effective action whose parameters are related to the observables directly. A modern formulation of the FRG approach to get an exact flow equation at the one-loop level is given by the Wetterich RG equation [10, 11, 12, 13, 14] which has the following form for one-component scalar field theory in d-dimensional space:

$$\partial_t \Gamma_k [\varphi] = \frac{1}{2} Tr \left(\frac{\partial_t r_k}{\Gamma_k^{(2)} [\varphi] + r_k} \right), \tag{1.1}$$

which we will consider written in the form

$$k\partial_k \Gamma_k [\varphi] = \frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \frac{k\partial_k \mathrm{r}_k(p)}{\mathrm{r}_k (p) + \Gamma_k^{(2)}[\varphi]}, \tag{1.2}$$

where the derivative parameter t is a dimensionless scale parameter related to the running scale as $t = \log\left(\frac{k}{\mu}\right)$, μ is a mass cut-off, $\Gamma_k^{(2)}[\varphi]$ is the Hessian of the running effective action, and $\mathrm{r}_k(p)$ is the so-called regulator function.

The main goal of the FRG approach is to relate the behaviour at different scales (here the running scale represents the momentum scale), i.e. smoothly relate the bare action $S_{\text{bare}} = \Gamma_\Lambda$ at the microscopic (or UV) scale, i.e. at $k = \Lambda$, to its effective value $\Gamma_{eff}[\varphi]$ at the macroscopic (or IR) scale, i.e. at $k = 0$ as illustrated in figure1.1 [15]. This requires that the regulator function $\mathrm{r}_k(p)$ to fulfill

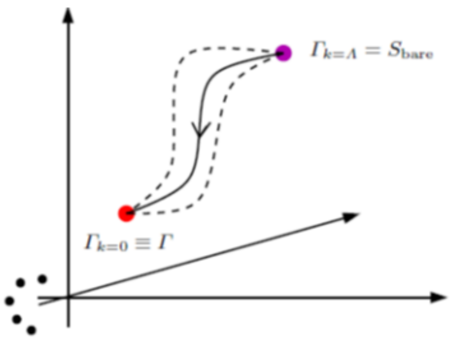

Figure 1.1: RG flow in the parameter space of the effective action $\Gamma_k[\varphi]$. The solid line shows the exact flow obtained by solving the RG equation with a regulator function $\mathrm{r}_k(p)$. The dashed lines show the RG flows obtained by solving the RG equation with two different regulator functions. From Ref. [15].

the following conditions:

1. $\mathrm{r}_{k\to 0}(p) = 0$. This condition implies that the quantum effective action of the original model has to be recovered, $\Gamma_k[\varphi] = \Gamma_{eff}[\varphi]$, at $k = 0$, where all fluctuations are taken into account.

2. $\mathrm{r}_{k\to\Lambda}(p) = \infty$. This divergence of the regulator implies that the all fluctuations are frozen and integrated out and the running action recovers only its bare (classical) value, $\Gamma_k[\varphi] = \Gamma_\Lambda \approx S_{\mathrm{bare}}$, at $k = \Lambda$.

3. $\mathrm{r}_k(p \to 0) > 0$. This condition ensures that the regulator function has an IR regularization. Two popular choices of the regulator functions, the exponential regulator $\mathrm{r}_k(p) = \alpha\, p^2/(e^{p^2/k^2} - 1)$ and the theta regulator $\mathrm{r}_k(p) = \alpha\left(k^2 - p^2\right)\Theta\left(k^2 - p^2\right)$ [16].

A general of the regulator function satisfying the previous conditions can be written as $\mathrm{r}_k(p) = p^2 r(p^2/k^2)$, where $r(p^2/k^2)$ is a dimensionless regulator shape function with dimensionless momentum argument. The RG running equation (1.1) can be integrated over k to get its connection to the effective action $\Gamma_{eff}[\varphi]$ at one-loop level as [15, 17, 18]:

$$\Gamma_{eff}[\varphi] = S_\Lambda[\varphi] + \frac{1}{2}\int \frac{d^d p}{(2\pi)^d} \ln(S_\Lambda{}^{(2)}[\varphi]) \;+\; O(\hbar^2). \tag{1.3}$$

The momentum integral term in the rigth-habd side of the above equation could be divergent at both IR and UV scales. The Pauli-Villars regularization approach can be used to avoid that by adding a momentum-dependent mass-like term $\frac{1}{2}\int \mathrm{r}_k(p)\,\varphi^2$ to the bare action $S_\Lambda[\varphi]$, so we can write the scale-dependent action as

$$\Gamma_k[\varphi] = S_\Lambda[\varphi] + \frac{1}{2}\int \frac{d^d p}{(2\pi)^d} \ln(\mathrm{r}_k(p) + S_\Lambda{}^{(2)}[\varphi]) \tag{1.4}$$

recovering the effective action in the IR limit if the conditions (i) and (iii) for the regulator functions are considered, and there is no IR divergences. It is noticed that for the condition (ii) of the regulator function, the effective action is approximately recovering the bare action S_{bare}, at $k = \Lambda$. The reason is that if condition (ii) is considered in the UV limit, the running action $\Gamma_k[\varphi]$ in Eq. (1.3) will recover the bare action up to a constant field-independent, but k-dependent, term which can be denoted as $\mathrm{V}_k(0)$. So, equation (1.3) can be written at the limit $k \to \Lambda$ as

$$\Gamma_{k\to\Lambda}\ [\varphi] = S_\Lambda[\varphi] + constant = S_\Lambda[\varphi] + \int d^d x\, \mathrm{V}_{k\to\Lambda}(0)\,. \tag{1.5}$$

Regarding the constant term of the potential $\mathrm{V}_k(0)$, there are cases where it has physical importance [18]. In low dimensional quantum mechanical systems it can represent the ground-state energy. In statistical field theory, this term could be associated with free energy in the case of flat space background. In the framework of the non-perturbative RG method, this term needs a special treatment, where it can play the role of the cosmological constant term in $d = 4$ dimensional curved space background. In this case this constant term could play a role in cosmic inflation and its FRG treatment can be used to investigate the cosmological constant problem [18, 19]. The RG running of the field-independent term and its cosmological implications in both minimal coupling and non-minimal coupling cases is studied in chapter 3 of this work.

The Wetterich equation (1.1) is an exact flow equation for the effective action up to the 1-loop level. It cannot be in general solved exactly and there is the need of approximation techniques. The most common techniques are: (i) finding an iterative solution, (ii) vertex expansion (VE), and (iii) derivative expansion (DE). The latter is based on expanding the effective action in powers of the field spatial derivatives. Such expansions contain all possible field terms preserving the model symmetries.

The Local Potential Approximation (LPA) can be obtained by considering the lowest order of the DE approximation technique. In the LPA the effective action can be introduced [20] at the continuous limit of the fields φ as following:

$$\Gamma_k[\varphi] = \int d^d x \left\{ \frac{1}{2} \partial_\mu \varphi_i(x) \partial_\mu \varphi_i(x) + U_k(\rho(x)) \right\}, \tag{1.6}$$

which is entirely determined by an effective potential $U_k(\rho)$. The index $i \in \{1, \ldots, N\}$ runs over the internal components of the field The simplicity of the LPA approach and the reliability of its results make it a formal alternative to the use of the mean-field approximation. The LPA procedure is based on choosing a specific functional space parametrized by a finite number of parameters; this functional space can be described by an ansatz for the effective action as given in (1.5). This leads to obtaining a set of differential equations that represent the RG flow for the considered theory. If we take the projection of the exact equation (1.1) on the ansatz functional (1.5) in the constant field configuration, i.e. $\varphi(x) = \varphi = constant$, the second derivative of the action $\Gamma_k{}^{(2)}[\varphi]$ will vanish and the space integration yields just a constant volume factor V as

$$\Gamma_k[\varphi] = V\ U_k(\rho). \tag{1.7}$$

The latter equation can be written to give the flow equation of the local potential $U_k(\rho)$ as

$$\partial_k U_k(\rho) = \frac{1}{V} \partial_k \Gamma_k[\varphi] = \frac{1}{2} Tr \int \frac{d^d p}{(2\pi)^d} \frac{\partial_k \mathrm{r}_k(p)}{\mathrm{r}_k(p) + \Gamma_{ij}{}^{(2)}[p, -p]}. \tag{1.8}$$

The second derivative (Hessian) of the potential evaluated at constant field is independent from the coordinate x and it is given by [20]

$$\Gamma_{ij}{}^{(2)}[p, -p] = \left[(p^2 + U^{(1)}(\rho))\delta_{ij} + \varphi_i \varphi_j U^{(2)}(\rho) \right]_{\varphi(x)=\varphi}, \tag{1.9}$$

where $U^{(1)}$ and $U^{(2)}$ are the first and second field derivatives of the local potential, respectively. By inserting Eq. (1.8) in Eq.(1.7) we can get

$$\partial_k U_k(\rho) = \frac{1}{2}\int \frac{d^d p}{(2\pi)^d}\partial_k \mathrm{r}_k(p)\left(\frac{1}{\mathrm{r}_k(p) + p^2 + U_k^{(1)}(\rho) + 2\rho U_k^{(2)}(\rho)} + \frac{N-1}{\mathrm{r}_k(p) + p^2 + U_k^{(1)}(\rho)}\right). \tag{1.10}$$

This equation is the LPA form that corresponds to the exact Wetterich equation (1.1). It is given in the constant value of the order parameter $\rho(x) = \rho$ which agrees with the constant field configuration condition mentioned before.

1.2 The Cosmic Inflation and the Initial Conditions Problem

The scenario of cosmic inflation is a main ingredient of the standard model of cosmology. It is assumed to play a crucial role in the evolution of the very early universe, specifically before the beginning of the Big Bang. The following questions emerge: What are the motivations of assuming the inflation model? How far does it give us a better understanding of the very early universe? To know the answer, it is important to realize that the standard cosmological model, or the Friedmann-Lemaitre-Robertson-Walker (FLRW) model, is not sufficient to answer everything we can observe. It does not explain the origin of the homogeneity and the isotropy of the observable universe. It predicts that the early universe was made of causally disconnected regions called space bubbles or casual patches. The size of these patches is determined by the distance that light can be covered in a given time and this is limited by the speed of light in the vacuum. According to that, these patches cannot have the time needed to interact with each other and hence derive the expanding homogeneous and isotropic universe that we observe. This issue is called the horizon problem. As a result, it is difficult to answer the question of why the Cosmic microwave background (CMB) is so uniform, The CMB is the relics of the very early radiation produced immediately after the Big Bang. About 380,000 years after it, the universe was cool enough to allow for the protons and electrons to combine forming the hydrogen atoms and hence decoupling the radiation photons forming the CMB radiation. This radiation was filling the whole cosmic space. The horizon problem implies that the different spots of this CMB radiation never were causally contacted and could not produce the observable highly uniform thermal CMB radiation.

However, the CMB is not perfectly uniform. This raised the problem of the small scale nonuniformity, where the intensity of the CMB has a tiny observable ripple (at the level of 1 part in 100,000). The inflation scenario can give a solution to these problems. In this scenario, the universe began so small that its uniformity easily established [24]. Then the inflation stretched this very small patch of space to be large enough to include the whole observable universe. By assuming this initial small patch was uniform enough, this uniformity will be extended very fast with the inflated cosmic space. Regarding to the nonuniformity ripples, inflation attributes them to quantum fluctuations existed in the very early universe as a direct result of quantum field theory results. Inflation makes general predictions of the spectrum these ripples and how their intensity varies with their wavelength. The measured data have so far agreed with inflation predictions.

The inflation scenario can also address other problems in the standard cosmological model. Among them, the fine-tuning problem is related to the mass density ρ_m of the observable universe. This density is measured by the ratio $\Omega = \frac{\rho_m}{\rho_c}$, where ρ_c is the critical density for which the universe is flat and obeys the Euclidian geometry. The measured value of this ratio is always very close to one. This seems as it was set very precisely to be like that, as any tiny deviation from this value will change significantly the fate of the universe. This fine-tuning problem, which is also called the flatness problem, has a solution in the inflation scenario where most of the inflationary models predict that $\Omega = 1$. A third problem that inflation can address is the magnetic monopoles problem. The Grand Unified Theory (GUT) in particle physics predicts the existence of exotic heavy magnetic monopoles when the universe was very hot and dense. It also predicts that these exotic particles are stable enough to have a measurable abundance in the recent universe. This of course is not what we observe and many attempts tried to find or detect these particles, without success so far. The period of inflation that occurs immediately below the temperature of magnetic monopole production would give a possible solution to this problem: magnetic monopoles would be separated from each other as the early universe expanded during inflation, possibly lowering their observable density by many orders of magnitude. This can suggest why it is very unlikely to observe the existence of these hypothetical particles The first scenario proposed to understand this inflation scenario was that of assuming a metastable false vacuum state suddenly moves to a true vacuum state via quantum tunneling [21].

This true vacuum state emerged as a bubble of space nucleated during the inflation and then expanded at the speed of light. This scenario gives satisfactory solutions to some important cosmological problems, such as the horizon problem, magnetic monopoles problem, and the flatness problem, however, it suffers from some problems in turn. One of these problems is defining the reheating mechanism appropriately. This scenario requires a very long exponential expansion of space during inflation to eliminate the magnetic monopoles and produce our observable universe. This prolonged exponential inflation will make the space bubbles to be very rare and never merging. This causes the collision between the bubble walls not being able to generate radiation. So, the bubble nucleation process will never be complete and the reheating will never have happened [17]. Another scenario of inflation is proposed to fix this problem. This scenario assumed that cosmic inflation is induced by a scalar field, called inflaton, whose vacuum expectation value (VeV) rolls slowly down an inflationary potential, and so this is called the slow-roll inflation model. This model can replace the old one with quantum tunneling of the false vacuum. This model allows to the bubble nucleation to get completed and the reheating could be produced from the oscillations of the field around the minimum of the inflationary potential as illustrated in figure 1.2 [22].

However, this slow-roll inflation scenario introduces an acceptable solution

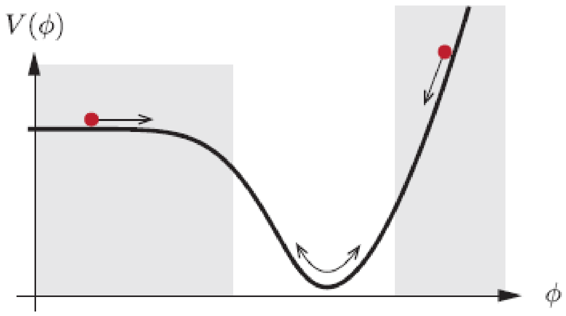

Figure 1.2: Slow-roll inflation potential. Inflation happens in the shaded regions of the potential and the VeV of the inflation field can oscillate around its potential minimum inducing the reheating. From Ref. [22].

to the problems of bubbles nucleation and the reheating-producing mechanism, but it still suffers from an initial conditions problem. One may then ask: why the inflation potential should be tuned initially in that straight shape, such that it provides a well-defined slow roll domain? One of the possible solutions to address this problem is studying the RG running of the inflationary potential and fixing its free parameters from the measuring data of cosmic inflation such as the measurements of the thermal fluctuations of the CMB radiation [23].

Chapter 2

Renormalization Group Running Induced Cosmic Inflation

2.1 RG Flow of Inflationary Models

The initial conditions problem of the slow-roll cosmic inflation can be investigated by assuming an inflationary scenario with a false vacuum state that moved to the slow-roll state via the RG flow of the inflaton scalar potential. In this scenario, the VeV of the inflaton scalar field is trapped in the false vacuum with concave potential at a very high energy scale, the Planck scale, which is higher than the inflation scale, the GUT scale. The RG running of the inflation potential will be less concave at lower energy scales (i.e. the GUT scale) and become eventually convex where it releases the VeV to initiate the slow-roll inflation. So, this scenario can justify the initial conditions of the slow-roll inflation model. This argues about the applicability of this RG flow mechanism to large classes of inflationary potentials with one or more concave regions. This mechanism favors the existence of particle physics origin of the chaotic large field inflation model and eliminates the need for large field quantum fluctuations at the GUT scale that can induce eternal inflation and hence multiverse scenario.

The running momentum scale in this approach is considered to be the inverse of the cosmological time [17], $k \sim t^{-1} = H(t)$, where $H(t)$ is the Hubble parameter. The RG evolution of this inflaton field potential is a consequence of its scale-dependent quantum fluctuations. So, this inflationary scenario allows the existence of quantum fluctuations below the scale of inflation, where the slow-roll process is otherwise considered to be "classical". However, below the scale of inflation, the quantum fluctuations are less important as the false vacuum has already disappeared. The task now is looking for scalar field potentials that can be chosen at the Planck scale, so that when it is evolved down to the cosmological scale ($k \sim k_{GUT} = 10^{16}$ GeV), via the RG running of its couplings, it becomes identical to the slow-roll potential.

From cosmological measurements [23] one can shown that the periodic natural inflationary model seems to provide better agreement than the simple quadratic large field inflationary potential (i.e. $V(\varphi) = \frac{1}{2}m^2\varphi^2$).

The acceptable inflationary models should satisfy the following conditions: (i) the potential has definite lower bounds; (ii) the model has Z_2 symmetry; and (iii) the model has more than one non-degenerate minima with different energy values.

The simplest model that satisfies these three conditions and is non-perturbatively renormalizable at UV scales in four dimensions is the Massive Sine-Gordon (MSG) model. Its potential reads as

$$V_{MSG}(\varphi) = \frac{1}{2}m^2\varphi^2 + u[1 - \cos(\beta\varphi)]. \tag{2.1}$$

The full Eucledian action of this potential is given by

$$S = \int d^4x \left[\frac{1}{2}\left(\partial_\mu\varphi\right)^2 + V_{MSG}(\varphi)\right]. \tag{2.2}$$

A suitable way to consider the RG evolution of the parameters of a given inflationary potential (or in general a given field theory), is the FRG method previously discussed and based on the Wetterich equation [10]. Furthermore, one can try to connect the running momentum scale (k) to the cosmological time (t) as mentioned before to induce time dependence of the potential associated with the momentum scale running. The FRG allows treating the effects of quantum fluctuations by studying the scale dependence of a model via a blocking construction, rescaling the couplings, and the successive elimination of the degrees of freedom that lie above a running momentum cutoff. In the following, we discuss the LPA approach adopted in [17].

The Wetterich RG equation (1.1) can be written for the effective running potential V_k in d-dimensional space as follows

$$k\partial_k V_k = \frac{1}{2}\int \frac{d^d p}{(2\pi)^d}\frac{k\,\partial_k\,\mathrm{r}_k(p^2)}{\mathrm{r}_k(p^2) + \ p^2 \ + \ \partial_\varphi{}^2 V_k}. \tag{2.3}$$

where $V_k = V_k(\varphi)$. Using the LPA form of this equation as given in (1.9), the running equation for the dimensionless quantities can be written as

$$(d - \frac{d-2}{2}\widetilde{\varphi}\,\partial_{\widetilde{\varphi}} + k\,\partial_k)\tilde{V}_k = \frac{1}{2}\int \frac{d^d p}{(2\pi)^d}\frac{k^{1-d}\,\partial_k\,\mathrm{r}_k(p^2)}{\mathrm{r}_k(p^2) + \ p^2 \ + k^{d-2}\,\partial_\varphi{}^2\tilde{V}_k}. \tag{2.4}$$

The dimensionful and dimensionless quantities are related as

$$\tilde{V}_k = k^{-d}V_k,\ \widetilde{\varphi} = k^{-(d-2)/2}\,\varphi, \tilde{u} = k^{-d}u, \tilde{m} = k^{-1}m \tag{2.5}$$

where u and m are the running free parameters of the considered MSG inflationary potential. In the case of spherical symmetry in the d-dimensional space, (2.4) can be written as

$$\left(d - \frac{d-2}{2}\widetilde{\varphi}\,\partial_{\widetilde{\varphi}} + k\,\partial_k\right)\tilde{V}_k = \alpha_d \int_0^\infty dp \frac{p^{d-1}}{k^{2d-3}} \frac{\partial_k\,\mathrm{r}_k(p^2)}{k^{-(d-2)}\,\mathrm{r}_k\,(p^2)\ +\ k^{-(d-2)}p^2\ +\ \partial_\varphi{}^2\tilde{V}_k} \tag{2.6}$$

with $\alpha_d = \frac{\Omega_d}{2(2\pi)^d}$ and Ω_d is the d-dimensional solid angle.

Introducing the change of variables $\mathrm{R}_k\left(p^2\right) = p^2 r(y)$, where $y = p^2/k^2$, then (2.6) can be written as

$$(d - \frac{d-2}{2}\tilde{\varphi}\partial_{\tilde{\varphi}} + k\,\partial_k)\tilde{V}_k = -\alpha_d \int_0^\infty dy \frac{r'\,(y)\ y^{(\frac{d}{2}+1)}}{[1 + r\,(y)]\ y\ +\ \partial_\varphi{}^2\tilde{V}_k}, \tag{2.7}$$

where $r'\,(y) = dr/dy$. By introducing the theta regulator function $r(y) = \left(\frac{1}{y} - 1\right)\Theta\,(1-y)$, also adopted in [16], we have $\int_0^\infty dy \frac{r'(y)\ y^{(\frac{d}{2}+1)}}{\left[1+r(y)\right]\ y\ +\ \partial_\varphi{}^2\tilde{V}_k} = -\frac{2}{d}\frac{1}{1+\partial_\varphi{}^2\tilde{V}_k}$, and equation (2.7) will take the form

$$\left(d - \frac{d-2}{2}\widetilde{\varphi}\,\partial_{\widetilde{\varphi}} + k\,\partial_k\right)\tilde{V}_k = \frac{2\alpha_d}{d}\frac{1}{1 + \partial_{\widetilde{\varphi}}{}^2\tilde{V}_k}. \tag{2.8}$$

To study the RG running of the MSG inflationary potential we have to substitute the expression (2.1) in (2.8). One gets

$$\frac{d}{2}\tilde{m}^2\tilde{\varphi}^2 + d\tilde{u}\ \left[1 - \cos\left(\tilde{\beta}\tilde{\varphi}\right)\right] - \frac{d-2}{2}\left[\tilde{m}^2\tilde{\varphi}^2 + \tilde{u}\tilde{\varphi}\tilde{\beta}\sin\left(\tilde{\beta}\tilde{\varphi}\right)\right] + k\Big\{\partial_k\left(\frac{1}{2}\tilde{m}^2\tilde{\varphi}^2\right)$$

$$+ (\partial_k\tilde{u})\left[1 - \cos\left(\tilde{\beta}\tilde{\varphi}\right)\right] + \tilde{u}\sin\left(\tilde{\beta}\tilde{\varphi}\right)\left[\tilde{\beta}\partial_k\tilde{\varphi} + \tilde{\varphi}\partial_k\tilde{\beta}\right]\Big\} = \frac{2\alpha_d}{d}\frac{1}{1 + \left[\tilde{m}^2 +\ \widetilde{\tilde{u}\beta^2}\cos\left(\tilde{\beta}\tilde{\varphi}\right)\right]}. \tag{2.9}$$

The left-hand side of this equation is a combination of periodic and non-periodic terms, while its right-hand side is a periodic function. By matching first the non-periodic terms, we get

$$\tilde{m}^2\tilde{\varphi}^2 - \frac{d-2}{2}\tilde{u}\,\tilde{\varphi}\,\tilde{\beta}\,\sin\left(\tilde{\beta}\tilde{\varphi}\right) + k\partial_k\left(\frac{1}{2}\,\tilde{m}^2\tilde{\varphi}^2\right) + \tilde{u}\,\sin\left(\tilde{\beta}\tilde{\varphi}\right)\tilde{\varphi}\,\partial_k\tilde{\beta} = 0. \tag{2.10}$$

By treating separately the polynomial and the trigonometric terms of the left-hand side of this equation we get the two differential equations

$$\left(2\tilde{m}^2 + k\partial_k\tilde{m}^2\right)\tilde{\varphi}^2 + k\,\tilde{m}^2\partial_k\tilde{\varphi}^2 = 0, \tag{2.11}$$

$$\left(-\frac{d-2}{2}\tilde{u}\,\tilde{\varphi}\,\tilde{\beta} + \tilde{u}\,\tilde{\varphi}k\partial_k\tilde{\beta}\right)\sin\left(\tilde{\beta}\tilde{\varphi}\right) = 0. \tag{2.12}$$

They have the solutions $\tilde{\beta}_k = \beta\, k^{\frac{d-2}{2}}$ and $\tilde{m}\,_k{}^2 = m^2\, k^{-2}$ for $\tilde{\beta}$ and $\tilde{m}$ respectively. These solutions show that the dimensionful parameters m and β remain unchanged with the RG running. The only dimensionful variable that is changed with the RG running is the Fourier amplitude u, which depends on the cutoff scale k. This allows the inflationary potential to be less concave and transit toward a convex potential when the scale k reduces its value from the high-energy (at Planck scale) value Λ to the low-energy inflationary value $k \sim k_{GUT}$. The RG flow equation for the Fourier amplitude can be obtained by treating the non-periodic terms of equation (1.18) as

$$-\cos\left(\tilde{\beta}\tilde{\varphi}\right)(d + k\partial_k)\,\tilde{u} = \frac{2\alpha_d}{d}\frac{1}{1 + \left[\tilde{m}^2 + \tilde{u}\tilde{\beta}^2\cos\left(\tilde{\beta}\tilde{\varphi}\right)\right]} \tag{2.13}$$

The periodic function in the right-hand side can be expanded as a Fourier series. We can keep the first cosine term only of this series as the left-hand side is a linear function of $\cos\left(\tilde{\beta}\tilde{\varphi}\right)$. So, the RG flow equation for $\widetilde{u}$ can be written as

$$(d + k\partial_k)\,\tilde{u}_k = -\frac{4\alpha_d}{d}\frac{1}{\tilde{\beta}^2\tilde{u}_k}\left(1 - \sqrt{\frac{\left(1 + \tilde{m}_k^{22}\right)}{\left(1 + \tilde{m}_k^2\right)^2 - (\tilde{\beta}^2\tilde{u}_k)^2}}\right), \tag{2.14}$$

which can be written in terms of the dimensionful variables as

$$k\partial_k u_k = -\frac{4\alpha_d}{d}\frac{k^{d+2}}{\beta^2\, u_k}\left(1 - \sqrt{\frac{(k^2\ +\ m^2)^2}{(k^2\ +\ m^2)^2 -\ (\beta^2 u_k)^2}}\right). \tag{2.15}$$

By assuming that $\frac{\beta^2 u_k}{k^2\ +\ m^2}\ \ll\ 1$, near the Gaussian fixed point at high-energy scale, $k\ \sim\ \Lambda$, then we can linearize the right-hand side of the equation (2.14) around this fixed point. So, the linearized RG equation reads as

$$k\partial_k u_k \approx \frac{2\alpha_d}{d}\ k^{d-2}\beta^2\ u_k \tag{2.16}$$

which can be related to the RG equation of massless sine-Gordon model. This equation has the following solution in $d = 4$ dimensions as

$$u_k = u_\Lambda\ \exp\left[\frac{\beta^2}{64\pi^2}\left(k^2 - \Lambda^2\right)\right], \tag{2.17}$$

where u_Λ is the initial value for the Fourier amplitude at the UV scale $\Lambda = \Lambda_p = 2.4{\times}10^{18}$ GeV, which is the Planck scale.

The strategy of the RG analysis here is matching the parameters of the inflation model against astrophysical observations. So, the value of the running Fourier amplitude u_k and also the value of the dimensionful parameters, β and m at the scale of inflation, i.e. the GUT scale, $k \sim k_{GUT} = 2 \times 10^{16}$ GeV, can be matched with the slow-roll parameters (u_0, β_0, m_0) that obtained from the data of observations.

2.2 Slow Roll Inflation and RG Running

Fixing the three parameters (u_0,β_0, m_0) of the MSG model at the scale of inflation requires studying the picture emerging for the inflationary period using this model. In this slow-roll study, it is conventional to represent the dimensionless variables given in the reduced Planck units ($c = G = \hbar = 1$). So, the dimensionless variables of the MSG model of inflation are given by

$$\hat{u} = u_0/{m_p}^4,\ \hat{\beta} = \beta_0 m_p,\ \hat{m} = \ m_0/m_p,\ \hat{\varphi} = \varphi/m_p, \tag{2.18}$$

where $m_p = \Lambda p = \ 2.4{\times}10^{18}$ GeV, and u_0, $\beta_{0,}$ and m_0 are the dimensionful couplings fixed at the scale of inflation, i.e. the GUT scale or below. In Planck units, the slow-roll conditions are the following

$$\epsilon = V^{'2}/(2V)^2 \ll 1,\ \eta = V^{''}/V \ll 1, \tag{2.19}$$

which have to be fulfilled by the considered inflationary potential to give a long enough exponential inflation with the e-fold number given by $N = -\int_{\varphi_i}^{\varphi_f} d\varphi \frac{V}{V'}$ with $50 < N < 60$. So, the slow-roll parameters ϵ and η are given in terms of the free parameters of the given inflationary points, as it is clear from Eq. (2.18). They can be fixed by the observations investigating the power spectra of scalar P_s and tensor P_T fluctuations. This can be characterized by their scale dependence as $P_s \sim k^{n_s-1}$, where k is the comoving wave number. So, slow-roll parameters can be obtained from expressions for the scalar tilt $n_s - 1 = 2\eta - 6\epsilon$ and for the tensor-to-scalar ratio $r = P_T/P_S \approx 16\epsilon$. These later quantities can be compared directly to the cosmological data, such as CMB data, then the slow-roll conditions and the parameters of the inflation model can be fixed directly using these data [24]. To establish this fixing of parameters from the cosmological observations data, let φ_i

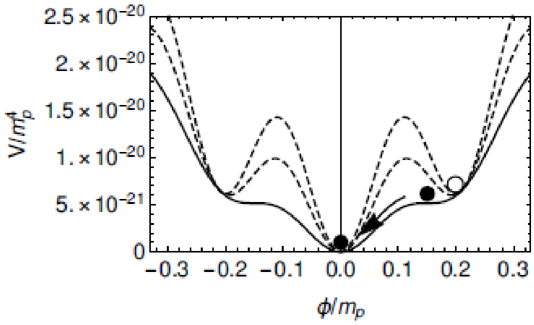

Figure 2.1: The potential of the MSG model (2.1) at different RG scales. The solid line is the potential at the scale of inflation ($k = 2 \times 10^{13}$ GeV) which is below $k_{GUT} = 2 \times 10^{16}$ GeV for $\hat{\beta} = 30$. The normalized parameters are $\hat{m} = 5.4 \times 10^{-10}$ and $\hat{u} = 1.5 \times 10^{-21}$ which yield the solid line that represents the potential over the whole inflationary period, where the VeV rolls down inducing inflation. The dashed lines represent the pre-inflationary UV values obtained by the solution (2.16). From Ref. [17].

denote the VeV of the inflation field at the onset of the slow-roll, where the RG running of the potential will induce the false vacuum to disappear and the slow-roll to begin. We can see that fixing the absolute value of the potential at $\varphi = \varphi_i$ is equivalent to choosing a particular normalization for the value of $u = u_{k,i}$. The scale of inflation can be fixed to the GUT scale k_{GUT} depending on the measured value of the tensor-to-scalar ratio r

The procedure of fixing the couplings from the CMB data has been discussed in more detail in [17, 25, 26]. An example of this fixing of the couplings and the associated RG running of the inflationary potential with decreasing k in the case of the considered MSG model is shown in 2D representation in figure 2.1 and in 3D representation in figure 2.2 [17]. These figures are obtained from the fitting to experimental data for large values of the frequency $\hat{\beta}$, i.e. at $\hat{\beta} \geq 30$.

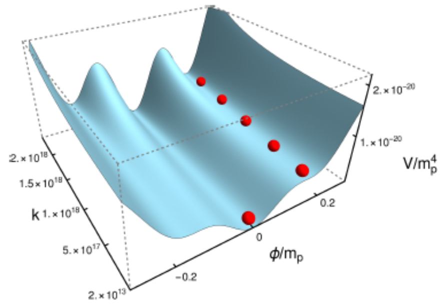

Figure 2.2: The RG running of the MSG potential from the Planck scale towards the inflation scale for $\hat{\beta} = 30$. The MSG potential is fixed at $k = 2 \times 10^{13}$ GeV by analyzing the CMB data for the slow-roll scenario. The VeV is denoted by a red ball which is trapped in a false vacuum at the Planck scale and the RG running releases it at the scale of inflation to roll down slowly to the real ground state of the vacuum. From Ref. [17].

2.3 The Massive Double Sine-Gordon Model

In this section, we will investigate the effects of including higher Fourier harmonics in the trigonometric part of the MSG model of inflation given in Eq. (2.1). The simplest way to do that is by adding the second harmonic term with frequency parameter $2\beta'$ and amplitude u' as

$$V(\varphi) = \frac{1}{2} m^2 \varphi^2 + u \left[1 - \cos(\beta\varphi)\right] + u' \left[1 - \cos(2\beta'\varphi)\right]. \tag{2.20}$$

We refer to the latter as the massive double sine-Gordon (MDSG) model. The task here is to study the RG running of this model and investigate its effects on the results discussed in the previous chapter of the MSG model. To establish that, we have to write the RG flow equation (2.8) for the MDSG potential in terms of the dimensionless variables given in Eq.(2.5). This can be obtained as

$$d\left(\tfrac{1}{2}\tilde{m}^2\tilde{\varphi}^2 + \ \tilde{u}\ \left[1-\cos\left(\tilde{\beta}\tilde{\varphi}\right)\right] + \tilde{u}'\ \left[1-\cos\left(2\tilde{\beta}'\tilde{\varphi}\right)\right]\right) - \tfrac{d-2}{2}\left(\tilde{m}^2\tilde{\varphi}^2 + \tilde{u}\ \tilde{\varphi}\tilde{\beta}\sin\left(\tilde{\beta}\tilde{\varphi}\right)\right)$$
$$+2\tilde{u}'\tilde{\varphi}\tilde{\beta}\sin\left(2\tilde{\beta}'\tilde{\varphi}\right) + k\Big\{\partial_k\left(\tfrac{1}{2}\ \tilde{m}^2\widetilde{\varphi}^2 + (\partial_k\tilde{u})\left[1-\cos\left(\tilde{\beta}\tilde{\varphi}\right)\right]\right) + \tilde{u}\sin\left(\tilde{\beta}\tilde{\varphi}\right)\left(\tilde{\beta}\partial_k\tilde{\varphi} + \tilde{\varphi}\partial_k\tilde{\beta}\right)$$
$$+ (\partial_k\tilde{u}')\left(1-\cos\left(2\tilde{\beta}'\tilde{\varphi}\right)\right) + 2\tilde{u}'\sin\left(2\widetilde{\beta'\varphi}\right)\left(\tilde{\beta}'\partial_k\tilde{\varphi} + \tilde{\varphi}\partial_k\widetilde{\beta'}\right)\Big\}$$

$$= \frac{2\alpha_d}{d}\frac{1}{1+\left[\tilde{m}^2 + \tilde{u}\tilde{\beta}^2\cos\left(\tilde{\beta}\tilde{\varphi}\right) + 4\tilde{\beta}'^2\tilde{u}'\cos\left(2\tilde{\beta}'\tilde{\varphi}\right)\right]} \tag{2.21}$$

The left-hand side of this equation is a sum of periodic and non-periodic terms, while the right-hand side is a periodic function only. So, by treating the non-periodic terms we get

$$\begin{aligned} &\tilde{m}^2\tilde{\varphi}^2 + k\partial_k\left(\frac{1}{2}\ \tilde{m}^2\tilde{\varphi}^2\right) - \frac{d-2}{2}\tilde{u}\tilde{\varphi}\tilde{\beta}\sin\left(\tilde{\beta}\tilde{\varphi}\right) + \tilde{u}\tilde{\varphi}\sin\left(\tilde{\beta}\tilde{\varphi}\right)\ \left(k\partial_k\tilde{\beta}\right) \\ &-(d-2)\,\tilde{u}'\tilde{\varphi}\tilde{\beta}'\sin\left(2\tilde{\beta}'\tilde{\varphi}\right) + 2\tilde{u}'\tilde{\varphi}\sin\left(2\tilde{\beta}'\tilde{\varphi}\right)\left(k\partial_k\tilde{\beta}'\right) = 0 \end{aligned} \tag{2.22}$$

This equation can be split into polynomial terms in $\tilde{m}^2$ and trigonometric terms in $\tilde{\beta}$ and $\tilde{\beta}'$ which give three running equations

$$\left(2\tilde{m}^2 + k\partial_k\tilde{m}^2\right)\tilde{\varphi}^2 + k\ \tilde{m}^2\partial_k\tilde{\varphi}^2 = 0, \tag{2.23}$$

$$\left(2\tilde{m}^2 + k\partial_k\tilde{m}^2\right)\tilde{\varphi}^2 + k\ \tilde{m}^2\partial_k\tilde{\varphi}^2 = 0, \tag{2.24}$$

$$\left(2\tilde{m}^2 + k\partial_k\tilde{m}^2\right)\tilde{\varphi}^2 + k\tilde{m}^2\partial_k\tilde{\varphi}^2 = 0. \tag{2.25}$$

These equations can be solved for $\tilde{m}^2, \tilde{\beta}$ and $\tilde{\beta}'$ as

$$\tilde{m}_k^2 = m^2\ k^{-2}, \tilde{\beta}_k = \beta\ k^{\frac{d-2}{2}}, \tilde{\beta}'_k = \beta'\ k^{\frac{d-2}{2}}, \tag{2.26}$$

which are very similar to the solutions that have been obtained before for $\tilde{m}_k^2$ and $\widetilde{\beta_k}$ in the case of the MSG model with the same feature that the dimensionful parameters m^2 and β stay unchanged during the RG running during changing the scale k. Similarly, as for the MSG model, the RG flow of the dimensionless Fourier amplitudes $\widetilde{u}$ and $\widetilde{u'}$ can be obtained by treating the non-periodic terms of (2.21) as follows:

$$
\begin{aligned}
&-\cos\left(\tilde{\beta}\tilde{\varphi}\right)(d+k\partial_k)\ \tilde{u}-\cos\left(2\tilde{\beta}'\tilde{\varphi}\right)\left(d+k\partial_k\tilde{\beta}'\right)(d+k\partial_k)\tilde{u}' = \\
&\frac{2\alpha_d}{d}\frac{1}{\tilde{m}^2+\tilde{u}\tilde{\beta}^2\cos\left(\tilde{\beta}\tilde{\varphi}\right)+4\left[\tilde{\beta}'^2\tilde{u}'\cos\left(2\tilde{\beta}'\tilde{\varphi}\right)\right]}.
\end{aligned}
\tag{2.27}
$$

In this case, the right-hand side of this equation can be written as a Fourier expansion

$$
\frac{2\alpha_d}{d}\frac{1}{1+\left[\tilde{m}^2+\tilde{u}\tilde{\beta}^2\cos\left(\tilde{\beta}\tilde{\varphi}\right)+4\left(\tilde{\beta}'\right)^2\tilde{u}'\cos\left(2\tilde{\beta}'\tilde{\varphi}\right)\right]}=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}C_{n,m}\cos(n\tilde{\beta}\tilde{\varphi})\cos\left(m\tilde{\beta}'\tilde{\varphi}\right).
\tag{2.28}
$$

We need to keep on the right-hand side are the terms including $\cos\left(\tilde{\beta}\tilde{\varphi}\right)$ and $\cos\left(2\tilde{\beta}'\tilde{\varphi}\right)$. This means we need only the coefficients $C_{1,0}$ and $C_{0,2}$ which take the following forms for the Double Massive sine-Gordon (DMSG) model:

$$
C_{1,0}=\frac{2}{\tilde{u}\tilde{\beta}^2}\left(1-\frac{1+\ \tilde{m}^2}{\sqrt{(1+\ \tilde{m}^2)^2-(\tilde{u}\tilde{\beta}^2)^2}}\right)
\tag{2.29}
$$

$$
C_{0,2}=\frac{2}{4\ \tilde{u}'\ \widetilde{\beta'}^2}\left(1-\frac{1+\ \tilde{m}^2}{\sqrt{(1+\ \tilde{m}^2)^2-16(\tilde{u}'\ \tilde{\beta}'^2)^2}}\right)
\tag{2.30}
$$

Using these equations with equation (2.28), we can get two separate RG flow equations for $\tilde{u}$ and $\tilde{u}'$ as

$$
(d+k\partial_k)\,\tilde{u}_k=-\frac{4\alpha_d}{d}\frac{1}{\tilde{u}_k\tilde{\beta}_k^2}\left(1-\frac{1+\ \tilde{m}^2}{\sqrt{(1+\tilde{m}^2)^2-(\tilde{u}_k\tilde{\beta}_k^2)^2}}\right),
\tag{2.31}
$$

$$(d + k\partial_k)\,\tilde{u}'_k = -\frac{2\alpha_d}{d}\frac{1}{4\,\tilde{u}'_k(\tilde{\beta}'_k)^2}\left(1 - \frac{1 + \tilde{m}^2}{\sqrt{(1+\tilde{m}^2)^2 - 16\left(\tilde{u}'\,(\tilde{\beta}'_k)^2\right)^2}}\right). \tag{2.32}$$

Notice that the two previous equations for $\tilde{u}_k$ and $\tilde{u}'_k$ are uncoupled, since we are linearizing near the UV Gaussian fixed point, see the discussion in [27].

Using the relations between the dimensionless and the dimensionful variables, equations (2.31) and (2.32) can be rewritten as

$$k\partial_k u_k = -\frac{4\alpha_d}{d}\frac{k^{d+2}}{\beta^2 u_k}\left(1 - \sqrt{\frac{(k^2+m^2)^2}{(k^2+m^2)^2 - (\beta^2 u_k)^2}}\right), \tag{2.33}$$

$$k\partial_k u'_k = -\frac{4\alpha_d}{d}\frac{k^{d+2}}{4\,\beta'^2 u'_k}\left(1 - \sqrt{\frac{(k^2+m^2)^2}{(k^2+m^2)^2 - 16\left(\beta'^2 u'_k\right)^2}}\right). \tag{2.34}$$

The first of these equations is the same as equation (2.15) of the RG flow of the running amplitude u_k in the case of the single frequency MSG model, while the second one differs for a factor 4. Again, the dimensionful parameters β and m here are also unchanged under the RG flow. Equations (2.33) and (2.34) can also be similarly linearized around their Gaussian fixed points in the parameter space by assuming $\beta^2 u_k/(k^2 + m^2) \ll 1$ and $4\,\beta'^2 u'_k/(k^2 + m^2) \ll 1$. The linearized equations are

$$k\partial_k u_k = \frac{\alpha_d}{d}k^{d-2}\beta^2 u_k \tag{2.35}$$

$$k\partial_k u'_k = \frac{4\,\alpha_d}{d}k^{d-2}\,\beta'^2 u'_k. \tag{2.36}$$

These equations are the linearized RG flow equations for the MDSG model. Their solutions can be obtained for $d = 4$ as

$$u_k = u_\Lambda\,\exp\left[\frac{\beta^2}{64\,\pi^2}\left(k^2 - \Lambda^2\right)\right], \tag{2.37}$$

$$u'_k = u'_\Lambda \exp\left[\frac{\beta^2}{16\,\pi^2}\left(k^2 - \Lambda^2\right)\right], \tag{2.38}$$

where u_Λ and u'_Λ are the initial values of these Fourier amplitudes at the Planck scale, and they could be possibly evaluated from the equations (2.37) and (2.38) by evaluating their values at the inflation scale using the observational data regarding the slow-roll model of inflation as discussed in [17]. We note the solution (2.37) is the same as (2.17) of the single amplitude MSG model. All of the three solutions have the same analytical form. This means that, both MSG and MDSG models have the same RG flow of their parameters and their potentials are running very similarly with changing the scale k. So, we can expect that, the MDSG potential will also become less concave and more flatten with decreasing k and eventually will tend to be convex and the VeV expectation of the inflation field will release to slow-roll inflation period.

From this analysis we conclude that in general we may expect that any inflationary model with concave regions may induce a slow-roll scenario of the cosmic inflation by applying the mechanism of the RG running. This is also expected to occur with any variant of the inflation model.

As an example, let change the sign of the Fourier amplitude of the potential or adding a constant term, as given in equations (2.39) and (2.40). This will give exactly the same results, and the different variants are still a viable models that can confront with the very recent Planck data of the CMB radiation:

$$V_{MSG}(\varphi) = \frac{1}{2}m^2\varphi^2 + u\left[\cos(\beta\varphi) - 1\,\right], \tag{2.39}$$

$$V_{MSG}(\varphi) = \frac{1}{2}m^2\varphi^2 + u\left[1 - \cos(\beta\varphi)\,\right] - V_0. \tag{2.40}$$

The constant term V_0 in equation (2.40) is often not studied in particle physics - however, in statistical field theory has an important role as the free energy.

In cosmology this term is the field-independent term and its treatment its non-perturbative renormalization running using the FRG method can be very helpful in investigating the problem of the cosmological constant. The importance here of the RG method is in providing us with a suitable way to get the emergency of the vacuum from the false state to the slow-roll inflation state. Since, the large field inflationary models appear to be reasonably supported by measurements data, they have the weakness of their large initial value of the VeV near the Planck scale. This produces some difficulties with getting a consistent mechanism of inflation. It is also difficult to support these models by a particle physics model because of the dominance of the very high quantum fluctuation between the GUT scale and the Planck scale. So, the RG running of the potential of these models eliminates the need for these very high fluctuations and left it affordable to be supported by a particle physics model even at the GUT scale at which the slow-roll inflation starts to happen. This is the topics of the next chapter.

Chapter 3

Renormalization of the Field-Independent Term and Its Cosmological Implications

3.1 FRG of The Field Independent Term

In considering the non-perturbative renormalization of a given field theory, the RG evolution of the field-independent scaling term requires special care. This is because of the construction of the non-perturbative approach implied by the Wetterich equation, which allows the bare action to be reproduced up to a constant field-independent term at the UV scale as given by (2.20).

In many relevant cases, the field independent term has important applications. For example, this term represents the ground state energy of a $0 + 1$-dimensional quantum anharmonic oscillator. In the higher dimensional case, like Einstein gravity with $d = 4$, this term can be associated with the cosmological constant term Λ appearing in Einstein equations and can play a role in cosmic inflation. In this chapter, we are interested in studying the RG running of the field-independent term as the cosmological constant in the non-perturbative analysis of Einstein gravity theory with asymptotic safety at the UV energy scale. In the standard model of cosmology (ΛCDM), the cosmological constant is static with a fixed value that remains constant over the history of the universe. This is necessary to describe the accelerated expansion of the recent observable universe. The energy value corresponding to this cosmological constant can be put in relation to what is called dark energy. So, extrapolating back in time to the early universe, this energy has a very small value in comparison to the Planck scale. It would be more natural for the cosmological constant to start in the early universe with a value of its energy density in the same order as the matter and radiation densities. This gives rise to the fine-tuning problem. To overcome this problem, the concept of quintessence is introduced which assumes a scalar field similar to inflaton with negative pressure but with a very large wavelength. This quintessence scalar field is described by

$$S = \int d^4x \sqrt{-g}\left[\frac{{m_p}^2}{2}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\,\partial_\nu\varphi - V(\varphi)\right]. \tag{3.1}$$

The equation of state of the quintessence is dynamic and given by

$$\omega = \frac{\frac{1}{2}\dot{\varphi} - \ V(\varphi)}{\frac{1}{2}\dot{\varphi} \ + \ V(\varphi)}, \tag{3.2}$$

which has been also used for the inflaton case. So, it does not matter whether one considers the quintessence or the cosmological constant; the field-independent term of the action is involved, and its value can be evaluated. The standard model of cosmology includes two periods of accelerated expansion: In the early universe when the universe doubled its size in $10^{-35}s$ during the inflation, and then the accelerated expansion of the observable recent universe.

The value of the field-independent term has to be fine-tuned in each period of accelerated expansion. Thus, a reliable theory should take into account the evolution of this term over time. This means the variation of the cosmological constant in time can be investigated by flow RG of the field-independent term when the momentum scale k is changing. The The non-perturbative RG analysis can be used to determine the evolution of the self-interaction potential from the UV scale (Planck scale or over the Planck scale) to the scale of inflation and towards the low-energy (IR) limit of the recent universe. The RG running of the constant field-independent term can be established with the asymptotic safety scenario of quantum gravity which is based on the existence of a non-Gaussian fixed point at the UV scale. In this context, the running cosmological constant can be realized using the Einstein-Hilbert truncation of the effective average action of a matter free theory

$$\Gamma_k = \frac{1}{16\pi G_k}\int d^4x\,\sqrt{-g}\,[R-2\Lambda_k]\,, \tag{3.3}$$

where the scale-dependent parameters G_k and Λ_k are the Newton coupling and the cosmological constant, respectively, and their corresponding dimensionless couplings are $g_k = k^2\mathrm{G}_k$ and $\lambda_k = k^{-2}\Lambda_k$. In this case, the scaling field-independent term is related to the scaling cosmological constant parameter as $\mathrm{V}_k(\varphi=0) = {m_p}^2\Lambda_k$, where ${m_p}^2 = \frac{1}{8\pi G}$ is the Planck mass. The RG running of these dimensionless couplings is given by [18]

$$k\partial_k g_k = \beta_{\mathcal{G}}, \qquad k\partial_k\lambda_k = \beta_\lambda. \tag{3.4}$$

The β-functions $\beta_{\mathcal{G}}$ and β_λ are given by

$$\beta_{\mathcal{G}} = (2+\eta_N)g_k, \beta_\lambda = (\eta_N-2)\ \lambda_k + \frac{g_k}{12\pi}\left[\frac{30}{1-2\lambda_k} - 24 - \frac{5}{1-2\lambda_k}\eta_N\right], \tag{3.5}$$

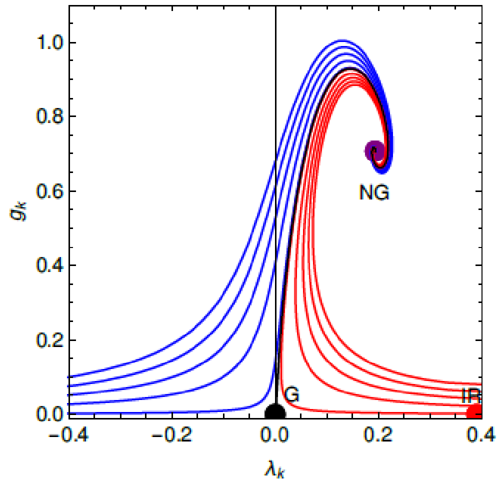

Figure 3.1: The RG flow of the couplings g_k and λ_k of the asymptotically safe quantum gravity theory based on the Einstein-Hilbert truncation given in equation (3.3). From Ref. [18].

where $\eta_N = G_k^{-1} k \partial_k G_k$ is called the anomalous dimension of Newton's constant. The numerical analysis of these β-functions gives rise of two fixed points in the parameter space of the couplings: The Gaussian fixed point at the origin $(\mathcal{G}_*, \lambda_*) = (0, 0)$, and a non-Gaussian fixed point $(\mathcal{G}_*, \lambda_*)$, where the particular numerical values of $\mathcal{G}_*$ and λ_* are found to be given by $(\mathcal{G}_*, \lambda_*) = (0.707, 0.193)$, as shown in figure 3.1 [18].

3.2 RG evolution of the constant term with minimal coupling

As mentioned before, assuming that the running momentum scale k is (proportional to) the reciprocal of the cosmic time t is the essential idea of using the RG flow to the value of the couplings at various energy scale. In this way the cosmological constant problem can be investigated using the non-perturbative approach of quantum gravity theory with asymptotic safety. So, the aim is to study the RG flow of the field-independent term in the presence of scalar fields, so that the coupling constants in Eq.(3.1) can b considered to be similar to the running parameters in Eq. (3.3).

There are two possible cases of including the scalar field in curved space: (i) minimal coupling; and (ii) non-minimal coupling of the scalar field with gravity. In this section the first case of minimal coupling is considered. The running effective average action is given by

$$\Gamma_k[\varphi] = \int d^4x \sqrt{-g} \left[\frac{1}{16\pi G_k} R - \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \, \partial_\nu \varphi - V_k(\varphi) \right], \tag{3.6}$$

where the scalar potential can be expanded as a polynomial in terms of field as

$$V_k(\varphi) = V_k(0) + \sum_{n=1}^{\infty} \frac{1}{(2n)!} \lambda_{2n,k} \varphi^{2n}. \tag{3.7}$$

This expansion refering to the potential V expresses that the latter should be bounded from below and an even function for symmetry reasons. If this expression is truncated at the quartic term, it can be written as

$$V_k(\varphi) = V_k(0) + \frac{1}{2} {m_k}^2 \varphi^2 - \frac{1}{4!} \lambda_{4,k} \varphi^4, \tag{3.8}$$

where $V_k(0) = \frac{2\Lambda_k}{16\pi G_k}$ is the field-independent term, which we are interested in studying its RG running. The RG equation for the scalar field potential $V_k(\varphi)$ can be given in the LPA approximation by using the Litim cutoff obtaining in d-dimensions

$$k\partial_k V_k(\varphi) = \mu_d k^d \frac{k^2}{k^2 + {\partial_\varphi}^2 V_k(\varphi)}, \tag{3.9}$$

with $\mu_d = \frac{1}{[(4\pi)^d \, \Gamma(\frac{d}{2}+1)]}$. The RG equation for the constant term $V_k(0)$ could in principle given using Eq. (3.9) in $d = 4$ dimensions by

$$k\partial_k V_k(0) = \frac{k^4}{32\pi^2}\frac{k^2}{k^2 + \partial_\varphi{}^2 V_k(\varphi)\,|_{\varphi=0}}. \tag{3.10}$$

Using the relations (2.5) we can write the dimensionless constant term in terms of dimensionless coupling as $\tilde{V}_k(0) = \frac{2\lambda_k}{16\pi g_k}$. Thus, the RG of $\tilde{V}_k(0)$ can be written as

$$k\partial_k \tilde{V}_k(0) = \frac{1}{32\pi^2}\left(\frac{1}{1+\tilde{m}_k{}^2}\right) - 4\,\tilde{V}_k(0). \tag{3.11}$$

From this equation one can write the RG equation for the dimensionless cosmological constant λ_k as

$$k\partial_k \lambda_k = \frac{1}{4\pi} g_k \left(\frac{1}{1+\tilde{m}_k{}^2}\right) - 2\,\lambda_k. \tag{3.12}$$

In the absence of quantum gravity effects, i.e. the Newton constant is assumed to be scale-independent, $\mathrm{G}_k = G$, the anomalous dimension is vanished, $\eta_N = G_k{}^{-1} k\partial_k \mathrm{G}_k = 0$, and the dimensionless coupling g_k has the trivial RG scaling $g_k \sim k^2$ as the solution of the RG equation $k\partial_k g_k = 2g_k$. In this approximation g_k diverges to infinity in the UV limit. The UV scaling of the dimensionless mass parameter is $\tilde{m}_k{}^2 \sim k^{-2}$, so it tends to vanish in the UV limit. This means the β-function in Eq. (3.12) is diverging: $\frac{1}{4\pi} g_k \left(\frac{1}{1+\tilde{m}_k{}^2}\right) \to \infty$ in the UV limit $k \to \infty$. Thus, in the UV limit the Gaussian fixed point here cannot be reached, although it is formally exist as discussed in the previous section. The reason of this is the presence of the constant term of the scalar potential. In other words, the β-function of the constant term should be vanishing if the all other couplings are equal to zero. This is not what is found with equation (3.12) and, at variance, it is what is found, with equation (3.3) of the Einstein-Hilbert truncation without any scalar field potential.

This shows that one has to apply a regularization by adding subtraction terms to restore the Gaussian fixed point in Eq.(3.12). These subtraction terms should not influence the existence of the non-Gaussian fixed point in the UV scale of the quantum gravity theory including scalar fields. To get an idea about the subtraction terms that should be considered to restore the Gaussian fixed point, we can investigate the solution of equation (3.10) at the UV scale, where $k^2 \gg \partial_\varphi{}^2 V_k\,(0)$ $\partial_\varphi{}^2 V_k\,(0) = \partial_\varphi{}^2 V_k\,(\varphi)\,|_{\varphi=0}$. In this limit the solution of equation (3.10) is

$$V_k\,(0) = V_\Lambda\,(0) + \frac{k^4 - \Lambda^4}{128\,\pi^2} \tag{3.13}$$

This solution indicates a "rampant" quartic divergence of $V_k\,(0)$ in the UV limit with large value of the cosmological constant Λ. This leads to a considerable change of $V_k\,(0)$ between the Planck scale and the GUT scale of the inflation, and this will require some fine-tuning of the inflation model at the Planck scale. Similar divergencies appear in other problems, such as the $d = 1$ quantum anharmonic oscillator where the field-independent constant term gives the ground state energy of the system. This system features a k^2 divergency of its ground state energy in the harmonic limit at the UV scale. In this case adding subtraction terms recovers the physical convergent ground energy and shows that the divergent term is spurious.In the case of the cosmological constant in $d = 4$ space, a similar subtraction scheme can be used to retain the Gaussian fixed point. It is important to note that the k^4 divergency in the latter case may be need more than one subtraction [18].

The following consideration can be imposed on the subtracted RG evolution equation of the constant term: (i) To retain the Gaussian fixed point, the RG evolution of the constant term $V_k\,(0)$ must vanish if the other coupling of the model are setting to be zero; (ii) The subtractions cannot influence the IR behavior of the RG flow, as they originated purely from the UV behavior of the regulator function used in formulating Wetterich equation where $\mathrm{R}_{k\to\Lambda}\,(p) = \infty$. The first subtraction can be applied by subtracting $\frac{k^4}{32\pi^2}$ in equation (3.10) as detailed in the following:

$$k\partial_k V_k\,(0) = \frac{k^4}{32\pi^2}\frac{k^2}{k^2 + \partial_\varphi{}^2 V_k\,(0)} \to \frac{k^4}{32\pi^2}\left[\frac{k^4}{32\pi^2}\;\frac{k^2}{k^2 + \partial_\varphi{}^2 V_k\,(0)} - 1\right]. \tag{3.14}$$

At the UV scale this equation can be written as

$$k\partial_k V_k\,(0) \to -\frac{k^2}{32\pi^2}\partial_\varphi{}^2 V_k\,(0)\,,\;\; k \to \infty, \tag{3.15}$$

which still has the quadratic term k^2 in its right-hand side in the UV limit. This subtraction also modifies the RG evolution of the cosmological constant as

$$k\partial_k\lambda_k = \frac{1}{4\pi}g_k\left(\frac{1}{1 + \tilde{m}_k{}^2} - 1\right) - 2\,\lambda_k. \tag{3.16}$$

Here, the β-function does not diverge in the UV limit but converge to a nonzero negative value if we keep in mind the scaling relations $\tilde{m}_k{}^2 \sim k^{-2}$ and $g_k \sim k^2$ in the UV limit. In equation (3.15) the RG evolution vanishes in the UV limit if the model couplings are vanishing: $\partial_\varphi{}^2 V_k\,(0) = 0$, therefore satisfying the first consideration imposed above. However, equation (3.13) shows that if k goes to infinity faster than the couplings go to zero, the β-function in equation (3.15) will not be vanich. So, the β-function does not approach uniformly to zero in the UV limit. This behavior implies that another subtraction is needed if we assume that the couplings should have uniform convergence to zero at k goes to infinity. This second subtraction can be applied as

$$k\partial_k V_k\,(0) = \frac{k^4}{32\pi^2}\left[\frac{k^4}{32\pi^2}\;\frac{k^2}{k^2 + \partial_\varphi{}^2 V_k\,(0)} - 1 + \frac{k^2}{32\pi^2}\partial_\varphi{}^2 V_k\,(0)\right]. \tag{3.17}$$

This RG evolution equation is still safe in the IR limit because of the k^4 prefactor and in the UV limit it can be written as

$$k\partial_k V_k(0) \to \frac{\left[\partial_\varphi{}^2 V_k(0)\right]^2}{32\pi^2}, k \to \infty. \tag{3.18}$$

This equation is vanishing uniformly in the UV limit if the couplings are set equal to zero. The corresponding RG evolution of λ_k is given by

$$k\partial_k \lambda_k = \frac{1}{4\pi} g_k \left(\frac{1}{1 + \tilde{m}_k{}^2} - 1 + \tilde{m}_k{}^2 \right) - 2\,\lambda_k, \tag{3.19}$$

from which we observe that the factor $g_k \left(\frac{1}{1+\tilde{m}_k{}^2} - 1 + \tilde{m}_k{}^2 \right)$ vanishes at $k \to \infty$ and the β function is vanishing in the UV limit.

If the scaling of the mass parameter is neglected, i.e. $m_k \to m$, the solution of the RG flow equation can be obtained as

$$V_k(0) \sim \frac{1}{32\pi^2} m^4 \log(k)\,, \quad k \to \infty, \tag{3.20}$$

which is diverging logarithmically in the UV limit. Therefore, applying a double-subtraction scheme here can alleviate the divergency of the constant (field-independent term) at the UV scale and retain the Gaussian fixed point of the running cosmological constant parameter.

3.3 RG evolution of the constant term with non-minimal coupling

In the previous section the case of minimal coupling of the scalar field with the curved background in $d = 4$ spacetime has been considered and the RG running of the cosmological constant parameter studied. In this section we are going to consider the non-minimal coupling case and investigate the RG evolution of the field-independent term and the cosmological constant parameter. To establish that, a further term of the non-minimal coupling $F(\varphi)R$ should be considered in the running effective action and in elaborating Wetterich equation [28]. We will assume here the non-minimal coupling $F(\varphi)R = \frac{1}{2}\xi R\varphi^2$ studied in [29], where ξ is the non-minimal coupling parameter running with the the scale k in the non-perturbative RG approximation. In [28] and [29], the β functions of the running potential $V_k(\varphi)$ and the running parameter ξ_k were derived and the fixed points determined numerically. In those references, the β-function for the field-independent term and hence the cosmological constant problem were not investigated in the case of the non-minimal coupling. This is what we will do in this section by investigating the influence of including the non-minimal coupling term on the RG evolution of the constant field-independent term.

By assuming that there is no wave function renormalization, the effective average action including the non-minimal coupling term can be written in the form

$$\Gamma_k[\varphi] = \int d^4x\sqrt{-g}\left[\frac{1}{16\pi G_k}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\,\partial_\nu\varphi - u_k(\varphi)\right] + {\Gamma_k}^{grav}[g], \qquad (3.21)$$

where ${\Gamma_k}^{grav}[g]$ is the purely gravitational part of the effective action.

The term $u_k(\varphi)$ is the field-dependent part of the scaling potential $V_k(\varphi)$, where $V_k(\varphi) = V_k(0) + u_k(\varphi)$. The field-independent term $V_k(0)$ is included in the gravitational term ${\Gamma_k}^{grav}[g]$, where $V_k(0) = \frac{2\Lambda_k}{16\pi G_k}$. The Wetterich equation for the effective action reads as

$$\partial_t\Gamma_k = \frac{1}{2}Tr\left(\frac{\partial_t R_k}{{\Gamma_k}^{(2)}[\varphi] + R_k}\right). \qquad (3.22)$$

The right-hand side of this equation can be obtained by extending equation (31) given in [28] to include the higher derivative term of R. This can be established by the $n = 0, 1, 2$ terms of the series given in [28]. Omitting details and comparing the φ- dependent terms, one gets for the RG evolution of the purely gravitational part

$$\partial_t \Gamma_k{}^{grav} = -\frac{1}{32\pi^2 M_k}\int d^4x\sqrt{-g}\,\left[k^6\left(1-\frac{\eta_k}{6}\right)+\frac{k^4}{3}\left(1-\frac{\eta_k}{4}\right)R+\frac{k^2}{3}\left(1-\frac{\eta_k}{2}\right)\Delta_g R\right]$$
$$\left[1-\frac{\xi_k R}{M_k}+\left(\frac{\xi_k R}{M_k}\right)^2-\dots\right], \tag{3.23}$$

where $M_k = k^2 + m_k{}^2$ and η_k is the anomalous dimension of the wave function renormalization. We neglect the latter in the considered case of an even potential with $\mathbb{Z}_2$ symmetry, We observe that the operator Δ_g corresponds to the mass squared in the case of the on-shell corrections, then Δ_g can be replaced by $m_k{}^2$. Furthermore, the terms containing $R^2, R^3, \dots$ must be omitted as they go beyond the considered Einstein-Hilbert truncation. So, replacing the gravitational action by Eq.(3.3) and considering the absence quantum gravity effects Eq.(3.23) reads as

$$k\,\partial_k V_k(0) = \frac{k^4}{32\pi^2 M_k}\left[k^2\left(1-\frac{\xi_k R}{M_k}\right)+\frac{R}{3}\left(1+\frac{1}{k^2}\right)\right], \tag{3.24}$$

where we used $\partial_t = k\partial_k$. This equation is the counterpart of equation (3.10) of the RG evolution of the field-independent term after including the non-minimal coupling. It can be rewritten for the dimensionless field-independent term $\tilde{V}_k(0)$ as

$$k\partial_k \tilde{V}_k(0) = \frac{1}{32\pi^2 \tilde{M}_k}\left(1-\frac{\xi_k R}{k^2\tilde{M}_k}+\frac{R}{3}\frac{1+k^2}{k^4}\right)-4\,\tilde{V}_k(0)\,. \tag{3.25}$$

For the dimensionless cosmological constant we get

$$k\partial_k \lambda_k = \frac{1}{4\pi\tilde{M}_k}g_k\left(1-\frac{\xi_k R}{k^2\tilde{M}_k}+\frac{R}{3}\frac{1+k^2}{k^4}\right)-2\,\lambda_k, \tag{3.26}$$

where $\tilde{M}_k = k^{-2}M_k = 1+\tilde{m}_k^2$ is a dimensionless mass-like parameter. One can observe the correspondence between the last two equations and their counterpart provided by equations (3.10) and (3.11) for the minimal coupling case given in [18].

In the case of the non-minimal coupling, the RG evolution equation for $\tilde{V}_k(0)$ and λ_k includes both the R and ξ_k. Thus, the evolution equations have explicit dependence on the curvature of the background space unlike the minimal coupling case, as expected. The UV behavior of the field-independent term can be investigated in the limit $k^2 \gg {m_k}^2$, where $M_k \sim k^2$, and the solution of equation (3.24) is given by

$$\begin{aligned} V_k(0) = V_\Lambda(0) + \frac{1}{128\,\pi^2}\left(k^4 - \Lambda^4\right) + \frac{1}{64\,\pi^2}\left({m_k}^2 - \xi_k R + \frac{R}{3}\right)\left(k^2 - \Lambda^2\right) \\ + \frac{2{m_k}^2 R}{96\,\pi^2}\ln\left(\frac{k}{\Lambda}\right). \end{aligned} \tag{3.27}$$

The Gaussian and the non-Gaussian fixed points of the RG flow for a scalar field on curved background have been investigated in [28, 29]. The existence of k^4, k^2 and logarithmic divergences for the running field-independent term is in a close agreement with that given in [30, 31], where a generalized Schwinger-DeWitt adiabatic renormalization is proposed in curved space which obtained RG running equations for the cosmological constant, as that one given by equation (14) in [31]. This equation can be also compared with Eq.(5.9) in [19], where the authors considered a non-minimal coupling case with unchanging coupling, i.e. $\xi_k = \xi$.

From equation (3.26) we see that the β-function diverges at the UV scale, where $g_k\left(1 - \frac{\xi_k R}{k^2 \tilde{M}_k} + \frac{R}{3}\frac{1+k^2}{k^4}\right) \to \infty$ if $k \to \infty$. Thus, the Gaussian fixed point cannot be reached.

Therefore we need to apply subtractions to restore the Gaussian fixed point, as was done in the case of the minimal coupling. We can start by applying the same first subtraction used in the case of the minimal coupling, i.e. by adding $-\frac{k^4}{32\pi^2}$ to equation (3.24). After taking the UV limit and then setting the couplings m_k and ξ_k equal to zero, we get

$$k\,\partial_k V_k(0) \to \frac{1}{32\pi^2}\left[2k^4 + \frac{R}{3}\left(1 + k^2\right)\right], \quad k \to \infty, \tag{3.28}$$

which is diverging in the UV limit. This violates the first consideration imposed on the RG evolution equation to restore the Gaussian fixed point in the UV limit. So, this suggests the first subtraction proposed in [18] is not applicable to restore the Gaussian fixed point in the case of the non-minimal coupling. However, we can get a similar subtraction method by carefully investigating the behavior of the β- function on the right-hand side of (3.24) in the UV limit where we get

$$k\partial_k V_k(0) \to \frac{1}{32\pi^2}\left[k^4 + \left({m_k}^2 - \xi_k R + \frac{R}{3}\right)k^2 + \frac{2{m_k}^2 R}{3}\right], \quad k \to \infty, \tag{3.29}$$

which explicitly has the same divergences obtained before. So, we can estimate a subtraction term consisting from the divergent terms in (3.29). We can then add a subtraction term $-\frac{1}{32\pi^2}\left[k^4 + \left({m_k}^2 - \xi_k R + \frac{R}{3}\right)k^2\right]$ to equation (3.24) as

$$k\,\partial_k V_k(0) = \frac{k^4}{32\pi^2 M_k}\left[k^2\left(1 - \frac{\xi_k R}{M_k}\right) + \frac{R}{3}\left(1 + \frac{1}{k^2}\right)\right] - \frac{1}{32\pi^2}\left[k^4 + \left({m_k}^2 - \xi_k R + \frac{R}{3}\right)k^2\right]. \tag{3.30}$$

In the UV limit one gets

$$k\partial_k V_k(0) \to \frac{1}{32\pi^2}\left[-2{m_k}^2 k^2 + 2{m_k}^2 R\xi_k - 3{m_k}^4\right], \quad k \to \infty, \tag{3.31}$$

which is vanishing in the UV limit even if the couplings m_k and ξ_k are set to zero. This satisfy the conditions imposed on the RG evolution of the field-independent constant term. The previous equation can be then considered the counterpart of

(3.15), obtained for the minimal coupling case after applying the first subtraction considered in [18]. The same non-uniformly vanishing approach in the UV limit can be seen again in Eq. (3.30) if k goes to infinity faster than m_k goes to zero. To avoid this, a second subtraction can be done as proposed in [18].

The second subtraction to be considered here is provided by the divergent term in the limit (3.31), i.e. adding the term $-\frac{1}{32\pi^2}\left(-2{m_k}^2k^2\right)$ to equation (3.30). In the UV limit this gives

$$k\partial_k V_k\left(0\right) \to \frac{1}{32\pi^2}\left[2{m_k}^2R\xi_k + {m_k}^4\right], \quad k \to \infty, \tag{3.32}$$

which can be solved as

$$V_k\left(0\right) \sim \frac{1}{32\pi^2}\left[2{m_k}^2R\xi_k + {m_k}^4\right]\ln\left(k\right). \tag{3.33}$$

We observe that, by setting $\xi_k = 0$ in this equation we can restore Eq. (3.18) of the minimal coupling. This implies that the double subtraction procedure proposed in [18] when applied to the present case with non-minimal coupling can give results comparable with those obtained for the minimal coupling case and can led to restore the UV Gaussian fixed point in the case of considering the Einstein-Hilbert truncated effective action.

Conclusions

The main focus of this thesis has been the study of how the Renormalization Group (RG) scaling could be useful in addressing some important cosmological problems related either to the recent universe or to the early universe. The motivation of that is the possibilty using RG techniques to study the behaviour of systems at different energy scales, which has been applied to quantum field theories to describe the fundamental interactions of the nature and to develop our understanding of critical phenomena and phase transitions of matter. The idea of relating the macroscopic large-scale properties of a system to its small-scale fluctuations has proven its great importance in many areas of physics. So, it is natural to investigate the applicability of this method to cosmological problems. As the gravitational force is the dominant in the cosmological phenomena, even in the early universe, the perturbative RG methods may turn on in several instances not very useful. This is due to the fact that the theory of gravity base on Einstein general relativity is not perturbatively renormalizable in its quantized formulation. So, it is important to treat its non-perturbative renormalizability or asymptotic safety behavior. This kind of non-perturbative investigation is also important in studying some other problems in high energy physics and condensed matter physics that cannot be understood perturbatively. The non-perturbative approach can be performed using the Functional RG (FRG) method based on the Wetterich equation of scaling the effective average action, as reminded in Chapter 1.

In Chapter 2, using symmetry considerations, we discussed how the Massive sine-Gordon (MSG) model can be adapted as a UV completion of the quadratic large field model of inflation. Studying the FRG evolution of the scalar potential of MSG model shows that any initial concave setting of this potential will become more convex decreasing the energy scale of the universe. This makes the inflationary potential to be more flat near the inflation energy scale (Grand Unified Theory scale) which allows the slow-roll inflation to begin. This gives a solution to the "initial conditions problem" and explains how the vacuum expectation value of the scalar field can exit from the trapping false vacuum to the true vacuum during the inflation. This also avoid the possible difficulties with the chaotic large field inflation, where the quantum fluctuations at the high energy scales of inflation or above can result in eternal inflation [17]. Also, this avoids the more problematic issue of small-field inflation, where the interactions homogenize the universe at scales larger than the horizon. The running parameters of the MSG model can be fixed against the measurements data by considering the slow-roll scenario of inflation.

These results are heuristic in the sense that we have to consider a specific inflation model in the beginning of the inflation period. If this is certainly a highly desirable objective, the main goal of this study has been not proposing a model of inflation, but rather to justify how the initial conditions of this model could be obtained from the metastable false vacuum state. Extending the same analysis to include higher Fourier harmonics in the MSG potential is considered as well in Chapter 2. Considering the second harmonic, and hence the Double Massive sine-Gordon (DMSG) model potential, we find that there is no any significant difference from the MSG model. The parameters of the DMSG model run very similarly as those of the MSG model. They have the same β-functions of the RG evolution. They have also the same solution and the degree of the divergency in the UV scale. This concludes that the DSGM model can be treated similarly in fixing its running parameters with the measurement data and its convexity is also increasing with decreasing the running scale. However, it might be still interesting to investigate the general case of including the all higher harmonics and investigate the influence of that on the RG evolution of the inflation potential.

Finally, in Chapter 3 we study the field independent term. The latter is often not considered in field theory, but at the same time is sometimes important in understanding several phenomena. In statistical field theory it plays the role of the free energy, while in the curved spacetime it represents the cosmological constant term. This correspondence motivates using the same subtractions to remove the spurious UV divergences. The truncated Einstein-Hilbert effective action is considered in some simple cases. By assuming asymptotic safe quantum gravity theory, the action is given in the case of a not running metric tensor, with minimal coupling and at the same time quantum gravity effects neglected. Even with these approximations, it gives an acceptable RG evolution of the cosmological constant. Using subtraction parameters it is possible to obtain non-divergent vacuum energy density in the UV limit. We then considered as original work the more complex case of the non-minimal coupling between the scalar field and the curved spacetime, which is investigated in Section 3.3. Although the RG evolution equations obtained in the non-minimal case are whown to have an explicit correspondence with the original equations of the minimal coupling case, the same subtractions seem to be not applicable here. Applying the same subtractions does not give RG evolution equations with vanishing β-functions in the UV limit if the all coupling are set to be zero. This is a necessary condition to restore the Gaussian fixed point of the considered quantum gravity theory in the UV limit. However, similar subtractions could be implemented by investigating the divergent terms of the modified RG evolution equation in the UV limit. So, this results a modified double subtraction scheme, which we propose and show to successfully work.

Similar results can be obtained for the field-independent term in the UV limit after applying this modified subtraction scheme. However, it appears that pushing this procedure to the simplest case beyond the Einstein-Hilbert truncation(3.3), i.e. by including R^2 terms in the effective action and in both sides of (3.23), is not able to restore the Gaussian fixed point in a similar way. So, this indicates that, the renormalization programe supposed in this work in the case of the non-minimal coupling can be considered as an approximation of more general quantum field theories in curved background space, e.g. those discussed in [19], where higher terms and derivatives of the curvature beyond Einstein-Hilbert truncation (3.3) are considered in order to get a more realistic renormalization scheme in studying quantum field theory in curved spacetime. In the case of a 4-dimensional space, the considered higher terms are those of the square of curvature scalar and Ricci tensor, i.e. R^2 and $R^{\mu\nu}R_{\mu\nu}$, and this needs different subtraction schemes, which we think to be very interesting to be studied in the future. This possible future investigations lies in the general program of exploring the role that the RG technique can play in studying the interactions between quantum fields and gravity at different scales, which is an essential task in the implmentation of any viable quantum gravity theory.

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